The Clebsch diagonal, the associahedra and motivic gravity

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Abstract

The associahedra appear in a line configuration space for the real Clebsch diagonal surface, which we relate to $\mathbf{e_6}$ in the magic star, with applications to mass generation.

Contents

1 Overview

Mach's principle for inertia in cosmology introduces non local degrees of freedom into the extended Standard Model, which is studied using the categorical combinatorics of the motivic approach [1][2][3][4][5]. Left and right handed neutrino states are distinguished by a braiding for suitable modular tensor categories for quantum computation, wherein the dimension of a qudit space generically replaces the cardinality of a set in classical logic. Non local chirality naturally breaks planar symmetries, just as Metatron's cube resolves a point in the centre of a hexagon into a three dimensional cube.

Scattering theory for massless particles employs the sequence of associahedra polytopes [6][7], which form a one dimensional operad. The vertices on the polytope in dimension d are labelled by either binary rooted planar trees with $d+2$ leaves or the dual chorded polygons with $d+3$ sides. The real points of the moduli space $\mathcal{M}_{0,n}$ of the Riemann sphere $\mathbb{CP}^1$ with n marked points is tiled by associahedra [8][9][10][11], because $\mathbb{RP}^1$ is represented by the index polygon. Figure 1 shows the associahedron in dimension 3. Edges and higher dimensional cells are labelled by trees of higher valency, whereby an edge between binary trees carries a tree with one collapsed internal edge.

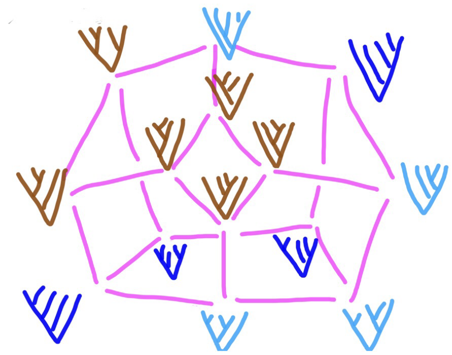

Figure 1: Associahedron A_6 with 14 vertices

We are especially interested in this three dimensional polytope A_6, indexed by a hexagon for six particles. It appears in the study of metric ribbon graphs for matrix models [12], and the associahedron in any dimension may be constructed as a product of those in dimension 3 or less [13]. Here it will define a boundary cell on a very special four dimensional polytope, helping us to understand the discrete geometry of the Higgs mechanism in motivic gravity. Figure 2 indicates the duality between (i) chorded polygons and trees and (ii) configurations of projective lines, as studied in the twistor string theory of [14].

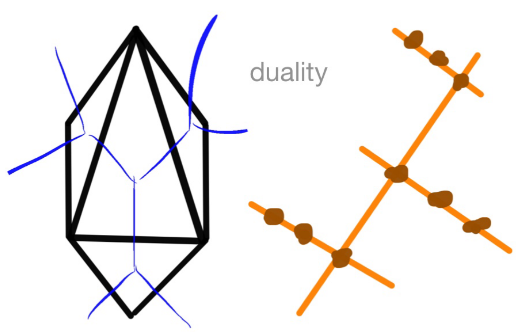

Figure 2: Nine edges or nine points

In figure 2, an edge on a tree turns into an orange point. There are six generic points that do not lie on the triple line, corresponding to the six leaves of the tree, where the root leaf is included. Note that there are six identical copies of A_6: one for each choice of root on an index hexagon.

In graph theory, a *star* is a collection of three lines in the real plane that all intersect at a point [15]. An arbitrary collection of lines in $\mathbb{R}^d$ is *star closed* if any pair defines a star, making it a root system. The ADE Lie algebras are defined by the extra condition that the root system does not decompose into perpendicular spaces. In the next section we introduce the magic star in the real plane. The Lie algebra $\mathbf{e_6}$ lies on the centre point of the $\mathbf{e_8}$ star.

The special four dimensional polytope is related to the real Clebsch diagonal surface [16][17]. Cayley knew that smooth cubics in $\mathbb{CP}^3$ contain exactly 27 lines, and Clebsch showed in 1871 that these lines in his surface may be made real. This Clebsch surface in $\mathbb{RP}^3$ is given by

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 = (x_1 + x_2 + x_3 + x_4)^3. \tag{1}$$

We want to look at configurations of six lines in the Clebsch surface [18], so that blow downs of the six lines into the projective plane $\mathbb{RP}^2$ define a configuration space with a very interesting action of $W(\mathbf{e_6})$. The Clebsch surface is the only cubic surface with exactly 10 points [17] at which three lines meet, and this is the maximal number of such points. The six line configuration space will define a polytope whose boundary is specified by a collection of ten A_6 cells along with five cubes.

Both 3-cubes and A_6 cells are fundamental in the motivic approach. The polytope A_6 defines the sheaf cohomology of $\mathbb{RP}^2$ in the sense that the good cover with 10 points may be extended by (i) one point at infinity in the diagram plane and (ii) three points to define this plane, in such a way that A_6 is the resulting diagram. That is, A_6 does not merely tile the real points of $\mathcal{M}_{0,6}$ with $60 = 5!/2$ copies of itself. It directly represents $\mathbb{RP}^2$, just as the pentagon represents the triangle good cover for $\mathbb{RP}^1$ when two points are added.

Now as quantum logicians, we don't like to start with the reals. A similar analysis of the Clebsch configurations has been carried out for finite fields [19]. Recall that the angles defining a root system satisfy $4\cos^2(2\pi/(p+1)) \in \{0, 1, 2, 3\}$, where the primes p lie in $\{2, 3, 5, 7, 11\}$. These are the Galois primes. The primes 2, 3 and 5 lead respectively to the Gaussian integers $\mathbb{Z}[i]$, the Eisenstein integers $\mathbb{Z}[\omega]$ for $\omega = \exp(2\pi i/3)$, and the golden integers $\mathbb{Z}[\phi]$ for $\phi = (1 + \sqrt{5})/2$ the golden ratio. The ribbon double numbers $\{4, 6, 10, 14, 22\}$ are also important, leading to numbers like $\rho = 2\cos\pi/10$, and correspond to higher dimensions in the algebraic approach. These extensions of $\mathbb{Q}$ are really the only numbers that we need.

The next section introduces the magic star and its algebras. Section 3 looks at the six line configurations and their relation to the associahedron, and in section 4 we discuss the application of all these ideas to mass generation in motivic quantum gravity.

2 The magic star and $\mathbf{e_6}$

Here is the big picture. The canonical example of a Lie algebra is $\mathbf{e_8}$, with 240 roots. Its Lie group $\mathbf{E_8}$ has 240 root dimensions, which we aim to build discretely [4] from ten copies of 24 dimensions, which is the dimension of the Leech lattice. The magic star of figure 3 [20][21][22][23] is a way of collecting the roots of an arbitrary Lie algebra onto the 13 points of the diagram, by projecting sets of roots from higher dimensions. For $\mathbf{e_8}$, there are 27 roots at the vertex of a triangle. These are the 27 dimensions of the exceptional Jordan algebra $J_3(\mathbb{O})$. At the centre point of the star lie the $72 = 3 \times 24$ roots of $\mathbf{e_6}$. The outer six points give the $\mathbf{a_2}$ plane of the star, so that each of these six points carries only one root. Each of the six copies of $J_3(\mathbb{O})$ has 24 off-diagonal dimensions.

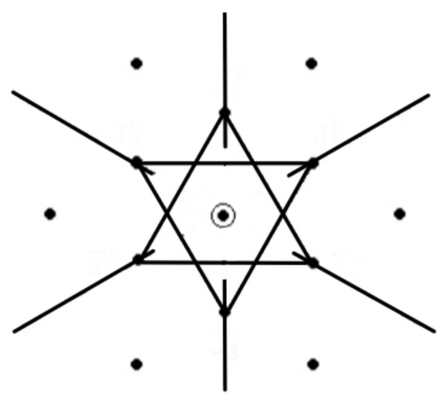

Figure 3: The 13 points of the magic star

In tiling the plane with intersecting triangles for conjugate Jordan algebras, we create a ten point tetractys tile. A discrete blowup in such a plane is defined as the replacement of a point by a tetractys tile. Our discrete simplices are defined as diagonal cuts on a cubic lattice, so that each point on the tetractys represents a set of paths on the cube. For example, three noncommutative paths of type aab lead to a middle point on an edge. After several blowups, the centre point of the magic star may be viewed as the diagram in figure 4, carrying a total of 78 paths. Here we have employed a magic triality and ignored the usual rules of geometry!

When the magic star is missing a central point, it is the root system for $\mathbf{g_2}$. In figure 4 we have replaced the central node with a three dimensional braiding of three lines. In a braided lattice, we kill all the singleton vertices, leaving only the 72 root paths for $\mathbf{e_6}$. A tetractys is filled with three A_5 pentagons [24][25], so that the six pentagons of A_6 come from two tetractys simplices.

The magic stars for $\mathbf{e_6}$, $\mathbf{e_7}$ and $\mathbf{e_8}$ correspond to the 3×3 Jordan algebras over $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$, with respective dimensions 9, 15 and 27. There are also 9, 15 and 27 vertices on the three special strongly regular graphs [15] specified by a parameter $u \in \{2, 3, 5\}$, such that the degree of the graph is $2u$ and there are $6u - 3$ vertices. The 9 vertex graph is given in figure 9 as three colored hexagons. These are the only non trivial regular graphs of triangles such that any fourth vertex is joined to only one vertex of a given triangle. Thus we

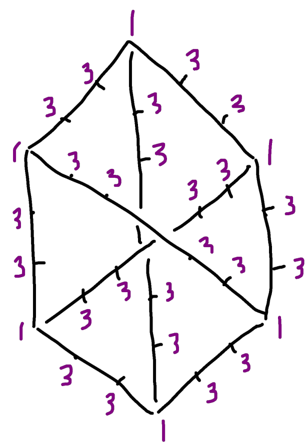

Figure 4: Six tetractys blowup of the central $\mathbf{e_6}$

like to associate $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ with the numbers i, ω and ϕ. Other important strongly regular graphs are the 5 vertex pentagon and 10 vertex Petersen graph, which appears below. The only strongly k-regular graphs of order k^2+1 have $k^2+1 \in \{5, 10, 50, 3250\}$.

We also want to multiply the whole $\mathbf{e_6}$ magic star on $J_3(\mathbb{C})$ by 6. Then there are a total of 432 roots, with 72 in the centre for $6 \times \mathbf{g_2}$. Each one of these roots will label a connected component of the configuration space for six lines in $\mathbb{RP}^2$, given by reflections in $W(\mathbf{e_6})$. For the stars of $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ leading up to $\mathbf{e_8}$, the centre point dimensions are respectively

$$3(1+1), \quad 3(3+1), \quad 3(9+1), \quad 3(27-3), \tag{2}$$

wherein we recognise spacetime metrics.

3 Blowdowns of the Clebsch diagonal

The six vectors in $\mathbb{R}^3$, or rather $\mathbb{Q}(\omega)^3$, that define the hexacode [15] are given by the columns of the identity and the 3×3 Fourier transform,

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & \omega & \overline{\omega} \\ 0 & 0 & 1 & 1 & \overline{\omega} & \omega \end{pmatrix}. \tag{3}$$

Here we have two mutually unbiased qutrit bases. A vector defines a line in $\mathbb{C}^3$, as in

$$x + y\omega + z\overline{\omega} = 0. \tag{4}$$

A similar equation shows up in the next section. A golden analog is the special set of six column vectors

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & x_1 & x_2 \\ 0 & 0 & 1 & 1 & y_1 & y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & -\phi & 1-\phi \\ 0 & 0 & 1 & 1 & 1-\phi & 2-\phi \end{pmatrix}, \tag{5}$$

viewed as points in $\mathbb{RP}^2$, which blow up to the Clebsch surface [18]. Following [18], we define the line configuration space in terms of coordinates $(x_1, x_2, y_1, y_2) \in \mathbb{R}^4$. Let D_{ijk} be a minor of the 3×6 matrix. In particular,

$$D_{456}(x_1, x_2, y_1, y_2) = x_1 y_2 - x_2 y_1 + x_2 - x_1 + y_1 - y_2. \tag{6}$$

Define a determinant map by

$$D_C \equiv x_1 x_2 y_1 y_2 \cdot D_{456}(\frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{y_1}, \frac{1}{y_1}). \tag{7}$$

Then the configurations of six generic lines are

$$C \equiv \{(x_1, x_2, y_1, y_2) \in \mathbb{R}^4 : \ D_{ijk} \neq 0, \ D_C \neq 0\}. \tag{8}$$

Let an open set in C be given by inequalities on the D_{ijk}. Such a set of triangles is called a *pentagonal set* [18] as shown in figure 5. There are 432 components to C, so that four copies of C would give 1728 sets, which shows up in the discrete approach to modular forms.

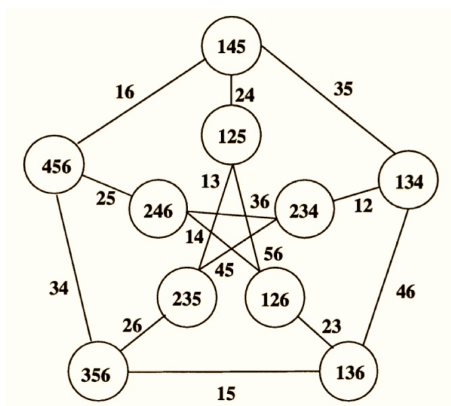

Figure 5: A pentagonal set on the Petersen graph

For the six lines in $\mathbb{RP}^2$ such that no three lines meet at a point, we have subtracted a subspace of fixed points under a certain involution from the full configuration space [18]. This subspace happens to be the configurations of six points on an $\mathbb{RP}^1$, which is the situation for the associahedra moduli space $\mathcal{M}_{0,6}$

[8]. The S^3 boundary of our open polytope in C is built with ten A_6 cells, called A_{ijk}, and five cubes $Z_{ab,cd,ef}$. The choice from figure 5 is [18]

$$A_{234}, A_{456}, A_{126}, A_{125}, A_{145}, A_{134}, A_{136}, A_{356}, A_{235}, A_{246}, \tag{9}$$

$$Z_{12,34,56}, Z_{13,25,46}, Z_{14,26,35}, Z_{15,36,24}, Z_{16,45,23}.$$

For example, A_{234} has six pentagons labelled 456, 126, 125, 145, 136, 356, along with three squares $Z_{12,34,56}$, $Z_{15,36,24}$, $Z_{16,45,23}$ coming from cubes. The cube $Z_{12,34,56}$ has the six faces 234, 356, 126, 456, 134, 125. That is, the opposite faces 234 and 134 share the 34, while the opposite faces 125 and 126 share 12, and the remaining pair shares 56, giving three axes 12, 34 and 56. Similarly, the pentagon 126 is opposite 125 on A_6, as shown in figure 6.

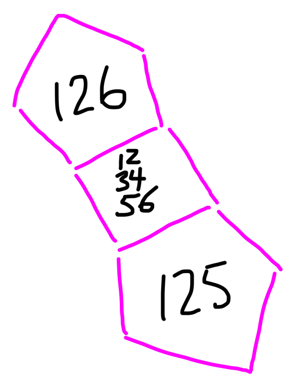

Figure 6: Three faces on A_{234} forming a decagon

The golden matrix (5) gives the fixed point for the S_5 action on the polytope, where 120 divides the order 2×25920 of the group $W(\mathbf{e_6})$. The space C has 432 components and has a transitive action of $W(\mathbf{e_6})$, where $432 = 2 \times 25920/120$. Each component of C is given by a pentagonal set, which has a double cover dodecahedron. We describe $\mathbf{e_6}$ following [18], where a set of six simple roots is denoted

$$r_{123}, r_{12}, r_{23}, r_{34}, r_{45}, r_{56}. \tag{10}$$

Define these using the six dimensional subspace of $\mathbb{R}^8$ spanned by $e_1, \cdots, e_5$ and $e \equiv e_6 - e_7 - e_8$, as in

$$r_{1j} = -e_{j-1} + r, \;\; r_{jk} = e_{j-1} - e_{k-1}, \;\; r_{1jk} = -e_{j-1} - e_{k-1}, \tag{11}$$

$$r_{ijk} = -e_{i-1} - e_{j-1} - e_{k-1} + r,$$

where $r = (e_1 + \cdots + e_5 + e)/2$ and the distinct indices satisfy $i, j, k \geq 2$. The simple roots along with r give an extended 7 point Dynkin diagram with obvious triality symmetry. The root system $\mathbf{g_2}$ uses r_{23}, r_{34}, r_{24}.

The group S_6 is generated by reflections $r_{12}, r_{23}, r_{34}, r_{45}, r_{56}$. Then $W(\mathbf{e_6})$ is generated by this S_6 along with the reflection r_{123}. Our S_5 acts on pentagonal sets determined by the indices $i, j \in \{2, 3, 4, 5, 6\}$. It depends on an outer automorphism σ of S_6 and is not simply a permutation subgroup. We take $\sigma(ij) \circ r$ as elements of S_5, and we are also interested in a 5×5 representation of $W(\mathbf{e_6})$, noting that the physical $\mathbf{a_2}$ algebras have a $5 + 3$ dimensional adjoint representation.

The ten triplets ijk from (9) correspond to the roots $\pm r_{ijk}$, and these 20 roots are in one orbit of S_6, containing 12 pentagonal sets. This is the orbit for configurations like figure 8. In total there are 27 elements in $\{r_{ijk}\} \cup r$ amongst the four orbit representatives, and a total of $432 = 27 \times 16$ sets. In figure 8 there are a total of 22 regions, including the ten interior pentagons and triangles indicated. In the projective space, the 12 outer regions pair up into six polygons, giving the final configuration of 6 pentagons and 10 triangles. Line numbers in figure 8 indicate the exact correspondence with the pentagonal set. In another orbit, there are 3 pentagons and 7 triangles, along with squares.

The simple Burkhardt group of order 25920 is $Sp_4(\mathbb{F}_3)/\pm 1$ and has four 5×5 matrix generators [26] with entries in $\mathbb{F}_4 = \{0, 1, \omega, \overline{\omega}\}/3$ and $\{1/3, 2/3\}$, such that complex conjugation is an outer automorphism. The complex generators are all diagonal. Amongst other things, the order 25920 counts the years in an Earth precession cycle. By Mach's principle, in a theory where the middle Earth scale is fundamental, this Earth spin may well have a basic numerical value.

In the notation of [8], the correspondence between trees and bracketed permutations in S_6 is shown in figure 7. The string 1234(56) represents a compactification component on $\mathcal{M}_{0,6}$ for which two points (56) are coming together. This 1234(56) pentagon is one cell in the boundary of 123456. The full boundary of 12345 is itself a pentagonal set of five edges, namely 12, 23, 34, 45, 51. Thus the marked associahedra form a complex. The 3 points in the compactification of $\mathbb{CP}^1$ for $\mathcal{M}_{0,4}$ were given [8] by brackets like (12)34 on a basic four leg tree diagram. Here is the geometry of spacetime.

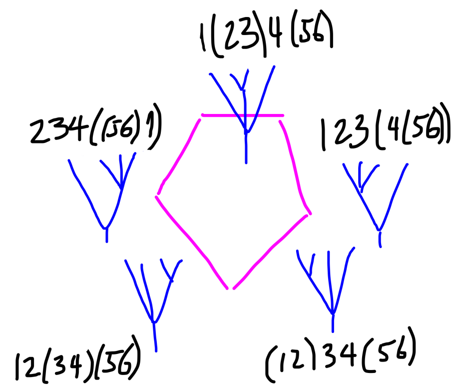

Figure 7: Five boundary faces of 1234(56)

4 Mass with motives

The creation of mass is about a pairing of two massless states, related to the minimal UV and maximal IR scales of quantum field theory, where the IR scale is cosmological. As is well known, the massless scattering theory is governed by associahedra polytopes. Here we look at how the space C may be used to study an $\mathbf{e_6} \otimes \mathbf{a_2}$ Higgs mechanism within multiple copies of $\mathbf{e_8}$. Three times 72 roots appear at the centre of a higher dimensional star.

We would like to see how the 27 roots of $\mathbf{e_8}$ in $J_3(\mathbb{O})$ correspond to the geometry of cubics. The simple group of order 25920 was studied long ago [16], even by Jordan himself, but more recently [27][4] the Jordan basis was approached using ideas from quantum information theory. Take the trit powers a, b, c in a product $U^a V^b W^c$, where

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \overline{\omega} \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad W = \omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{12}$$

This U and V phase space is associated to the six columns of (3), and all 81 such matrices determine a basis for the Jordan algebra in [27]. Here the trits a, b and c denote the paths leading to a tetractys simplex. The V^b matrices may represent the neutrino and photon, while mixed ω and $\overline{\omega}$ matrices are naturally associated to the π^0.

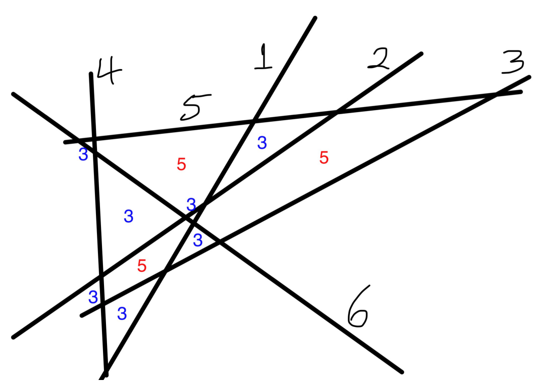

Figure 8: Six lines on a pentagon triangle orbit

A *double six* on the Clebsch surface [17] is a set of 12 out of 27 lines, denoted by a matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \end{pmatrix}, \tag{13}$$

such that the line a_{ij} meets the five lines whose indices do not intersect with ij. That is, given a set of six lines, we can find another set so that altogether there are 30 points of intersection amongst the 12 lines. A *tritangent* plane is a plane that intersects a cubic surface in a triangle. When all 27 lines are real, there are 15 tritangent planes, and these may be labelled by the missing 15 lines in our set.

On a general cubic there are 45 tritangents [16]. These come from 45 points in $\mathbb{CP}^4$ described by coordinates $(X_1, \cdots, X_6)$ such that the sum $\sum_{i=1}^{6} X_i = 0$, as follows. Given a permutation $(ijklmn)$ in S_6, there are 30 points (ij, kl, mn) of the form

$$X_i = X_j = 1, \quad X_k = X_l = \omega, \quad X_m = X_n = \overline{\omega} \tag{14}$$

and 15 points (ij) of the form

$$X_i = 1, \quad X_j = -1, \quad X_k = X_l = X_m = X_n = 0. \tag{15}$$

These 45 points are also defined [16] by the respective equations

$$(X_i + X_j) + \omega(X_k + X_l) + \overline{\omega}(X_m + X_n) = 0, \tag{16}$$

$$X_i + X_j = 0.$$

Permuting the role of the coordinates, we define complementary curves called *Jordan primes*,

$$(ij, kl, mn) \mapsto (x_i + x_j) + \omega(x_m + x_n) + \overline{\omega}(x_k + x_l) = 0, \tag{17}$$

$$(ij) \mapsto x_i - x_j = 0.$$

For example, the point (12) gives the curve $x_1 = x_2$, which contains six points of the form $(12, kl, mn)$, $(ij, 12, mn)$ or $(ij, kl, 12)$ and another six of the form (kl). In general, fixing a point P out of the 45, there are always 12 out of 44 remaining points that lie in the Jordan prime for P.

Consider the decomposition $44 = 12 + 32$ [16]. For the 12 points B, the line PB lies in a P-plane on a set of three such points, and there are 270 such planes. Each P-plane is uniquely the intersection of Jordan primes for two points Q and R, and this set of 5 points is called a 5-cell. Altogether there are 27 such 5-cells, and there are 72 ways of choosing six distinct 5-cells out of 27 with a total of 30 vertices. Given a set of six, there is a *double six* of 5-cells on 30 points, just as there is a double six of lines on the Clebsch surface. The remaining 32 out of 44 points, for any P, lie on 240 lines with three points on each line. An example is the line $(ij), (jk), (ki)$.

A parameter 6/27 appears as a phase δ in low energy lepton mass matrices, along with the basic arithmetic $\pi/12$ phase for neutrinos [2]. Such Koide operators

$$\sqrt{M} = \mu \begin{pmatrix} \sqrt{2} & \delta & \overline{\delta} \\ \overline{\delta} & \sqrt{2} & \delta \\ \delta & \overline{\delta} & \sqrt{2} \end{pmatrix} \tag{18}$$

in $J_3(\mathbb{C})$ belong to the group algebra on C_3 generated by ω, and are diagonalised by the 3×3 Fourier transform of (3), which acts in a triangle of the magic star. On the tetractys tiles of section 2, there are 6 out of 27 trit paths on the centre point of a simplex. Mass scales within the Higgs mechanism then occur in, say, a cubic form for three copies of $\mathbf{E_6}$ on 3×3 algebras, as given in [27], going as $\mu_1 \mu_2 \mu_3$ on matrix triples with equal diagonals.

5 Comments

The matrix (5) is equivalent [18] to the triality representation of the icosidodecahedron,

$$\begin{pmatrix} -1/\phi & 1/\phi & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1/\phi & 1/\phi \\ 1 & 1 & -1/\phi & 1/\phi & 0 & 0 \end{pmatrix}. \tag{19}$$

The ubiquitous scale factor of ϕ is essential for a norm of -1 in $\mathbb{Z}[\phi]$, needed for Lorentzian metrics. In the full theory, where the reals are not fundamental, ϕ appears in infinite dimensions from the limit of Fibonacci ratios. The modular group generator T generalises to the operator

$$\begin{pmatrix} 1 - F_n + F_{n-1} & F_n \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1-\phi & \phi \\ 0 & 1 \end{pmatrix}, \tag{20}$$

which has an eigenvector $(-1/\phi, 0)$. The eigenvector $(-1/\phi, 1)$ from

$$\begin{pmatrix} 1 & -1 \\ -\phi^3 & 0 \end{pmatrix} \tag{21}$$

introduces the other basic scaling of ϕ^{-3} in $\phi \mapsto 2 - \phi$.

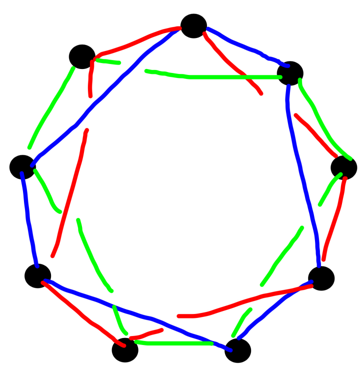

Figure 9: A strongly regular graph on 9 points

Three hexagons in figure 9 are turned into three cubes on a degree 6 graph, for a total of 27 edges. In [19] it is shown that six line configurations with an S_5 action, along with 27 lines, survive over $\mathbb{F}_p$ so long as 5 is a quadratic residue mod p. The first prime to satisfy this condition is 11, with $4^2 = 5$. The projective plane over $\mathbb{F}_{11}$ has 133 points, which is the total dimension of $\mathbf{e_7}$. On the magic star [20][22][23] we build 126 roots for $\mathbf{e_7}$ from the line of points 72+27+27 through conjugate star tips. Recall that Galois primes dictate physical dimensions. Ribbon graphs double the 11 strands of the magnetic braid group B_{11} to 22, just as the magnetic dimension 14 arises as a double of the prime 7, which has a projective plane of order 57. Motivic gravity is telling us how to build up physical manifolds from number theory.

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