The area and volume of a $J = Q = 0$ black hole

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Abstract

The present note addresses a paper by DiNunno & Matzner, in which the authors claim that 1) the volume of a $J = Q = 0$ black hole as measured in "Schwarzschild coordinates" vanishes and 2) the volume itself is coordinate-dependent. We refute these statements as elementary conceptual mistakes, which originate from a basic misunderstanding of general covariance in the context of the gauge theory of General Relativity.

In face of the widespread misconceptions regarding the geometric structure of black holes (and quite likely that of other exotic objects such as wormholes), as exemplified by a fairly recent work by DiNunno & Matzner[1], I purport to analyze in more careful terms the rather elementary problem of calculating areas and volumes in the context of General Relativity (GR), in a manner that may be readily understood by even undergraduates with some very basic knowledge of calculus and differential geometry. We focus our attention on the one-parameter family of Kerr-Newman vacuum solutions given by $J = Q = 0$, whose interior regions are commonly (but, as we shall briefly see, inaccurately) referred to as "Schwarzschild black holes".

The starting point of the discussion is the statement that a spacetime $(M, \mathbf{g}_{Kug})$ is equipped with the (4-)metric field

$$\mathbf{g}_{Kug} = -\left(1 - \frac{r_S}{R(r)}\right) c^2 dt^2 + \left(1 - \frac{r_S}{R(r)}\right)^{-1} dR(r)^2 + R^2(r)\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{1}$$

The first thing to be stressed for our purposes is this: *it doesn't matter where this field came from.* All that matters is that we are given a metric, and that's all that we're going to need. Next we remark that, even though this expression for the metric $\mathbf{g}_{Kug}$ is often referred to as being in Schwarzschild "coordinates", it should be stressed that the coordinate basis (ct, r, θ, φ) is nothing other than the ordinary (Cartesian time+)spherical coordinates which are very familiar to any physics undergraduate. Finally, we need to pick a specific form of $R(r)$ in order to be able to calculate anything, and at this point physics makes a brief cameo. Historically, at the dawn of GR, two different expressions were proposed[2],[3]: independently of each other, Hilbert and Droste stipulated simply $R = R_{HD} := r$, while Schwarzschild used in his original solution

$R = R_S := \left(r^3 + r_S^3\right)^{\frac{1}{3}}$; from the last expression, we can immediately see that Schwarzschild placed the origin of his spacetime at the event horizon (which in this case can be specified simply by $R = r_S$,$t =$some constant), so that, clearly, his solution does not admit black holes (i.e., a region *interior* to the event horizon). As another aside, we might as well mention that, since Schwarzschild's exterior solution is a vacuum everywhere and everywhen, but it contains regions of nonvanishing Riemann curvature[4], it is obviously unphysical - *unless* we consign an appropriate amount of gravity-causing stress-energy-momentum to the event horizon itself, since the solution is not defined there. But we emphasize, once more, that this argument is a physical digression dependent upon the gauge-theoretical conventions of GR as well as the interpretation that it assigns to geometrical entities such as connections, curvatures, torsions, etc. For the purpose of defining a certain region in a (metric) spacetime, and proceed to calculate related quantities such as areas and volumes, it is utterly irrelevant to know whether curvature is or can be interpreted as gravitational field strength, or whatever; *all that matters is fixing what the metric is, and what the region is.*

With all that in mind, let us proceed to compute the area of a "Schwarzschild" black hole (or, more accurately, of its event horizon), as given by the Hilbert-Droste (HD) solution. It is fairly obvious, from our description of the event horizon, that we may obtain the expression for a restricted 2-metric of our surface by replacing $dt = dr = 0$,$r = r_S$ in the HD solution - namely, $\mathbf{g}^{(2)} = r_S^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right) \equiv r_S^2 \mathbf{g}_{\mathbb{S}^2}$. Since the invariant volume-element of the surface is simply $\sqrt{\left|\mathbf{g}^{(2)}\right|} d\theta d\varphi$, we find with no difficulty that

$$A_S := \int_0^{\pi} \int_0^{2\pi} r_S^2 \sin\theta d\theta d\varphi = 4\pi r_S^2 \tag{2}$$

So, we conclude the black hole area is just that of ordinary $\mathbb{S}^2$ in units of r_S, which is easy to swallow. Now, for the traumatic part: the 3-metric of the volume is obtained from the substitution of $dt = 0$ in the HD solution, yielding $\mathbf{g}^{(3)} = \left(1 - \frac{r_S}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$ - but there's a catch now: for $0 < r < r_S$, we see that $1 - \frac{r_S}{r} < 0$; because of this, the black hole has an effective indefinite spatial $1+2$ metric with signature $[-,+,+]$, which means its volume-element must be defined as $\sqrt{-\left|\mathbf{g}^{(3)}\right|} dr d\theta d\varphi$, just like it is done in special relativity[5]. That said, we have

$$\begin{aligned} V_S \quad : \quad &= \int_0^{r_S} \int_0^{\pi} \int_0^{2\pi} \left(\frac{r_S}{r} - 1\right)^{-\frac{1}{2}} r^2 \sin\theta dr d\theta d\varphi \\ &= \frac{4\pi r_S^3}{3} \times \int_0^1 \frac{3x^2}{\sqrt{\left(\frac{1}{x} - 1\right)}} dx = \frac{4\pi r_S^3}{3} \times \frac{15\pi}{16} \end{aligned} \tag{3}$$

And this result, that the volume of the black hole is just a little under three times that subtended by $\mathbb{S}^2$ in units of r_S, while surprising, is still nonvanishing

- in contradistinction to the claim of [1], which seems all the more unfounded in that the authors also write down the above expression for $\mathbf{g}^{(3)}$ immediately before. The reasoning forwarded to substantiate this mistake is an appeal to the Kruskal-Szekeres map - which leads me to my next point. It is important to note that coordinate (or *metric*) singularities - such as the locus of events with $r = r_S$ in $\mathbf{g}_{Kug}$ - are quite common; you're welcome to assess this by my claim that the z-axis is a singular locus already in the basis of spherical coordinates. However, if, for some reason, you don't like them, you can "get rid" of them; to illustrate this point, we can introduce the coordinate transform $(ct, r, \theta, \varphi) \mapsto (v_\pm, r, \theta, \varphi)$ with $v_\pm$ given by $v_\pm := ct \pm r_S \ln\left|\frac{r}{r_S} - 1\right|$, so that $d\left[ct\left(v_\pm, r\right)\right] = dv_\pm \mp \left(\frac{r}{r_S} - 1\right)^{-1} dr$; it is easy to see that, after a little algebra, substitution of this into the HD metric gives

$$\begin{aligned}
\mathbf{g}_{HD} &= -\left(1 - \frac{r_S}{r}\right) dv_\pm^2 \pm 2\frac{r_S}{r} dv_\pm dr + \left(1 + \frac{r_S}{r}\right) dr^2 + r^2 \mathbf{g}_{\mathbb{S}^2} \qquad (4) \\
&= -dv_\pm^2 + dr^2 + r^2 \mathbf{g}_{\mathbb{S}^2} + \frac{r_S}{r}\left(dv_\pm \pm dr\right)^2, \\
\mathbf{g}_{HD}^{-1} &= -dv_\pm^2 + dr^2 + \frac{1}{r^2}\mathbf{g}_{\mathbb{S}^2}^{-1} - \frac{r_S}{r}\left(dv_\pm \mp dr\right)^2 \qquad (5)
\end{aligned}$$

(beware the begginer's trap: $2\frac{r_S}{r} dv_\pm dr$ really means $\frac{r_S}{r} dv_\pm \otimes dr \oplus \frac{r_S}{r} dr \otimes dv_\pm$, and so forth.) These are the famous Eddington-Finkelstein[1] (EF) "coordinates" (by this point, you should be able to tell the reason for the "": there's really just *one* Eddington-Finkelstein coordinate - namely, the "time(s)" $v_\pm$), and you can readily check that in this basis $\mathbf{g}_{HD}$, as well as its inverse $\mathbf{g}_{HD}^{-1}$, are indeed free of the $r = r_S$ metric singularity - even though the "z-axis" one persists. Introducing the EF scheme is more than just pedagogical, though: for, even as there is an abundance of other "coordinates" in the literature - Gullstrand-Painlevé, Lemaître, Novikov, isotropic, Kruskal-Szekeres,... -, the principle of general covariance (which, perhaps you haven't been told, is a purely mathematical ingredient of differential geometry that is independent of any physics) stipulates that they are all each equally as good to describe the region they cover - so, w.l.o.g., one can keep to a given scheme - say EF - and completely ignore the others. In any case, in any analysis of $v_\pm$ the very first thing anyone should notice is that it doesn't cover the entire region covered by the metric, thanks to the function $r_S \ln\left|\frac{r}{r_S} - 1\right|$, which is two-to-one - which is why we define two of them. One may think, at first, that this need for two "times" rather than just one is some sort of deficiency of the EF scheme; however, since differential geometry warrants us the use of *coordinate charts*, we can always cover our manifolds with an *atlas* of such charts, no matter how many - provided only we use them correctly[5].

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[1] Notice that another popular "recipe" is given as $v_{MTW\pm} := ct \pm \left(r + r_S \ln\left|\frac{r}{r_S} - 1\right|\right)$, according to which we obtain $\mathbf{g}_{HD} = -\left(1 - \frac{r_S}{r}\right)\left(dv_{MTW\pm}\right)^2 \pm 2dv_\pm^{MTW} dr + r^2 \mathbf{g}_{\mathbb{S}^2}$; cf. [4],[6],[7].

Another thing that one may wonder about is the "vanishing" of the event horizon; well, it's not that this one particular region of spacetime vanished - it's merely that you defined it away with the introduction of $v_{\pm}$, so that you don't have to stare at the ugly face of the $r = r_S$ singularity. Here, we return to the reasoning of DiNunno & Matzner: they'd have us believe that, in order to find the 3-metric for the volume of the black hole in the EF scheme, we have to put $dv_{\pm} = 0$ in (4). If this sounds innocent enough to you, consider this: you're assigned the problem of computing the area of $\mathbb{S}^2$ in Euclidian space; starting with the 3-metric $dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$ in the ordinary spherical coordinates, you notice that you can get the invariant volume-element for the surface by putting $dr = 0$,$r = 1$, and then repeating the procedure we've previously delineated, thus obtaining $A_{\mathbb{S}^2} = 4\pi$. You're not sure of this, though, so you cross-check the calculation by going to Cartesian coordinates, in which the 3-metric reads $dx^2 + dy^2 + dz^2$; repeating the very same procedure, you put $dx = 0$,$x = 1$, but strangely your final result is that $A_{\mathbb{S}^2} \neq 4\pi$. From this you brilliantly conclude that the volume of a beach ball depends on the coordinate system you use to calculate it, then you sign and deliver your paper to the teacher. Joking aside, it's clear that the correct procedure is to put $dv_{\pm} = \pm\left(\frac{r}{r_S} - 1\right)^{-1} dr$ in the metric when using the EF scheme - then we go on to, "surprisingly", obtain the exact same thing we computed in the "Schwarzschild" "coordinates".

I hope this is of some help for those willing to learn GR; in particular, the more ambitious student is encouraged to match its wits with the generic three-parameter $(\{(M, J, Q)\} \simeq \{(r_S, a, r_Q)\})$ Kerr-Newman family of solutions, for which, in Boyer-Lindquist "coordinates", the metric reads

$$\begin{aligned} \mathbf{g}_{KN} &= -\frac{\Delta}{\rho_{BL}^2}\left(cdt - a\sin^2\theta d\varphi\right)^2 + \frac{\rho_{BL}^2}{\Delta}dr^2 + \\ &+\rho_{BL}^2\left[d\theta^2 + \frac{\sin^2\theta}{\rho_{BL}^4}\left[\left(r^2 + a^2\right)d\varphi - acdt\right]^2\right], \end{aligned} \tag{6}$$

$$\begin{aligned} \mathbf{g}_{KN}^{-1} &= -\frac{1}{\Delta\rho_{BL}^2}\left[\left(r^2 + a^2\right)cdt + ad\varphi\right]^2 + \frac{\Delta}{\rho_{BL}^2}dr^2 + \\ &+\frac{1}{\rho_{BL}^2}\left[d\theta^2 + \frac{1}{\sin^2\theta}\left(d\varphi + a\sin^2\theta cdt\right)^2\right] \end{aligned} \tag{7}$$

with the ancilla

$$\rho_{BL}^2(r, \theta) \ : \ = r^2 + a^2\cos^2\theta \tag{8}$$

$$\Delta(r) \ : \ = r(r - r_S) + a^2 + r_Q^2 \tag{9}$$

After that, you can have a showdown with it in Kerr-Schild[2] "coordinates"

$$\begin{aligned}\mathbf{g}_{KN} &= -dv_{KS}^2 + dr^2 + r^2\mathbf{g}_{\mathbb{S}^2} + \frac{\rho_{KS}^2\left(\rho_{KS}r_S - r_Q^2\right)}{\left(\rho_{KS}^4 + r^2a^2\cos^2\theta\right)} \times \\ &\left\{dv_{KS} + \left[\frac{\rho_{KS}\left(r\sin^2\theta dr + r^2\sin\theta\cos\theta d\theta\right)}{\rho_{KS}^2 + a^2} + \right.\right. \\ &\left.\left. + \frac{a\left(r^2\sin^2\theta d\varphi\right)}{\rho_{KS}^2 + a^2} + \frac{\left(r\cos^2\theta dr - r^2\sin\theta\cos\theta d\theta\right)}{\rho_{KS}}\right]\right\}^2\end{aligned} \tag{10}$$

with the ancilla

$$\rho_{KS}^2(r,\theta)\left[\rho_{KS}^2(r,\theta) - r^2 + a^2\right] := r^2a^2\cos^2\theta \tag{11}$$

and see what is different. For reference, and perhaps motivation, I leave the result for the area: $A_{KN} = 4\pi r_+^2 \times \frac{\left(r_+^2 + a^2\right)}{r_+^2}$.

References

[1] B. S. DiNunno, R. A. Matzner, Gen. Relativ. Gravit., 2010, 42 (1), 63-76; arXiv:0801.1734 [gr-qc]

[2] S. Antoci, D.-E. Liebscher, Astron. Nachr., 2001, 322 (3), 137-142; arXiv:gr-qc/0102084

[3] I. Mol, "Revisiting the Schwarzschild and the Hilbert-Droste Solutions of Einstein Equation and the Maximal Extension of the Latter"; arXiv:1403.2371 [math-ph]

[4] C. W. Misner, K. S. Thorne, J. A. Wheeler, *Gravitation*. Freeman & Co., 1973

[5] B. F. Schutz, *Geometrical Methods of Mathematical Physics*. Cambridge University Press, 1980

[6] A. S. Eddington, Nature, 1924, 113 (2832), 192

[7] D. Finkelstein, Phys. Rev., 1958, 110 (4), 965-967

[8] R. P. Kerr, A. Schild in: *Atti del Convegno sulla Relativita Generale: Problemi dell'Energia e Onde Gravitazionali*. G. Barbèra Editore, 1965, pp. 1-12. Reprinted in R. P. Kerr, A. Schild, Gen. Relativ. Gravit., 2009, 41 (10), 2485-2499

[2] In spite of my best efforts to find a simple expression that immediately reduces to the EF case as written above, unfortunately the literature on the subject is rather technical, with nomenclature all over the place, so I settled on the original expression (adapted to include charge) as written in eq. 5.8 of [8].