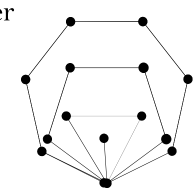
Our objective is to demistify prime gaps in the integers. We will show that the explicit range of prime gaps in the integers is bounded from below by two and above by the expression $2p_{n-1}$, valid for gaps beginning $(p_n^2 - 1) - p_{n-1}$. This upper bound theoretically becomes necessarily greater than empirical observation within empirically verified range, enabling explicit closure on prime gap issues. These results confirm the prime pattens conjecture and the Prime Inter-Square Conjecture (PISC) *Legendre's conjecture.*

Table 1. Evaluating $2p_{n-1}$ establishes the third column, "Theoretical Max Gap." Evaluating $(p_n^2 - 1) - p_{n-1}$. is equivalent to ("Column two" minus one ("- 1")) minus (-) "the preceeding row's Column one," and is computed in the last column. Wikipedia has a nice chart that provides further data. It is remarkable that going forward, the formula necessarily over-limits the prime gaps, that is, the empirical results will not approach the the maximum as they do for these early cases, but nevertheless using this formula easily proves the PISC in all cases by the mere fact that 2n + 1 > 2n. This because, *Proof:* the difference between perfect squares is $2n+1$, and the prime gap upper bound is $2p$ where $n \geq p$; ∴ a prime appears between all perfect squares. ■.

Prime Number	Prime Squared	Theoretical Max gap	Related empirical Gap? Which?	Empirical Maximal Gap	Preceeding Prime	Theoretical Range Start
2	4	2	Yes 2	2	3	2
3	9	4	Yes 3	4	7	6
5	25	6	Yes 4	6	23	21
7	49	10	No	8	89	43
11	121	14	Yes 6	14	113	113

First we report the source of prime numbers. Primes are the prequel to the story of integers closing natural numbers under subtraction and rationals closing integers under division. Prime numbers close the union of circular modules to infinite linear addition. Finite modular arrithmetic is axiomatically closed under binary addition and multiplication. This does not automatically extend to linear settings. For example, consider the natural numbers raised to the one half or second power. To the half power (square root), while maintaining closure under multiplication, only the subsets defined by the fibbonacci sequence and natural squares show additive closure. By the pythagorean theorem the fibbonacci sequence holds infinitely, but outside of this sequence and perfect square summands, binary addition is not a closed operation. $e.g.\ \sqrt{3} + \sqrt{5} = \sqrt{8}.\ But\ \sqrt{3} + \sqrt{6} \neq \sqrt{9}.$

Similarly, to the second power, only the pythagorean tripples are closed under binary addition, but binary multiplication persists. Prime numbers were axiomatically invented to close a linear space under both addition and multiplication. Any set of numbers 1 to n requires new primes greater than n in addition to granting existing primes their composite extensions up to $2n$ to keep additive closure.



Viewed *in vitro*, each prime introduces a factor progression that is coprime by definition to all other progressions. Viewing them incrementally combining succsessive primes generates a palindromic cycle of prime gaps of primorial length each iterartion. We calculate theoretical and asymptotic seasonal statistics based on *in vitro* analysis.

Prime Seasons. Although all prime factor progressions are ubiquitous from the origin, most positions are redundant with a factor's progression. The combined progression of two and three we call the *tonic progression* ¢(2,3). This accounts for all composite numbers other than, *atonal composites* of the form $6k \pm 1$. Interested only in *atonal* composite numbers for the purposes of addressing the variability observed in prime gaps, we consider only composite numbers of this form. By this approach, a seasonal topology emerges in the numberline, such that after a prime's emergence, the next unique value is its' square. Thereafter, two paralell progressions (1 and 5 mod 6) containing all elements with p as least factor appear for all factor progressions greater than 3.

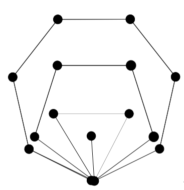
$$Generic\ odd\ orbit\ mod\ 6.\ (p^2,\ p^2+2pk,\ \ p^2+4pk\ )\begin{cases}(1,5,3), if\ 6k+1\}\\(1,3,5), if\ 6k-1\}\end{cases}.$$

Definition: <u>(Well ordered) *seasons*,</u> intervals between prime squares, remarkable for their constant asymptotic composite density as a result of a known finite set of prime factors serving as least factor in all composites.

Organizing Principles

The elemental, *tonic* progression that grounds the natural numbers is the combined orbits of two and three, ⌈¢(2,3)⌉ = 4/6. Which leave only $6k \pm 1$ as prime emergence sites thereafter. We are certain that prime pairs will forever appear no matter how big natural numbers get because we know how composite number orbits work. For *atonal* orbits, i.e. of primes five or greater, there are two types of orbits, one for primes born $6k + 1$ and the other for $6k - 1$ primes. The first composite number with p as least factor is always p^2 since no smaller number can multiply with p to obtain anything greater. We call the period after the prime's emergence but without contributing composites as the least factor the *launching* period. The launching period has length p^2-p. After the square, the orbit has period $6p$ in natural positions $6k \pm 1$.

$$Generic\ prime\ orbit\ mod\ 6.\ (p, 2p, 3p, 4p, 5p, 6p\ )\ \begin{cases}(1,2,3,5,0)\ \ if\ \ 6k+1\}\\(5,4,3,2,1,0)\ if\ \ 6k-1\}\end{cases}.$$



This means, from a composite orbit point of view, there are only four unique prime numbers, $\mathbb{P} = \{2, 3, 6k\text{-}1, 6k\text{+}1\}$. Any prime a multiple of ten different from another prime, has an identical orbit mod ten. In order to manifest the larger given the smaller, we add the appropriate number of tens between each orbit entry. For example, {7, 14, 21, 28, 35, 42, 49, 56, 63, 77} add $10x \rightarrow$ {17, 34, 51, 68, 85, 102, 119, 136, 153, 177…}, or $30x \rightarrow$ {37, 74…}. Viewing it this way is helpful because the orbits, being orbits, repeat *ad infinitum*, therefore bigger numbers are composed of recycled old numbers. For instance, from nine to twenty four, no new orbits enter, and the basic element of six simply repeats three times without causing any problems. At the end of their launching period, the twin prime alumni composite orbits enter. But these orbits have progressively lower frequency with resepct to the basic element. So the highest frequency *atonal* orbit, five, occupies one of each twin prime position every $6p$, that is 30 numbers. We might consider 30 the fundamental unit as equally as six. We would have 30 ± {1, 7, 11, 13} as the template for prime emergence in place of $6k \pm 1$, itself in place of $2k + 1$.

Now, each prime that emerges from a candidate position begins an orbiting cycle of its own centered around a multiple of six. At first, 30 the product of five and six, is greater than five squared. So five's composite orbit begins before a full cycle has finished in its' mod 6 orbit. For five, the launching period (1) $p(p\text{-}1) = 20$, is less than the orbiting period.(2) $6p = 30$. For seven, the two periods are equal, both 42. For primes with k greater than one, the launching period grows increasingly in excess of the orbiting period, causing an increasing number of cycles with existing orbits persisting as they are before new composite orbits enter the number line.

Using transparent combination over the reciporical natural numbers or reciporical primes converge asymptotically to one. With increasing seasons the ratio of new composite prime orbits density, i.e., orbit of the greatest prime such that our observation range is greater than p_s^2, to the number of prime numbers appearing over the same inverval approaches zero. So at exotic parts of the natural number domain, the ratio of novel orbit entries to prime numbers goes to zero. Viewed in this light, the Landau problems become obvious conclusions.

The premise here considers the numberline as the concatenation of independent prime cycles, with new primes emerging when needed to maintain linearity as governed by the axioms of addition and multiplication. This is explicitly depicted by representing natural numbers in a residue number system format using all the primes and in the orbit graph. This means that integer infinitude is an illusion simmilar to infinite space on a stage, formed by differently arranging finite entities to simulate lower and lower frequency objects related to previous objects in a scale free form.

Atonal composite regularities .

$$\forall\, p \in 6k \pm 1.$$

$$1. \boldsymbol{p(x')}: x' = 6p_n x \pm p_n \; [x' \geq p_n^2]$$

$$\mathbf{2.}\; H = [0, 3k + 3N], \qquad (x - 2pH)^2 + y^2 = p^2 :$$

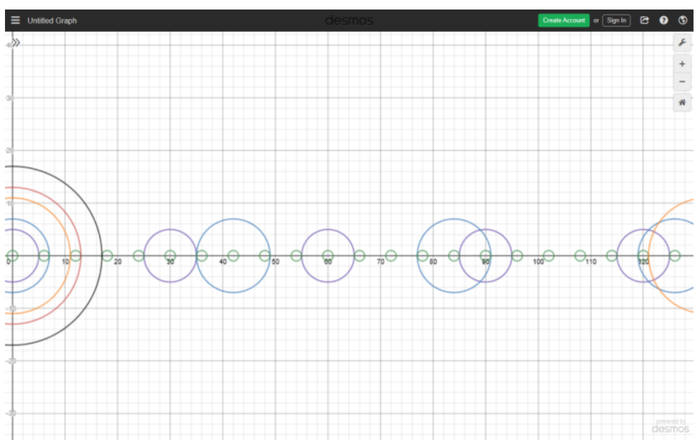


Figure 1 Partial graph of *equation 2* above depicting atonal composite numbers. These numbers appear only at the intersections of the relatively sparse non-green circles. All circles keep a rigid periodic appearance forcing precession of their gaps. Technically each circle kisses an adjacent circle identical to itself all throughout, representing the prime factor progression. Here most redundancies are removed for clarity. It is clear that the green circles are safe to continue *ad infinitum.* Nevertheless, a simple proof is this: Arithmetic progressions (1) $p+np$ have at most one prime number, (2) $p + np'$ have unlimited prime numbers. This is true because gaps sized p, despite occasional interference, are never the part of an orbit because they run paralell to the only progression matching their size (1). Hence, if in (2) we set $n = 1$ and $p' = 2$, we have p an arbitrary prime greater than 2 confirmed to have infinite appearances such that $p + 2$ is prime. ■.

Season, p_s	Range	[¢] Composite Density	Δ [¢] New composite rate	Gap Cap (1) $p_{s+2} - 1/$ (2) $2p_s$	Completed Periods / Season
A, 2	4-8	½	1/2	4	2 (2.000)
B, 3	9-24	2/3	1/6	6	15/6 (2.500)
C,.5	25-48	11/15	2/30	10	23/30 (0.767)
D, 7	49-120	27/35	4/105	12 / 14	71/210 (0.338)
E, 11	121-168	61/77	8/385	16 / 22	47/2310 (0.020)
F, 13	169-288	809/1001	16/1001	18 / 26	119/30030 (.004)

With the tonic progression, primorial 6, the symmetry around a prime gap of three consecutive composites, presents as $g_n = 4$. With primorial 30, the 30 ±5 positions are removed from prime candidacy. This is low frequency modification of the tonic progression. Therefore every cycle, of 30 integers, primes do not emerge because of five's orbit in two out of ten opportunities, two of thirty integers.. All factor orbits of greater cardinality independently reduce prime emergence by pregoessively less (2/14 opportunities, in 42 integers for seven, and then 2/11 opportunities, 2/66 integers, for eleven etc.). This appears like an unpredictable when each factor orbit is indistinguishable from the others, as if it were a unified centrally dictated production. But it is impossible to miss the mechanical rigidity when they are viewed independently. This conceptualization is justified because its' predictions match observations and permit meaningful deductive conclusions with zero uncettainty. The computations become precise when statistical principles, or equivalently, the principle of inclusion-exclusion is applied.

Transparent Combination

Conceptual Equation 1a. $¢(\mathrm{p},\mathrm{p}') = \{¢(p) \cup ¢(p')\} \setminus \{¢(p) \cap ¢(p')\}$

Numerical Equation 1b: $[¢] = ([¢](p) + [¢](p')) - ([¢](p) * [¢](p'))$

Prime	Reciporical	Partial sum	Partial product	Composite Density ¢, (¢ equidense period)	Δ¢ New Composite Density	1/Δ¢ Entries per new composite
2	1/2	-	-	1/2	1/2	2
3	1/3	5/6	1/6	2/3	1/6	6
5	1/5	13/15	2/15	11/15	1/15	15
7	1/7	92/105	11/105	27/35	4/105	26 1/4
11	1/11	332/385	27/385	61/77	8/385	48 1/8
13	1/13	870/1001*	61/1001*	809/1001*	16/1001*	62 9/16*

Prime	Season Range	Cycles in season	Configuration recycle period	Unique composites per cycle	Length of equidense period	Equidense periods per cycle	Total Composites per cycle
2	4-8	2	2	1	2	1	1
3	9-24	2.5	6	1	3	2	4
5	25-48	23/30 (≈2/3)	30	2	15	2	22
7	49-120	71/210 (≈1/3)	210	8	35	6	162
11	121-168	47/2310 (≈2/65)	2310	48	77	30	1830
13	169-288	119/30030 (≈3/757)	30030	480	1001	30	24270

Table 2. The last column in bold is particularly relevant for addressing intuition failures.. This "ratio of primes to new composites" measures the number of primes that enter as a function of the number of new composites that enter relative to before the latest prime factor progression is considered.

Prime	Total Composites per Equidense period	Overall Prime Density	New Composite Density	New Entitiy Density	**Ratio of primes to new composites**
2	1	1/2	1/2	1	**1**
3	2	1/3	1/6	1/2	**2**
5	11	4/15	1/15	1/3	**4**
7	27	8/35	4/105	4/15	**6**
11	61	16/77	8/385	8/35	**10**
13	809	"192/1001"	"16/1001"	16/77	**12**

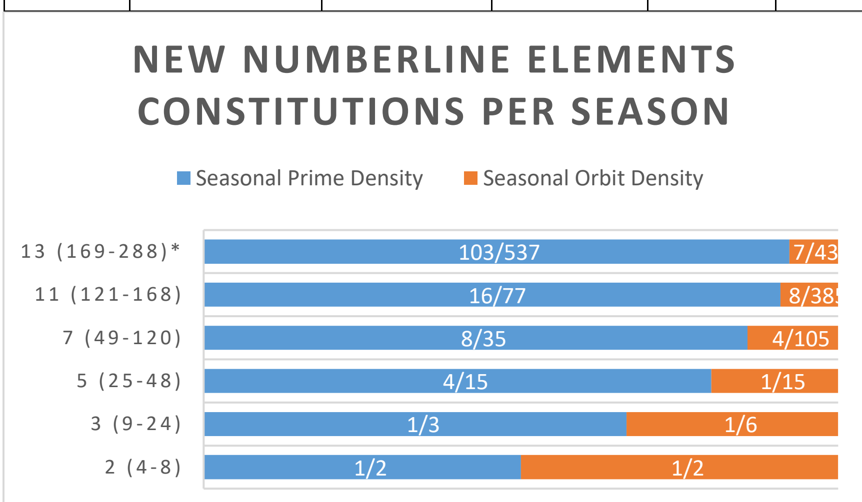


Figure 2. Some popular for profit spreadsheet software takes it upon itself to change the value of small fractions, we leave it in the document as an error to warn other researchers because the software does not signal that it is taking liberties, requiring the user to be aware of the lost precision. The appropriate ratios are {192/1001, and 16/1001}, notice how the simple pattern is veiled by the liberties taken by the software..*

We see empirically that composites due to a factor greater than three contribute by filling gaps at $6p \pm p$. This means that the longest possible gap is that formed when the $6p \pm 1$ positions are occupied by the greatest two factor orbits and the remaining are all factors of $6p$. A practical way to understand why this is true is the fact that between any composite number x such that $p(x)$ is greater than three occurs exclusively when x equals $6p_n \pm p_n$, hence for every appearance there is a gap of length $4p_n$ where there is a drought.

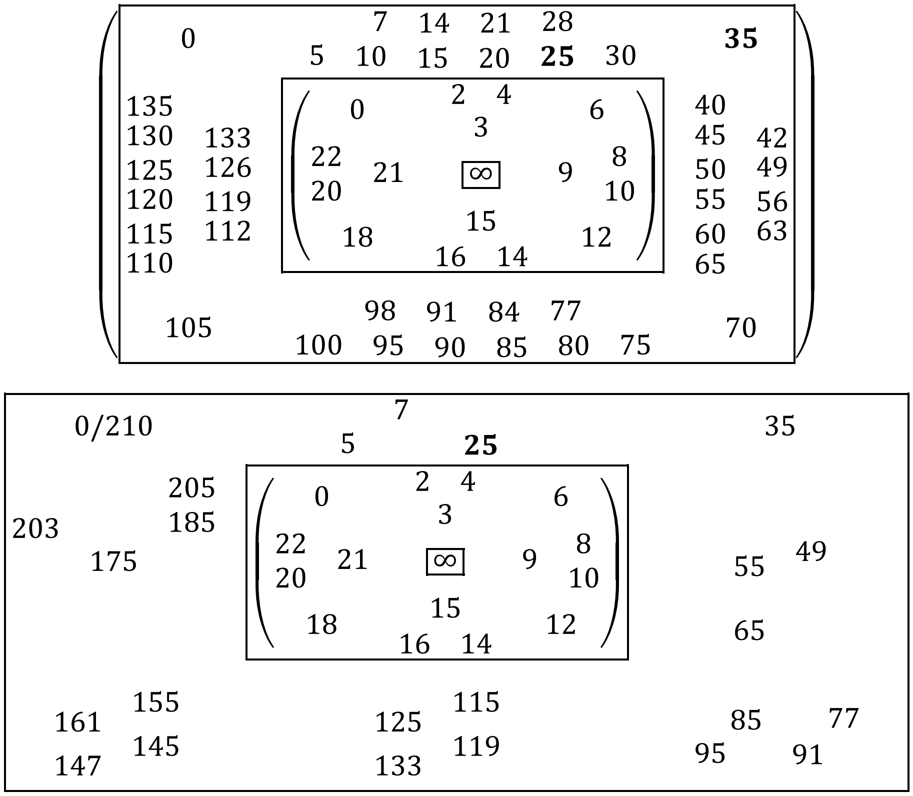
The theoretical implication is that maximal gap length increases as a step function. This function steps with a radius of the preceeding prime on the squares of prime numbers. Practically, due to the exponentially growing primorial with each additional prime, relative to the square of the prime introducing its' factor orbit, the theoretical limit to prime gaps is never seen for factor orbits greater than eleven. We see that the data match the premise in table 1. Since the source of prime gaps is explicitly known, and we have a closed formula for its' upper bound, we have addressed the PISC.

Regarding the lower bound, constructing *pa¢kets* explicitly proves the prime patterns conjecture. The tonic progression fits unoccluded within all combinations of orbits greater than three. This is exactly the microstructure of the number line explicitly detailed, The criterion for prime *k*-tuple admissibility are precisely the complement of pa¢ket formation. Since by definition the pa¢kets persist *ad infinitum* so too do all admissible $k-$ tuples.

Periodicity of factor *packets*: All finite sets of factors produce a palindromic list of composites that repeats at the product of the factors *ad infinitum.* Therefore, putting the beginning zero over any number sharing the factors indicated, the differences in the packet identify composites in that order indefinitely. We equally allow both conventions of indicating the difference between consecutive entries or the absolute difference from the initial zero since it is clear by the resulting numbers which is being used, and indicate that it is a packet rather than *k-tuple* presentation with a terminal ":|", e.g. (0, 2, 1, 1, 2):|. If we were looking only at one prime at a time, we would see an orbit progression ¢(*p*) = (0, *p*):|.

The sub-primes, two and three, form the *tonic progression packet* ¢(2,3)= (0,2,1,1,2):|. The next pair form dominant progression packet ¢(5,7) = (0,**5**,2,**3**,4,**1**,5,**1**,4,**3**,2,**5**):|. We can summarize all pairs of factors with a difference of two as a generalization of this one. The general packet for such factors has a tongue twisting description: palindromic alternating antiparalell parity progression. (0, *q*, *2*, *q-2*, …*1*, *q*, *1*, …, *q-2*, *2*, *q)*. For another concrete example, ¢(11,13) = (0, **11**, 2, **9**, 4, **7**, 6, **5**, 8, **3**, 10, **1**, 11, **1**, 10, **3**, 8, **5**, 6, **7**, 4, **9**, 2, **11**):|. The density of these packets of *composites* / *total numbers*, where $b = q+2$ is $\frac{b+q-1}{bq}$. So ¢(11,13)=23/143.

Now, if we do represent in the *k-tuple* notation convention by adding the elements and replacing the summand by the sum each time. For example, ¢(2,3) = (0, 2, 3, 4, 6):|. This packet becomes remarkable for the fact that it's complement is the twin prime *2- tuple* (0, 2). If two and three were the only prime numbers, twin prime infinitude would be immediately apparent, with primes emerging between the four and six and zero and two. This is obscured as we advance in the number systems. We see a transition from the modular context notation depicting the prime candidate positions in an integral context, to the modular packet context, to the infinite linear context: (-2, 0, 2) ⟶ (0, 2, 0) $\to \begin{matrix} 0 & 2 & 4 \\ 1 & 3 & 5 \end{matrix}$:| respectively for ¢(2), For ¢(2,3): (-6, -4,-3,-2, 0, 2, 3, 4, 6). We see these generalize



$$¢(5,7) = (0,5,2,3,4,1,5,1,4,3,2,5){:}| \;\rightarrow \begin{pmatrix} 0 & \left\{ \begin{matrix} 5 & 10 & 15 & 20 & 25 & 30 \\ 7 & 14 & & 21 & 28 & \end{matrix} \right. & 0 \\ \vdots & & \vdots \\ 210 & \begin{matrix} 215 & 220 & 225 & 231 & 238 \\ 217 & 224 & 230 & 235 & 240 \end{matrix} & 245 \equiv 35 \end{pmatrix}$$

where the top makes some effort of maintaining their relations but the bottom does not.

We now understand that the orbits of primes continue indefinitely precisely at their expected positions. The prime k=tuple admission condition is exactly the negation of a prime orbit. Namely, a prime k-tuple is admissible if and only if the desired progression of residues does *not* span the equivalence classes of any prime less than k. But this is *the **only way*** under canonical conditions *that there **cannot*** be infinitely many occasions for a k – tuple. The twin prime k – tuple, (0,2) is admissible because ¢(2,3) is (0,2,3,4,6):|, so it sneaks in at $6k \pm 1$. Now ¢(5,7) has relatively prime positions that accommodate the twin prime *2*-tuple in many ways. Indeed, all ¢-packets of primes greater than three contain such gaps, we know this expllicitly from the preceeding section. There is no meaningful uncertainty regarding the status of the twin prime or prime pattern conjectures. Its veracity is explicit and in the very definitions but obscured by impressions from the language.

We have shown the explicit range of prime gaps in the integers is bounded from below by two and above by the expression, $2p_{n-1}$, valid with infemum $(p_n^2 - 1) - p_{n-1}$. Thus satisfying the stated goal of this report.