Trigonometric Tutorial: Pythagorean Theorem, Rectangular Coordinates of Circular Arc Points, Chord Lengths of Arcs, and Key Calculus Features of the Cosine and Sine Functions

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Abstract

Trigonometry studies the properties of the cosine and sine functions, which relate a contiguous arc of the unit-radius circle centered on the origin of coordinates to the rectangular Cartesian coordinates of the arc's endpoints. Since the Pythagorean theorem underlies the concept of Cartesian coordinates, this tutorial commences with a plane-geometry recapitulation of that theorem. In the non-calculus treatment of the cosine and sine, their demonstrable properties are encompassed by the unit length of unit-radius circle vectors and the "angle-addition formula" which relates the rectangular coordinates of the endpoints of two immediately successive arcs of the unit-radius circle to the rectangular coordinates of the endpoints of the combined contiguous arc. Those properties are insensitive, however, to simultaneous single-parameter rescaling of all of the arc lengths involved, and so don't unambiguously characterize the cosine and sine functions of directed arc length. Unambiguous determination of the cosine and sine hinges on whether their derivatives with respect to directed arc length are well-defined, which presents no issues for arcs of the unit-radius circle. In fact the cosine and sine functions fascinatingly are the real and imaginary parts of the hyper-well-behaved exponential function of imaginary argument.

Review of the Pythagorean theorem in plane geometry

Some plane geometry texts gloss over the Pythagorean theorem *without mentioning its centrality to Cartesian coordinates*, or *emphasizing* that it *follows* from the equality of the ratios of the corresponding side lengths *of three particular similar right triangles*. It therefore seems worthwhile *to reprise its demonstration here*.

Given a right triangle whose two legs have lengths denoted l_1 and l_2, and whose hypotenuse has length denoted h, we construct the the line segment perpendicular to its hypotenuse from its right-angle vertex. This line segment, whose length we denote p, *divides the right triangle into two more right triangles, each of which is similar to the original right triangle because the angles are the same*. The intersection point of this line segment with the hypotenuse divides the hypotenuse into two line segments: we denote as s the length of the hypotenuse line segment which intersects the leg of length l_1; the remaining hypotenuse line segment, whose length of course is $(h-s)$, intersects the leg of length l_2. Because the three right triangles are similar, the following equalities of the ratios of their corresponding side lengths hold,

$$s/l_1 = p/l_2 = l_1/h \text{ and } p/l_1 = (h-s)/l_2 = l_2/h, \tag{1a}$$

where the last equality turns out to be redundant; we ignore it. Solving the remaining equalities for p yields,

$$p = s\,l_2/l_1 = l_1 l_2/h = (h-s)l_1/l_2, \tag{1b}$$

which can in turn be solved for s and $(h-s)$ in terms of l_1, l_2 and h, with the results,

$$s = (l_1)^2/h \text{ and } (h-s) = (l_2)^2/h. \tag{1c}$$

Adding the two equalities of Eq. (1c) to eliminate s yields,

$$h = \big((l_1)^2 + (l_2)^2\big)/h \;\Rightarrow\; h^2 = (l_1)^2 + (l_2)^2, \text{ which is the Pythagorean theorem.} \tag{1d}$$

The rectangular coordinates of a unit-length vector after its planar rotation

Trigonometry studies the properties of the functions $\cos\theta$ and $\sin\theta$ which are defined as being respectively the x and y coordinates of the x-direction unit vector $(1,0)$ *after its counterclockwise rotation about* $(0,0)$ *by the angle* θ, i.e.,

$$R(\theta)(0,1) = (\cos\theta,\, \sin\theta), \tag{2a}$$

which *as well* means that $(\cos\theta,\, \sin\theta)$ is the $(x,\, y)$ point *of the unit-radius circle centered on* $(0,0)$ which is arrived at *after counterclockwise traversal of that circle by arc length* θ*, starting from the circle's point* $(1,0)$. Familiar *simple examples of* Eq. (2a) *are*,

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$$R(0)(1,0) = (1,0) = (\cos(0),\, \sin(0)),\;\; R(\pi/4)(1,0) = \big((2)^{-\frac{1}{2}},\, (2)^{-\frac{1}{2}}\big) = (\cos(\pi/4),\, \sin(\pi/4)),$$

$$R(\pi/3)(1,0) = \big((1/2),\, (3^{\frac{1}{2}}/2)\big) = (\cos(\pi/3),\, \sin(\pi/3)),\;\; R(\pi/2)(1,0) = (0,1) = (\cos(\pi/2),\, \sin(\pi/2)), \tag{2b}$$

$$R(\pi)(1,0) = (-1,0) = (\cos\pi,\, \sin\pi),\;\; R(3\pi/2)(1,0) = (0,-1) = (\cos(3\pi/2),\, \sin(3\pi/2)).$$

Since $(\cos\theta,\, \sin\theta)$ *always lies on the unit-radius circle centered on* $(0,0)$*, it has the basic property*,

$$\cos^2\theta + \sin^2\theta = 1. \tag{2c}$$

The counterclockwise traversal of the unit-radius circle centered on $(0,0)$ by θ *from an arbitrary starting point* $(\cos\theta_0,\, \sin\theta_0)$ on that unit-radius circle *is naturally defined as*,

$$R(\theta)(\cos\theta_0,\, \sin\theta_0) \stackrel{\text{def}}{=} R(\theta_0+\theta)(1,0) = (\cos(\theta_0+\theta),\, \sin(\theta_0+\theta)). \tag{2d}$$

Since $(\cos\theta_0,\, \sin\theta_0) = R(\theta_0)(1,0)$, imposition of Eq. (2d) *makes the operator* $R(\theta)$ *angle-additive*, i.e.,

$$R(\theta)R(\theta_0)(1,0) = R(\theta_0+\theta)(1,0) = R(\theta+\theta_0)(1,0) = R(\theta_0)R(\theta)(1,0), \tag{2e}$$

which remains the case when $(1,0)$ *in* Eq. (2e) *is replaced by the arbitrary unit-radius circle starting point* $(\cos\theta_i,\, \sin\theta_i)$ (the reader may wish to fill in the demonstration).

By *drawing* the representations of the two successive counterclockwise traversals by θ_1 and θ_2 of the unit-radius circle *from the appropriate two successive starting points* $(1,0)$ *and* $(\cos\theta_1,\, \sin\theta_1)$, it can be *worked out*, with the aid of $\cos^2\theta_1 + \sin^2\theta_1 = 1$, that the following *trigonometric angle-addition formula* holds,

$$\begin{gathered} R(\theta_2)R(\theta_1)(1,0) = (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2,\, \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2) = \\ R(\theta_1+\theta_2)(1,0) = (\cos(\theta_1+\theta_2),\, \sin(\theta_1+\theta_2)). \end{gathered} \tag{2f}$$

One way to work out Eq. (2f) is to note that the point where the line segment representing $\sin(\theta_1+\theta_2)$ intersects the line segment representing $\cos\theta_2$ divides both of those line segments into two line segments; the lengths of all four resulting line segments can be worked out in terms of $\cos\theta_1$, $\sin\theta_1$, $\cos\theta_2$ and $\sin\theta_2$, and those particular four lengths are also fairly simply related to $\cos(\theta_1+\theta_2)$ and $\sin(\theta_1+\theta_2)$ (it is useful to take note that $\cos^2\theta_1 + \sin^2\theta_1 = 1$ however). One sees from the Eq. (2f) *result*, however, *that* $R(\theta)$ *is a linear operator*. By combining *the linearity of* $R(\theta)$ *with its angle-additivity noted in* Eq. (2e), *one arrives at the* Eq. (2f) *angle-addition formula result by the following alternative route*,

$$\begin{gathered} R(\theta_2)R(\theta_1)(1,0) = R(\theta_2)(\cos\theta_1,\, \sin\theta_1) = \cos\theta_1 R(\theta_2)(1,0) + \sin\theta_1 R(\theta_2)(0,1) = \\ \cos\theta_1 R(\theta_2)(1,0) + \sin\theta_1 R(\theta_2)R(\pi/2)(1,0) = \cos\theta_1 R(\theta_2)(1,0) + \sin\theta_1 R(\pi/2+\theta_2)(1,0) = \\ \cos\theta_1(\cos\theta_2,\, \sin\theta_2) + \sin\theta_1(\cos(\pi/2+\theta_2),\, \sin(\pi/2+\theta_2)) \end{gathered} \tag{2g}$$

By counterclockwise traversal of the unit-radius circle from the starting point $(1,0)$ by arc length $(\pi/2+\theta_2)$, and then by counterclockwise traversal of this circle from $(1,0)$ by arc length $(\pi/2-\theta_2)$ it can be seen that,

$$(\cos(\pi/2+\theta_2),\, \sin(\pi/2+\theta_2)) = (-\cos(\pi/2-\theta_2),\, \sin(\pi/2-\theta_2)) = (-\sin\theta_2,\, \cos\theta_2), \tag{2h}$$

which upon insertion into Eq. (2g) yields,

$$\begin{gathered} R(\theta_2)R(\theta_1)(1,0) = \cos\theta_1(\cos\theta_2,\, \sin\theta_2) + \sin\theta_1(-\sin\theta_2,\, \cos\theta_2) = \\ (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2,\, \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2) = \\ R(\theta_1+\theta_2)(1,0) = (\cos(\theta_1+\theta_2),\, \sin(\theta_1+\theta_2)), \end{gathered} \tag{2i}$$

which is exactly the same as the Eq. (2f) trigonometric angle-addition formula result that involved manipulations whose motivation was much less transparent. It is readily verified that Eq. (2i) properly adheres to $\cos^2(\theta_1+\theta_2)+\sin^2(\theta_1+\theta_2)=1$ provided that $\cos^2\theta_j+\sin^2\theta_j=1$, $j=1,2$, as is required by Eq. (2c). *Non-calculus trigonometry is encompassed by* Eq. (2c) *and the* Eq. (2i) *trigonometric angle-addition formula*, which both tolerate a simultaneous single-parameter rescaling $\theta\to k\theta$ of all θ which are involved, and thus *don't uniquely determine* $(\cos\theta, \sin\theta)$ *as a function of* θ. *Unambiguous determination* of $\cos\theta$ and $\sin\theta$ *involves obtaining their derivatives*, whose calculation *is known to be rendered well-defined by the requirement that* $\lim_{\delta\theta\to 0}(\sin\delta\theta/\delta\theta)=1$. The *arc length* $|\delta\theta|$ *of a sufficiently short smooth arc* is always well approximated by the length *of the straight-line-segment chord which joins the two ends of that arc*. Therefore we next obtain *the length of the chord of an arc of the unit-radius circle which is traversed from the starting point* $(1,0)$, and verify that the ratio of $|\sin\delta\theta|$ to that chord length approaches unity in the limit that the corresponding arc length $|\delta\theta|\to 0$.

Length of the chord of an arc of the unit-radius circle

Consider the arc of the unit-radius circle which starts at $(1,0)$ and has arc length $|\theta|$. We denote the length of the straight-line-segment chord which joins the two ends of that arc as $\mathrm{chl}\,\theta$. Of course that chord length $\mathrm{chl}\,\theta$ *is less than the arc length* $|\theta|$ *of the arc, but it turns out to be greater than* $|\sin\theta|$; $\mathrm{chl}\,\theta$ is readily obtained by application of the Pythagorean theorem,

$$\mathrm{chl}\,\theta=\big((\sin\theta)^2+(1-\cos\theta)^2\big)^{\frac{1}{2}}\geq|\sin\theta|;\text{ moreover, }\mathrm{chl}\,\theta=(2(1-\cos\theta))^{\frac{1}{2}}. \tag{3a}$$

The arc length of a sufficiently short smooth arc always approaches its corresponding chord length, i.e.,

$$\lim_{\delta\theta\to 0}(\mathrm{chl}\,\delta\theta/|\delta\theta|)=1. \tag{3b}$$

Inverting the Eq. (3a) result for the chord length $\mathrm{chl}\,\theta$ of a unit-radius circle's arc yields,

$$\cos\theta=1-\tfrac{1}{2}\mathrm{chl}^2\,\theta\quad\Rightarrow\quad|\sin\theta|=\big(1-\cos^2\theta\big)^{\frac{1}{2}}=\big(\mathrm{chl}^2\,\theta-\tfrac{1}{4}\mathrm{chl}^4\,\theta\big)^{\frac{1}{2}}=\mathrm{chl}\,\theta\big(1-\tfrac{1}{4}\mathrm{chl}^2\,\theta\big)^{\frac{1}{2}}. \tag{3c}$$

Eq. (3b) in conjunction with the Eq. (3c) result that $|\sin\theta|=\mathrm{chl}\,\theta\big(1-\frac{1}{4}\mathrm{chl}^2\,\theta\big)^{\frac{1}{2}}$ yields,

$$\lim_{\delta\theta\to 0}(|\sin\delta\theta|/|\delta\theta|)=1, \tag{3d}$$

and since $(|\sin\theta|/|\theta|)=(\sin\theta/\theta)$ when $0<|\theta|<\pi$, Eq. (3d) implies that,

$$\lim_{\delta\theta\to 0}(\sin\delta\theta/\delta\theta)=1. \tag{3e}$$

Unambiguous characterization of the cosine and sine functions

With Eq. (3e) in hand, *we next apply it together with* Eqs. (2i) *and* (2c) *to obtain* $d\cos\theta/d\theta$ *and* $d\sin\theta/d\theta$. We write Eq. (2i) in a form conducive to taking the limits which define $d\cos\theta/d\theta$ and $d\sin\theta/d\theta$,

$$\cos(\theta+\delta\theta)=\cos\theta\cos\delta\theta-\sin\theta\sin\delta\theta\ \text{ and }\ \sin(\theta+\delta\theta)=\sin\theta\cos\delta\theta+\cos\theta\sin\delta\theta. \tag{4a}$$

Combining $d\cos\theta/d\theta=\lim_{\delta\theta\to 0}((\cos(\theta+\delta\theta)-\cos(\theta))/\delta\theta)$ and $d\sin\theta/d\theta=\lim_{\delta\theta\to 0}((\sin(\theta+\delta\theta)-\sin(\theta))/\delta\theta)$ with Eq. (4a) yields,

$$\begin{aligned}d\cos\theta/d\theta&=\lim_{\delta\theta\to 0}((\cos(\theta+\delta\theta)-\cos(\theta))/\delta\theta)=\lim_{\delta\theta\to 0}[\cos\theta((\cos\delta\theta-1)/\delta\theta)-\sin\theta(\sin\delta\theta/\delta\theta)]\ \text{ and}\\ d\sin\theta/d\theta&=\lim_{\delta\theta\to 0}((\sin(\theta+\delta\theta)-\sin(\theta))/\delta\theta)=\lim_{\delta\theta\to 0}[\sin\theta((\cos\delta\theta-1)/\delta\theta)+\cos\theta(\sin\delta\theta/\delta\theta)].\end{aligned} \tag{4b}$$

The two *key limits* on the right sides of Eq. (4b) are (1) $\lim_{\delta\theta\to 0}(\sin\delta\theta/\delta\theta)=1$ from Eq. (3e), and (2) $\lim_{\delta\theta\to 0}((\cos\delta\theta-1)/\delta\theta)$. Since $(\cos\delta\theta-1)=(\cos^2\delta\theta-1)/(\cos\delta\theta+1)=-\sin^2\delta\theta/(\cos\delta\theta+1)$, we obtain,

$$\lim_{\delta\theta\to 0}((\cos\delta\theta-1)/\delta\theta)=\lim_{\delta\theta\to 0}[-(\sin\delta\theta)(\sin\delta\theta/\delta\theta)/(\cos\delta\theta+1)]=0. \tag{4c}$$

The Eq. (4c) and (3e) limits *are what is needed to evaluate the right sides of* Eq. (4b), which yield,

$$d\cos\theta/d\theta=-\sin\theta\qquad\text{and}\qquad d\sin\theta/d\theta=\cos\theta. \tag{4d}$$

A certain linear combination of $\cos\theta$ and $\sin\theta$ very usefully *turns out to be an exponential*. The hypothesis below is built to make its left and right sides agree at $\theta = 0$ (Eq. (2b) shows that $(\cos(0), \sin(0)) = (1, 0)$),

$$\cos\theta + \beta \sin\theta = \exp(\gamma\theta), \tag{5a}$$

Differentiating the two sides of Eq. (5a) with respect to θ yields,

$$\beta\cos\theta - \sin\theta = \gamma\exp(\gamma\theta) = \gamma(\cos\theta + \beta\sin\theta), \tag{5b}$$

which implies that $\beta = \gamma$ and $\gamma^2 = -1$, so $\beta = \gamma = \pm i$. Thus,

$$\cos\theta \pm i\sin\theta = \exp(\pm i\theta). \tag{5c}$$

The *two* signs of $\pm i$ in fact *are redundant*; their *effect is already accounted for when* $\theta \to -\theta$.

This *exponential version* of trigonometry *readily yields the* Eq. (2i) *angle-addition formula*; on one hand,

$$\exp(i\theta_1)\exp(i\theta_2) = \exp(i(\theta_1 + \theta_2)) = \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2), \tag{5d}$$

whereas on the other hand,

$$\begin{gathered}\exp(i\theta_1)\exp(i\theta_2) = (\cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) + i\sin(\theta_2)) = \\ (\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)) + i(\sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2)).\end{gathered} \tag{5e}$$

Eqs. (5d) and (5e) together produce the Eq. (2i) *angle-addition formula*, namely,

$$\cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) \quad \text{and} \quad \sin(\theta_1 + \theta_2) = \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2). \tag{5f}$$

As well as the angle-addition formula, exponential trigonometry yields Eq. (2c) since,

$$1 = \exp(0) = \exp\big((i\theta) + (-i\theta)\big) = \exp(i\theta)\exp(-i\theta) = (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = \cos^2\theta + \sin^2\theta. \tag{5g}$$

Proceeding now *to the Taylor expansions of trigonometric functions*, that of $\exp(i\theta)$ *is elementary*,

$$\begin{gathered}\exp(i\theta) = \textstyle\sum_{k=0}^{\infty}(i)^k(\theta)^k/k! = \sum_{n=0}^{\infty}(i)^{2n}(\theta)^{2n}/(2n)! + \sum_{n=0}^{\infty}(i)^{2n+1}(\theta)^{2n+1}/(2n+1)! = \\ \textstyle\sum_{n=0}^{\infty}(-1)^n(\theta)^{2n}/(2n)! + i\sum_{n=0}^{\infty}(-1)^n(\theta)^{2n+1}/(2n+1)!\,.\end{gathered} \tag{6a}$$

Since $\exp(i\theta) = \cos\theta + i\sin\theta$, the Taylor expansions of $\cos\theta$ and $\sin\theta$ *can now be read off from* Eq. (6a),

$$\cos\theta = \textstyle\sum_{n=0}^{\infty}(-1)^n(\theta)^{2n}/(2n)! \quad \text{and} \quad \sin\theta = \sum_{n=0}^{\infty}(-1)^n(\theta)^{2n+1}/(2n+1)!\,. \tag{6b}$$

It is readily verified that the Eq. (6b) Taylor expansions of $\cos\theta$ and $\sin\theta$ *also follow from their derivatives—which are given by* Eq. (4d)—*in conjunction with the* Eq. (2b) *fact that* $(\cos(0), \sin(0)) = (1, 0)$.