A computing method about how many ‘comparable’ pairs of elements exist in a certain set

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Abstract. Given two sets, one consisting of variables representing distinct positive n numbers, the other set ‘a kind of power set’ of this n-element set. I got interested in the fact that for the latter set, depending on the values of two elements, it can occur that not every pair of elements is ‘comparable’, that is to say, it is not always uniquely determined which of two elements is smaller. By proving theorems in order to go ahead with our research, we show a table which describes for how many ’comparable’ cases exist, for several n’s.

Keywords. power set, comparable, strict totally ordered set, strict partially ordered set, tuple, number of partitions, mutinomial coefficient

0.Introduction

A little ago I read a paper [1], and got interested in two sets. One was a set consisting of variables representing distinct positive n numbers. The other set was ‘a kind of power set’ of this n-element set. My interest was that for the former set every pair of elements is ‘comparable’ [2], that is to say, it is always uniquely determined which of two elements is smaller, but for the latter set, depending on the values of two elements, it can occur that not every pair of elements is ‘comparable’, that is to say, it is not always uniquely determined which of two elements is smaller. In this paper at first we define a set of n variables representing distinct positive numbers and its power set. Next we define another set, ’a kind of power set’, whose each element is the summation of each element of the power set.

For this new set we divide every pair of elements into two sets, the one a set consisting of comparable pairs, that is to say, uniquely determined which of the two is smaller, the other a set consisting of not comparable pairs, not uniquely determined which of the two is smaller depending on the values of the two. We compute how many elements exist in each set, for n=2 and n=3. Even if n grows larger we can compute it, but the total number of the elements of the two sets increases geometrically, so it is not practical to enumerate one by one.

Then with the aid of definitions of functions and theorems, we show a method to compute how many such elements exist in each set for larger n's.

1. A set consisting of n variables representing distinct positive numbers, and its power set

Suppose that variables $a_1,a_2,...,a_n$ represent positive numbers satisfying $a_1>a_2>...>a_n>0$. Now a set is defined consisting of these n variables as follows.

$$E(n) := \{a_1, a_2, \ldots, a_n\} \quad (1)$$

For this E(n), every pair of elements is comparable [2], that is to say, it is always uniquely determined which of two elements is smaller. This set is a strict totally ordered set, because any two elements of E(n) are not the same with each other.

We get the following set, via (1)'s power set, by summing up each element of which.

$$\begin{aligned} F(n) := \{&a_1, a_2, \ldots, a_n, a_1{+}a_2, a_1{+}a_3, \ldots, a_{n-1}{+}a_n, \\ &a_1{+}a_2{+}a_3, a_1{+}a_2{+}a_4, \ldots, a_{n-2}{+}a_{n-1}{+}a_n, \\ &a_1{+}a_2{+}a_3{+}a_4, \ldots, {+}a_{n-2}{+}a_{n-1}{+}a_n, \ldots, a_1{+}a_2{+}\ldots{+}a_{n-1}{+}a_n\} \end{aligned} \quad (2)$$

2. Many sets that are derivative from F(n)

The set (2) has 2^n-1 elements, for we exclude 0 out of this set intentionally, because 0 is evidently smaller than any other elements.

If the values of n variables are specifically given, this set is a totally ordered set, but now not given. So this set is not a totally ordered set, but a strict partially ordered set, for not every pair of elements is comparable, that is to say, it is not always uniquely determined which of any two elements is smaller and there is no pair consisting of the same elements, if there are not specific value assignments for n variables.

(e.g.)

For n=8, pairs of elements such as (a_1,a_2+a_3), (a_2,a_3+a_5) and $(a_2+a_5, a_3+a_6+a_8)$ are not comparable.

If $a_1=10, a_2=7$ and $a_3=4$, then $a_1=10$ and $a_2+a_3=7+4=11$, so $a_1=10<11 =a_2+a_3$. But if $a_1=10$, $a_2=5$ and $a_3=4$, then $a_1=10$ and $a_2+a_3=5+4=9$, so $a_1=10>9=a_2+a_3$. In this way it is not uniquely determined which of two elements is smaller.

In contrast, pairs of elements such as (a_1,a_3) and (a_2+a_4,a_5+a_7) are comparable, it is uniquely determined which of two elements is smaller.

Now we define a set as follows.

$$S(n) : \text{a set consisting of pairs of elements of } F(n) \tag{3}$$

From now on, we research for how many elements of S(n), how many pairs of elements of F(n), are comparable, it is uniquely determined which of any two elements is smaller, and how many many elements of S(n) are not comparable without specific value assignments for n variables, so we define two more sets;

$$C(n) : \text{a set of the two pairs of } S(n) \text{ uniquely determined which is smaller.} \tag{4}$$

$$D(n) : \text{a set of the two pairs of } S(n) \text{ not uniquely determined which is smaller.} \tag{5}$$

The number of elements of these sets for any n are what we are aiming at.

Lemma 1

$$S(n) = C(n) \cup D(n),\ C(n) \cap D(n) = \emptyset$$

$$|S(n)| = |C(n)| + |D(n)| = (2^n - 1)\ (2^{n-1} - 1)$$

Proof.

As mentioned above, for any element of S(n) there is only two possible cases, for one it is uniquely determined which is smaller, for the other not, and these cases are mutually exclusive.

$|F(n)| = 2^n - 1$, so $|S(n)| = {}_{2^n-1}C_2 = (2^n - 1)\ (2^{n-1} - 1)$.

□

Now look at two elements of F(n), that are also regarded as an element of S(n). Any element of S(n) has two expressions. For $s_1,s_2 \in F(n)$, the element of S(n) consisting of these two elements of F(n) can be expressed in two ways, the one (s_1,s_2) and the other (s_2,s_1). We choose one of the two expressions randomly and call it (s_1,s_2). By (2) (3), (s_1,s_2) can be uniquely expressed as the below equations.

$$s_1 = \sum_{i=1}^{n} s_{1i}a_i, \quad s_2 = \sum_{i=1}^{n} s_{2i}a_i$$
$$\text{where } s_{1i}= 0 \text{ or } 1 \text{ , and } s_{2i}= 0 \text{ or } 1. \tag{6}$$

Now we start the comparison of s_{1i} and s_{2i}.

Compare s_{11} and s_{21}. If $(s_{11},s_{21})=(1,0)$ or $(0,1)$ then we end up the comparison, else compare s_{12} and s_{22}. If $(s_{12},s_{22})=(1,0)$ or $(0,1)$ then we end up the comparison, else compare s_{13} and s_{23}........

In this way we continue the comparison repeatedly until we get $(s_{1i},s_{2i})=(1,0)$ or $(0,1)$ for the first time, which necessarily occurs because s_{1i} is not equal to s_{2i}.

Now if $(s_{1i},s_{2i})=(1,0)$ then we denote s_1 by A and denote s_2 by B, else if $(s_{1i},s_{2i})=(0,1)$ then we denote s_1 by B and denote s_2 by A.

(e.g.)

When n=5, if the pair is (a_2, a_3), then $A=a_2$ and $B=a_3$. And for the case of $(a_3,\ a_3+a_5)$, we define $A=a_3+a_5$ and $B=a_3$.

From now on we express an element of S(n) in the order of A, B.

For simplicity I classify elements of S(n) into equivalent classes by the subtraction of B from A. Two elements of S(n), the values of $A-B$ of which are equal to each other, belong to the same equivalent class. So (a_2, a_3) and $(a_2+a_5,\ a_3+a_5)$ belong to the same equivalent class.

(e.g.)

For (a_2, a_3), $A-B=a_2-a_3$; and for $(a_2+a_5,\ a_3+a_5)$, $A-B=(a_2+a_5)-(a_3+a_5)$ $=a_2-a_3$; the same as each other.

Now we designate a representative of each equivalent class, the way is as follows. For the equivalent class containing (a_2, a_3) and $(a_2+a_5,\ a_3+a_5)$, (a_2, a_3) is its representative. We designate the most simple element of S(n).

Note that for such an equivalent class that contains (a_2+a_5,a_2), $(a_5,0)$ is to be exceptionally selected as the representative for convenience, although 0 does not belong to F(n).

Furthermore we can reduce any representative to an n-vector as the following.

Given a representative (s_1,s_2) the expression of which is the same as (6) and we prepare an n-dimensional zero vector (0,0,...,0).

If $(s_{1i},s_{2i})=(1,0)$, then move A to the i-th component of the n-dimensional vector, and if $(s_{1i},s_{2i})=(0,1)$, then move B to the i-th component of the n-dimensional vector; we perform these two operations repeatedly for i=1 to n.
(e.g.)
Where n=4, for (a_2, a_3) we obtain a new 4-vector (0,A,B,0).

We denote by P(n), the set of these reduced representatives, note that in any element of P(n), A appears in advance of B.
(e.g.)
For n=3, P(3)={(A,A,B),(A,A,0),(A,B,A),(A,B,B),(A,B,0),(A,0,A), (A,0,B),(A,0,0),(0,A,A),(0,A,B),(0,A,0),(0,0,A)}.

Then out of P(n), we pick up sets in which there exist k(1≤k≤n) non-zero components as a new set, which we donote by P(n,k).
(e.g.)
P(3,1)={(A,0,0),(0,A,0),(0,0,A)},
P(3,2)={(A,A,0),(A,B,0),(A,0,A),(A,0,B),(0,A,A),(0,A,B)}.
P(3,3)={(A,A,B),(A,B,A),(A,B,B)}.

Next define two functions as below.

$$P_C(n) : \text{a set of the elements of } P(n) \text{ that are representatives of the elements of } C(n). \quad (7)$$

$$P_D(n) : \text{a set of the elements of } P(n) \text{ that are representatives of the elements of } D(n). \quad (8)$$

Lemma 2
$P(n)=\bigcup_{k=1}^{n} P(n,k)$, $\bigcap_{k=1}^{n} P(n,k)=\emptyset$
$P(n) = P_C(n) \cup P_D(n)$, $P_C(n) \cap P_D(n) = \emptyset$
$|P(n)|=\sum_{k=1}^{n}|P(n,k)|$
$|P(n)|=|P_C(n)|+|P_D(n)|$

Proof.
Any element of P(n) has i(1≤i≤n) non-zero components.
And any representative of the elements of S(n) belongs to C(n) or D(n).
So this Lemma 2 follows, according to (6) and (7).

□

Furthermore define two functions as below.

$$\begin{aligned}
&\mathrm{P_C(n,k)} : \text{a set of the elements of } \mathrm{P_C(n)} \\
&\qquad \text{which have k non}-\text{zero components.} \qquad (9)\\
&\mathrm{P_D(n,k)} : \text{a set of the elements of } \mathrm{P_D(n)} \\
&\qquad \text{which have k non}-\text{zero components.} \qquad (10)
\end{aligned}$$

Lemma 3

$\mathrm{P_C(n)}=\bigcup_{k=1}^{n} \mathrm{P_C(n,k)}$, $\bigcap_{k=1}^{n} \mathrm{P_C(n,k)}=\emptyset$,
$\mathrm{P_D(n)}=\bigcup_{k=1}^{n} \mathrm{P_D(n,k)}$, $\bigcap_{k=1}^{n} \mathrm{P_D(n,k)}=\emptyset$,
$|\mathrm{P_C(n)}|=\sum_{k=1}^{n}|\mathrm{P_C(n,k)}|$,
$|\mathrm{P_D(n)}|=\sum_{k=1}^{n}|\mathrm{P_D(n,k)}|$

Proof.

Any element of $\mathrm{P_C(n)}$ or $\mathrm{P_D(n)}$ has $\mathrm{i}(1\leq \mathrm{i}\leq \mathrm{n})$ non-zero components.
So this Lemma 3 follows, according to (9) and (10).

□

Lemma 4

$\mathrm{P(n,1)}=\mathrm{P_C(n,1)}$, $\mathrm{P_D(n,1)}=\emptyset$

Proof.

$\mathrm{P(n,1)}=\{(\mathrm{A},0,\ldots,0),(0,\mathrm{A},0,\ldots,0),\ldots,(0,\ldots,0,\mathrm{A})\}$.

Any two-pair of S(n)'s elements that has a element of $\mathrm{P(n,1)}$ as its representative belongs to C(n). So $\mathrm{P(n,1)}=\mathrm{P_C(n,1)}$, and so $\mathrm{P_D(n,1)}=\emptyset$

□

3. Examples for simple n's

Here we show our computations for n=2 and n=3.

3.1 n=2's case;

$\mathrm{E(2)}=\{\mathrm{a_1, a_2}\}$
$\mathrm{F(2)}=\{\mathrm{a_1, a_2, a_1{+}a_2}\}$
$\mathrm{S(2)}=\{(\mathrm{a_1, a_2}), (\mathrm{a_1{+}a_2, a_1}), (\mathrm{a_1{+}a_2, a_2})\}$
$\mathrm{C(2)}=\{(\mathrm{a_1, a_2}), (\mathrm{a_1{+}a_2, a_1}), (\mathrm{a_1{+}a_2, a_2})\}$
$\mathrm{D(2)}=\emptyset$
$\mathrm{P(2)}=\{(\mathrm{A,B}),(\mathrm{A},0),(0,\mathrm{A})\}$
$\mathrm{P_C(2)}=\{(\mathrm{A,B}),(\mathrm{A},0),(0,\mathrm{A})\}$,$\mathrm{P_D(2)}=\emptyset$
$\mathrm{P(2,1)}=\{(\mathrm{A},0),(0,\mathrm{A})\}$,$\mathrm{P_C(2,1)}=\{(\mathrm{A},0),(0,\mathrm{A})\}$,$\mathrm{P_D(2,1)}=\emptyset$
$\mathrm{P(2,2)}=\{(\mathrm{A,A})\}$,$\mathrm{P_C(2,2)}=\{(\mathrm{A,A})\}$,$\mathrm{P_D(2,2)}=\emptyset$
$|\mathrm{C(2)}|=3$

$|D(2)|=0$

3.2 n=3's case;

$E(3)=\{a_1, a_2, a_3\}$

$F(3)=\{a_1, a_2, a_3, a_1+a_2, a_1+a_3, a_2+a_3, a_1+a_2+a_3\}$

$S(3)=\{(a_1, a_2), (a_1, a_3), (a_1, a_1+a_2), (a_1, a_1+a_3), (a_1, a_2+a_3),$
$(a_1, a_1+a_2+a_3), (a_2, a_3), (a_2, a_1+a_2), (a_2, a_1+a_3), (a_2, a_2+a_3),$
$(a_2, a_1+a_2+a_3), (a_3, a_1+a_2), (a_3, a_1+a_3), (a_3, a_2+a_3), (a_3, a_1+a_2+a_3),$
$(a_1+a_2, a_1+a_3), (a_1+a_2, a_2+a_3), (a_1+a_2, a_1+a_2+a_3), (a_1+a_3, a_2+a_3),$
$(a_1+a_3, a_1+a_2+a_3), (a_2+a_3, a_1+a_2+a_3)\}$

$C(3)=\{(a_1, a_2), (a_1, a_3), (a_1, a_1+a_2), (a_1, a_1+a_3),$
$(a_1, a_1+a_2+a_3), (a_2, a_3), (a_2, a_1+a_2), (a_2, a_1+a_3), (a_2, a_2+a_3),$
$(a_2, a_1+a_2+a_3), (a_3, a_1+a_2), (a_3, a_1+a_3), (a_3, a_2+a_3), (a_3, a_1+a_2+a_3),$
$(a_1+a_2, a_1+a_3), (a_1+a_2, a_2+a_3), (a_1+a_2, a_1+a_2+a_3), (a_1+a_3, a_2+a_3),$
$(a_1+a_3, a_1+a_2+a_3), (a_2+a_3, a_1+a_2+a_3)\}$

$D(3)=(a_1, a_2+a_3)$

P(3)={(A,A,B),(A,A,0),(A,B,A),(A,B,B),(A,B,0),(A,0,A),(A,0,B),
(A,0,0),(0,A,A),(0,A,B),(0,A,0),(0,0,A)}

$P_C(3)$={(A,A,B),(A,A,0),(A,B,A),(A,B,0),(A,0,A),(A,0,B),
(A,0,0),(0,A,A),(0,A,B),(0,A,0),(0,0,A)},$P_D(3)$={(A,B,B)}

P(3,1)={(A,0,0),(0,A,0),(0,0,A)},$P_C(3,1)$={(A,0,0),(0,A,0),(0,0,A)},$P_D(3,1)=\emptyset$

P(3,2)={(A,A,0),(A,B,0),(A,0,A),(A,0,B),(0,A,A),(0,A,B)},

$P_C(3,2)$={(A,A,0),(A,B,0),(A,0,A),(A,0,B),(0,A,A),(0,A,B)},$P_D(3,2)=\emptyset$

P(3,3)={(A,A,B),(A,B,A),(A,B,B)},$P_C(3,3)$={(A,A,B),(A,B,A)},

$P_D(3,3)$={(A,B,B)}

$|C(3)|=20$

$|D(3)|=1$

4. First step to generalization.

In the above-mentioned computations two equations hold as follows.

$$\begin{aligned}|C(2)| &= (2^{2-1}-1)\,|P_C(2,1)| + |P_C(2,2)|\\ |C(3)| &= (2^{3-1}-1)\,|P_C(3,1)|\\ &\quad -((1/2)^{2-1})\,|P(3,2)| + (2^{3-2})\,|P_C(3,2)| + P_C(3,3) \qquad (11)\end{aligned}$$

These (11) are easy to check out, for 3=(2−1)×2+1,
20=(4−1)×3−(1/2)×6+2×6+2.

Generally for any n, we show the following theorem.

Theorem 1

$$
\begin{aligned}
&\text{If } n > 2 \text{ then} \\
&|C(n)| = (2^{n-1}-1)\,|P_C(n,1)| \\
&\qquad + \sum_{k=2}^{n-1} \left(-(1/2)^{k-1}\,|P(n,k)| + 2^{n-k}\,|P_C(n,k)|\right) \\
&\qquad + |P_C(n,n)| \qquad (12)
\end{aligned}
$$

Proof.

Any element of C(n) has a representative that is an element of particular $P_C(n,i)(1 \leq i \leq n)$ (ref (7), (8)).

When k=1,

by Lemma 4, $P(n,1)=P_C(n,1)$, $P_D(n,1)=\emptyset$. And for any element of $P_C(n,1)$, there being $n-1$ zero components, so there is $2^{n-1}-1$ elements of C(n), commonly having this element of $P_C(n,1)$ as their representative. The reason why not '2^{n-1}' but '$2^{n-1}-1$' is that 'B' is not included in a set of $P_C(n,1)$ as a component. For example when n=4, (0,A,0,0) is an element of $P_C(4,1)$, there are '2^{4-1}'=8 pairs whose representative is (0,A,0,0), (a_2,0) is one of them but not included in C(4). Thus for any n we obtain that the number of the C(n)'s elements the representative of which belongs to C(n,1) is

$$(2^{n-1}-1)\,|P_C(n,1)| \qquad (13)$$

When $2 \leq k \leq n-1$,

As above, $P_C(n,k)$ is a complementary set of $P_D(n,k)$ in $P(n,k)$. Now we pick up the elements of $P_C(n,k)$ whose non-zero components are all A's. As for any element of $P(n,k)$, its leftmost non-zero component is A, which is the only required condition, so there are totally 2^{k-1} elements that have $n-k$ non-zero components at the same location.

So the number of the elements of $P_C(n,k)$, whose non-zero components are all A's, is

$$(1/2)^{k-1}\,|P(n,k)| \qquad (14)$$

So the number of the elements of $P_C(n,k)$, whose non-zero components are mixture of A's and B's, is

$$|P_C(n,k)| - (1/2)^{k-1}\,|P(n,k)| \qquad (15)$$

Similarly as when k=1, there are $2^{n-k}-1$ elements of C(n) whose representative is a element related to (14), and 2^{n-k} elements of C(n) related to (15). The total number of these elements is

$$\begin{aligned}&(2^{n-k}-1)(1/2)^{k-1}\,|P(n,k)|\\&+(2^{n-k})(|P_C(n,k)|-(1/2)^{k-1}\,|P(n,k)|)\\&=-(1/2)^{k-1}\,|P(n,k)|+2^{n-k}\,|P_C(n,k)|\end{aligned} \tag{16}$$

When k=n,

There is only one element of C(n) for any element of $P_C(n,n)$. The number of the C(n)'s elements for k=n is

$$|P_C(n,n)| \tag{17}$$

(13), (16) and (17) complete the proof of Theorem 1.

□

However there remains a significant problem. If n grow larger it will become very harder to compute the numbers of elements.

We will try to create a new method in order to compute |C(n)| more easily.

5. Second step to generalization.

Now reductively we map P(n,k) into $P^*(k)$, a set of k-vectors, by picking up non-zero components of the elements of P(n,k).

(e.g.)

(A,0,0,B,0)∈ P(5,2) is mapped into (A,B)∈ $P^*(2)$.

Evidently $|P^*(k)|=2^k$.

Note that if and only if an element of P(n,k) belongs to $P_D(n,k)$, then $P^*(k)$ has at least one sequence of B's that is longer than its previous sequence of A's, for example (A,A,B,B,B).

This is because if and only if for an element of S(n) it is not uniquely determined which of the two consisting numbers is smaller, then an element of P(n,k), the representative of this element of S(n), has at least one sequence of B's that is longer than its previous sequence of A's.

Now an element of P(n,k) is the representative of an element of S(n). If it is uniquely determined which of the two consisting numbers of this element of S(n) is smaller, we also call this situation 'comparable', and if not uniquely determined, we also call this situation 'not comparable', thereafter.

We divide $P^*(k)$ into three sets, $C^*(k)$,$M^*(k)$ and $D^*(k)$ as follows.

$$
\begin{aligned}
&C^*(k) : \text{a set of elements of } P^*(k) \\
&\quad \text{whose original element of } P(n,k) \text{ is comparable,} \\
&\quad \text{even if B is newly added on its right side.} && (18) \\
&M^*(k) : \text{a set of elements of } P^*(k) \\
&\quad \text{whose original element of } P(n,k) \text{ is comparable,} \\
&\quad \text{but if B is newly added on its right side,} \\
&\quad \text{it will become not comparable.} && (19) \\
&D^*(k) : \text{a set of elements of } P^*(k) \\
&\quad \text{whose original element of } P(n,k) \\
&\quad \text{is already not comparable.} && (20)
\end{aligned}
$$

Theorem 2

$|D^*(k)| = 2|D^*(k-1)| + |M^*(k-1)|$

Proof.

Any element of $P^*(k)$ is made from a element of $P^*(k-1)$ by adding A or B on its right side.

When $(x_1, \dots, x_{k-1}) \in D^*(k-1)$,

$(x_1, \dots, x_{k-1}, A) \in D^*(k)$ and $(x_1, \dots, x_{k-1}, B) \in D^*(k)$.

When $(x_1, \dots, x_{k-1}) \in M^*(k-1)$,

$(x_1, \dots, x_{k-1}, A) \in C^*(k)$ and $(x_1, \dots, x_{k-1}, B) \in D^*(k)$.

When $(x_1, \dots, x_{k-1}) \in C^*(k-1)$,

$(x_1, \dots, x_{k-1}, A) \in C^*(k)$ and

$((x_1, \dots, x_{k-1}, B) \in C^*(k)$ or $(x_1, \dots, x_{k-1}, B) \in M^*(k))$.

□

Now define one more function.

$$
\begin{aligned}
&F_B(k) : \text{a set consisting of elements of } C^*(k) \text{ ,} \\
&\quad \text{whose the } k\text{−th component is B ,} \\
&\quad \text{and all elements of } M^*(k). && (21)
\end{aligned}
$$

(e.q.)

For k=6, (A,B,A,A,B,B) and (A,A,A,A,A,B) are sets of $F_B(6)$, because (A,B,A,A,B,B) $\in M^*(6)$, and (A,A,A,A,A,B) $\in C^*(6)$ ending with B.

We show a theorem related to (21).

Theorem 3

$|C^*(k)| = |M^*(k-1)| + |C^*(k-1)| + |F_B(k-1)|$

Proof.

Here we apply the process of Theorem 2's proof.

When $(x_1, \ldots, x_{k-1}) \in D^*(k-1)$,

$(x_1, \ldots, x_{k-1}, A) \in D^*(k)$ and $(x_1, \ldots, x_{k-1}, B) \in D^*(k)$.

When $(x_1, \ldots, x_{k-1}) \in M^*(k-1)$,

$(x_1, \ldots, x_{k-1}, A) \in C^*(k)$ and $(x_1, \ldots, x_{k-1}, B) \in D^*(k)$.

When $(x_1, \ldots, x_{k-1}) \in C^*(k-1)$,

$(x_1, \ldots, x_{k-1}, A) \in C^*(k)$ and

$((x_1, \ldots, x_{k-1}, B) \in C^*(k)$ or $(x_1, \ldots, x_{k-1}, B) \in M^*(k))$.

It is evident that no element of $D^*(k-1)$ is to be mapped into a element of $C^*(k)$ and that any element of $M^*(k-1)$ is to be mapped into a element of $C^*(k)$ if and only if added by A on the right side.

It is not easy for $C^*(k-1)$. Added by A on the right side, any element of $C^*(k-1)$ is to be mapped into a element of $C^*(k)$. But added by B, as above, there is two cases, a case mapped into a element of $C^*(k)$ and a case of $M^*(k)$, the former case is what we are interested in right now.

We will research any element of $C^*(k-1)$, by adding B on the right side of which, an element of $C^*(k)$ is to be made.

Now we divide its (k-1)-vector expression into some sequences of consecutive A's and consecutive B's in order. For example, (A,A,B,A,B,A,A,A,B) is divided into (A,A,B),(A,B), similarly (A,B,A,A,B,B,A,A,A) is divided into (A,B),(A,A,B,B),(A,A,A).

For the elements of $C^*(k-1)$, in the rightmost sequence the number of A's is more than that of B's by 2 or over, and in all the other sequences the number of A's is more than or equal to that of B's. This is because its element of $P^*(k)$ after adding B on the right side still belongs to $C^*(k)$. Hence by replacing rightmost A of the rightmost sequence with B, if and only if an element of $C^*(k-1)$ is such, this element changes to an element of $M^*(k-1)$, or an element of $C^*(k-1)$ whose vector expression ends with B.

This is the very $F_B(k-1)$, according to (21).

□

Now we define a function G as follows.

$$\begin{aligned}&G(j_1, j_2, \ldots, j_m) : \text{the number of cases}\\ &\quad \text{that means how variously}\\ &\quad \text{an m}-\text{tuple } (j_1, j_2, \ldots, j_m)\\ &\quad (\text{such that } j_1 \leq j_2 \leq \ldots \leq j_m)\\ &\quad \text{can be permuted.}\end{aligned} \tag{22}$$

(e.g.)
G is expressed as a multinomial coefficient as follows.

$G(1,2,3)=\binom{3}{1,1,1}=3!/(1!1!1!)=6$,
$G(1,2,2)=\binom{3}{1,2}=3!/(1!2!)=3$,
$G(1,2,3,5,8,10)=\binom{6}{1,1,1,1,1,1}=6!/(1!1!1!1!1!1!)=720$,
$G(2,2,3,3,3,5)=\binom{6}{2,3,1}=6!/(2!3!1!)=60$.

We must be careful of the multiplicity of each component of $(j_1, j_2, \ldots, j_m)$ in order to compute precisely.

Theorem 4

$$|F_B(k)| = \sum_{2 \leq j_1 \leq j_2, \leq \ldots \leq j_m, j_1 + \ldots + j_m = k} (G(j_1, j_2, \ldots, j_m)[j_1/2][j_2/2] \ldots [j_m/2]) \tag{23}$$

Proof.
By the definition (21) we know that an element of $|F_B(k)|$ consists of some sequences of consecutive A's and consecutive B's in order, and in each sequence the number of A's is more than or equal to that of B's and always ends with B, we have only to compute how many such elements exist.

If the length of a sequence is l, then there are totally [l/2] cases for this sequence. For example, if l=9, the cases are (8-1),(7-2),(6-3) and (5-4), in what the number of A's and B's are described in order, there are totally 4=[9/2] cases , similarly if l=12, the cases are (11-1),(10-2),(9-3),(8-4),(7-5) and (6-6), there are totally 6=[12/2] cases.

Now we set focus on a particular situation that there are m sequences and their lengths are $j_1 \leq j_2 \leq \ldots \leq j_m$. Then $2 \leq j_1$ and $j_1 + \ldots + j_m = k$, and for this particular case there is the number of cases which is how variously we can permute this m-tuple $(j_1, j_2, \ldots, j_m)$. This corresponds to $G(j_1, j_2, \ldots, j_m)$ (22).

So (23) follows.

□

(e.g.)
When k=5,
if $(j_1)=(5)$, then G(5)[5/2]=(1!/1!)×2=2,
and if $(j_1,j_2)=(2,3)$, then G(2,3)[2/2][3/2]=(2!/(1!1!))×1×1=2,
so $|F_B(5)|$=2+2=4.
When k=7,
if $(j_1)=(7)$, then G(7)[7/2]=(1!/1!)×3=3,
if $(j_1,j_2)=(2,5)$, then G(2,5)[2/2][5/2]=(2!/(1!1!))×2×1=4,
if $(j_1,j_2)=(3,4)$, then G(3,4)[3/2][4/2]=(2!/(1!1!))×1×2=4,
and if $(j_1,j_2,j_3)=(2,2,3)$,
then G(2,2,3)[2/2][2/2][3/2]=(3!/(2!1!))×1×1×1=3,
so $|F_B(7)|$=3+4+4+3=14.

Now let us denote p(n) as the number of partitions of the number n [3], that is to say, the number of all possible partitions of the natural number n as the sum of other integers, sorted, for instance, in non-increasing order, and denote p(n,2) as the number of partitions of n that does not include 1 as their component, in other words, any component of a partition is larger than or equal to 2. Then the number of m-tuples of (23) is equal to p(k,2). Also, it is well known that p(n)=p(n−1)+p(n,2), so the above-mentioned number of m-tuples is equal to p(k)−p(k−1).

Theorem 5

$$
\begin{aligned}
&|P_C(n,k)| \\
&\quad = {}_nC_k(|C^*(k)|+|M^*(k)|) \qquad (\text{when } n \neq k) \\
&\quad = {}_nC_k(|C^*(k)|+|M^*(k)|)-1 \quad (\text{when } n = k)
\end{aligned}
$$

Proof.

By the definition (18) through (20), both $C^*(k)$ and $M^*(k)$ correspond to the case of C(n). And $D^*(k)$ corresponds to the case of D(n).

For an element of $C^*(k)$ or $M^*(k)$, its corresponding $P_C(n,k)$'s element has k non-zero components, so this kind of elements of $P_C(n,k)$ are ${}_nC_k$ in number. But note that there is only one exception, $(A,A,\ldots,A)\in C^*(k)$, which does not belong to $P_C(n,k)$. So if n=k, the right side of the equation is smaller than the other cases, by 1.

□

6. Computing Results

By using what have been defined and proved, we can obtain C(n) as follows.

First we make Table 1 for $|C^*(k)|$,$|M^*(k)|$,$|D^*(k)|$,$|F_B(k)|$,$|P^*(k)|$ for k=1 to 10, after setting initial and trivial values for k=1. On this process we use Theorem 2, Theorem 3 and Theorem 4.

If we have known the values for k−1, then we can compute $|C^*(k)|$ by Theorem 3, $|D^*(k)|$ by Theorem 2, $|F_B(k)|$ by Theorem 4 and $|P^*(k)| = 2^k$, all values for k.

Table 1:

k	1	2	3	4	5	6	7	8	9	10
$\|C^*(k)\|$	1	1	3	4	9	14	28	47	85	155
$\|M^*(k)\|$	0	1	0	2	1	5	5	14	23	38
$\|D^*(k)\|$	0	0	1	2	6	13	31	67	148	319
$\|F_B(k)\|$	0	1	1	3	4	9	14	24	47	89
$\|P^*(k)\|$	1	2	4	8	16	32	64	128	256	512

Second we make Table 2 for $|\mathrm{F(n)}|$,$|\mathrm{S(n)}|$,$|\mathrm{C(n)}|$,$|\mathrm{D(n)}|$,$|\mathrm{P(n)}|$,$|\mathrm{P_C(n)}|$, $|\mathrm{P_D(n)}|$,$|\mathrm{P(n,1)}|$,$|\mathrm{P_C(n,1)}|$,$|\mathrm{P_D(n,1)}|$,...,$|\mathrm{P(n,n)}|$,$|\mathrm{P_C(n,n)}|$,$|\mathrm{P_D(n,n)}|$ for n=1 to 10.

Having created Table 1, we can compute the values for Table 2 as below, after setting initial and trivial values for n=1, 2 and 3, $|\mathrm{P_C(n,k)}|$ and $|\mathrm{P_D(n,k)}|$ by Theorem 5, $|\mathrm{P_C(n)}|$ and $|\mathrm{P_D(n)}|$ by Lemma 2, $|\mathrm{C(n)}|$ by Theorem 2, $|\mathrm{S(n)}|= (2^n-1)\ (2^{n-1}-1)$, $|\mathrm{D(n)}|$ by Lemma 1, all values for n. Note that values of $|\mathrm{C(n)}|/|\mathrm{S(n)}|$ are rounded off to three decimal places.

Table 2:

n	(=|E(n)|)	1	2	3	4	5	6	7	8	9	10
|F(n)|		1	3	7	15	31	63	127	255	511	1023
|S(n)|		0	3	21	105	465	1953	8001	32385	130305	522753
	|C(n)|	0	3	20	95	399	1588	6164	23590	90215	343350
	|C(n)|/|S(n)|	-	1.000	0.952	0.905	0.858	0.813	0.770	0.728	0.692	0.657
	|D(n)|	0	0	1	10	66	365	1837	8795	40090	179403
|P(n)|		0	3	12	39	120	363	1092	3279	9840	29523
	$\|\mathrm{P_C(n)}\|$	0	2	11	33	94	264	739	2068	5789	16207
	$\|\mathrm{P_D(n)}\|$	0	0	1	6	26	99	353	1211	4051	13316
|P(n,1)|	$\|\mathrm{P_C(n,1)}\|$	0	2	3	4	5	6	7	8	9	10
	$\|\mathrm{P_D(n,1)}\|$	0	0	0	0	0	0	0	0	0	0
|P(n,2)|	$\|\mathrm{P_C(n,2)}\|$	-	1	6	12	20	30	42	56	72	90
	$\|\mathrm{P_D(n,2)}\|$	-	0	0	0	0	0	0	0	0	0
|P(n,3)|	$\|\mathrm{P_C(n,3)}\|$	-	-	2	12	30	60	105	168	252	360
	$\|\mathrm{P_D(n,3)}\|$	-	-	1	4	10	20	35	56	84	120
|P(n,4)|	$\|\mathrm{P_C(n,4)}\|$	-	-	-	5	30	90	210	420	756	1260
	$\|\mathrm{P_D(n,4)}\|$	-	-	-	2	10	30	70	140	252	420
|P(n,5)|	$\|\mathrm{P_C(n,5)}\|$	-	-	-	-	9	60	210	560	1260	2520
	$\|\mathrm{P_D(n,5)}\|$	-	-	-	-	6	36	126	336	756	1512
|P(n,6)|	$\|\mathrm{P_C(n,6)}\|$	-	-	-	-	-	18	133	532	1596	3990
	$\|\mathrm{P_D(n,6)}\|$	-	-	-	-	-	13	91	364	1092	2730
|P(n,7)|	$\|\mathrm{P_C(n,7)}\|$	-	-	-	-	-	-	32	264	1188	3960
	$\|\mathrm{P_D(n,7)}\|$	-	-	-	-	-	-	31	248	1116	3720
|P(n,8)|	$\|\mathrm{P_C(n,8)}\|$	-	-	-	-	-	-	-	60	549	2745
	$\|\mathrm{P_D(n,8)}\|$	-	-	-	-	-	-	-	67	603	3015
|P(n,9)|	$\|\mathrm{P_C(n,9)}\|$	-	-	-	-	-	-	-	-	107	1080
	$\|\mathrm{P_D(n,9)}\|$	-	-	-	-	-	-	-	-	148	1480
|P(n,10)|	$\|\mathrm{P_C(n,10)}\|$	-	-	-	-	-	-	-	-	-	192
	$\|\mathrm{P_D(n,10)}\|$	-	-	-	-	-	-	-	-	-	319

7. Discussion and computation

We have computed for n=1 to 10 as in Table 2.

According to Table 2, when n=10, |C(n)|=343350, |D(n)|=179403, |S(n)|=522753, and |C(n)|/|S(n)| which is so called, the ratio of 'comparable', is $0.6568\cdots$. This ratio decreases monotonously with n, provided that n≤10.

As mentioned above, we can obtain |C(n)| for any n, even much more than 10 with the aid of Table 1 of the advanced version. We can continue to compute $|F_B(k)|$ (23) according to Theorem 4, and recreate Table 1 for larger k's.

But there remains one drawback that the computations are quite tough, for $|F_B(k)|$ includes floor functions and tuples so can't be described in a simple form. We can't dispense with $|F_B(k)|$ for any k(1≤k≤n-1). Therefore if n grows larger, almost certainly it takes time and trouble to compute |C(n)|.

However by the method we have presented, |C(n)|, |D(n)| and |C(n)|/|S(n)| can be computed for any n in a recursive manner, without enumerating all pairs of elements one by one.

In the near future we want to find out easier method to compute them.

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