Tutorial: Continuous-Function on Closed Interval Basics, with Mean-Value and Taylor Theorem Upshots

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Abstract

This tutorial explores the relation of the local concept of a function's continuity to its global consequences on closed intervals, such as a continuous function's unavoidable boundedness on a closed interval, its attainment of its least upper and greatest lower bounds on that interval, and its unavoidable assumption on that closed interval of all of the values which lie between that minimum and maximum. In a nutshell, continuous functions map closed intervals into closed intervals. It is understandable that verifying this local-to-global fact involves subtle and very intricate manipulation of the least-upper-bound/greatest-lower-bound postulate for the real numbers. In conjunction with the basic inequality properties of integrals, this continuous-function fact immediately implies the integral form of the mean-value theorem, which is parlayed into its differential form by the fundamental theorem of the calculus. Taylor expansion and its error estimation are further developments which are intertwined with these fascinating concepts.

The basic properties of a continuous function on a closed interval

The crucial basic properties of a continuous real function defined on a closed interval are that it *is bounded*, *attains* its least upper and greatest lower bounds, *and also assumes all values between that maximum and minimum.* To *establish these properties* involves *intricate manipulation* of the least-upper-bound postulate.

We begin with a real function $f(x)$ defined as a real number at every point of the positive-length closed interval $[a, b] = \{x|a \le x \le b,\ a < b\}$, *but we assume that $f(x)$ is unbounded above on* $[a, b]$. We then show that *this assumption implies the existence of a point $c \in [a, b]$ where $f(x)$ cannot be continuous.* To carry out that demonstration, we select an infinite set of *distinct* points $\{x_0, x_1, x_2, \ldots\}$ of $[a, b]$ such that $f(x_0) \ge 1$, $f(x_1) \ge 2f(x_0)$,..., $f(x_n)) \ge 2f(x_{n-1})$, $n = 1, 2, \ldots$. We next consider the following sequence of nested subsets $\{x_0, x_1, x_2, \ldots\} \supset \{x_1, x_2, x_3, \ldots\} \supset \{x_2, x_3, x_4, \ldots\} \supset \cdots$ of $[a, b]$, together with *the decreasing sequence* of the least upper bounds of each of these nested subsets, $r_0 \stackrel{\text{def}}{=} \sup\{x_0, x_1, x_2, \ldots\} \ge r_1 \stackrel{\text{def}}{=} \sup\{x_1, x_2, x_3, \ldots\} \ge r_2 \stackrel{\text{def}}{=} \sup\{x_2, x_3, x_4, \ldots\} \ge \cdots$, and also the greatest lower bound (in fact limit) $c = \inf\{r_0, r_1, r_2, \ldots\}$. Since $\{x_0, x_1, x_2, \ldots\} \subset [a, b]$, and $[a, b]$ is a *closed* interval, $r_0 \stackrel{\text{def}}{=} \sup\{x_0, x_1, x_2, \ldots\}$ is an element of $[a, b]$. Likewise r_1, r_2, ... are all elements of $[a, b]$, so the point $c = \inf\{r_0, r_1, r_2, \ldots\}$ is as well an element of the *closed* interval $[a, b]$. We note that *since x_0, x_1, x_2, ... are chosen so that no two of them can be equal, $r_n > a$ for all $n = 0, 1, 2, \ldots$,* where a is the left endpoint of $[a, b]$.

The *decreasing sequence* of least upper bounds $r_0 = \sup\{x_0, x_1, x_2, \ldots\} \ge r_1 = \sup\{x_1, x_2, x_3, \ldots\} \ge \ldots$ can exhibit *two distinct behaviors vis-à-vis its greatest lower bound c*: (1) starting from a $k \ge 0$, $r_k = c$, $r_{k+1} = c, \ldots,$ *or* (2) for *all* of the $l = 0, 1, \ldots$, $r_l > c$. In *both cases $f(x)$ is unbounded above in every positive-length open interval* $(c - \delta,\ c + \delta) \stackrel{\text{def}}{=} \{x|c - \delta < x < c + \delta,\ \delta > 0\}$ *around c, so $f(x)$ isn't continuous at $x = c$.* We now show for cases (1) and (2) successively that *given an arbitrary $\delta > 0$ and an arbitrary nonnegative integer m*, there exists an $x \in (c - \delta,\ c + \delta)$ for which $f(x) \ge 2^m$.

In case (1), let $n = \max(k, m)$. Then $c = r_n$, where $r_n = \sup S_n$, and $S_n \stackrel{\text{def}}{=} \{x_n, x_{n+1}, x_{n+2}, \ldots\}$. For all $x \in S_n$, $f(x) \ge 2^n \ge 2^m$. Because $c = \sup S_n$, there *exists* an $x \in S_n$ such that $x \le c$ and $c - x < \delta$, *so that $x \in (c - \delta,\ c + \delta)$ and also $f(x) \ge 2^m$.*

In case (2), there *exists* a nonnegative integer k such that for all nonnegative integers l which satisfy $l \ge k$, it is still true that $r_l > c$, but it is *also* true that $r_l < c + (\delta/2)$ because *c is the greatest lower bound (in fact the limit) of the decreasing sequence $r_0 \ge r_1 \ge r_2 \cdots$ of least upper bounds.* We let $n = \max(k, m)$, and note that for all $x \in S_n \stackrel{\text{def}}{=} \{x_n, x_{n+1}, x_{n+2} \ldots\}$, $f(x) \ge 2^n \ge 2^m$, and since $r_n = \sup S_n$, there *exists* an $x \in S_n$ such that $x \le r_n$ and $(r_n - x) \le (\delta/2)$. Since $r_n < c + (\delta/2)$ and $x \le r_n$, $x \le c + (\delta/2)$. Also, since $r_n > c$ and $r_n \le x + (\delta/2)$, $c \le x + (\delta/2) \Rightarrow x \ge c - (\delta/2)$, which together with $x \le c + (\delta/2)$ implies that $x \in (c - \delta,\ c + \delta)$. So *in case* (2) *as well as case* (1) there exists an $x \in (c - \delta,\ c + \delta)$ for which $f(x) \ge 2^m$.

Thus if a function $f(x)$ that is defined as a real number everywhere on the positive-length closed interval $[a, b]$ *is unbounded above on that interval, there exists a point $c \in [a, b]$ where $f(x)$ cannot be continuous*; this lemma is obviously readily extended to the case that $f(x)$ is *unbounded below* on $[a, b]$.

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To show that a *continuous* real function $f(x)$ on a closed interval $[a,b]$ *attains* its least upper bound $\sup_{x\in[a,b]} f(x)$ on that interval *again involves selecting a set of distinct points* $\{x_0, x_1, x_2, \dots\} \subset [a,b]$, *but here the distance by which* $f(x_0)$ *is below* $\sup_{x\in[a,b]} f(x)$ *is required to be less than unity, and the distance by which* $f(x_n)$ *is below* $\sup_{x\in[a,b]} f(x)$ *is required to be half or less of the distance by which* $f(x_{n-1})$ *is below* $\sup_{x\in[a,b]} f(x)$, $n = 1, 2, \dots$. It *is initially supposed that* $f(x)$ *is strictly below* $\sup_{x\in[a,b]} f(x)$ *at every* $x \in [a,b]$, *since if that wasn't the case, no argument would be needed*. Exactly as in the previous discussion one considers the point $c = \inf\{r_0, r_1, r_2, \dots\}$, where $r_0 \stackrel{\text{def}}{=} \sup\{x_0, x_1, x_2, \dots\} \ge r_1 \stackrel{\text{def}}{=} \sup\{x_1, x_2, x_3, \dots\} \ge r_2 \stackrel{\text{def}}{=} \sup\{x_2, x_3, x_4, \dots\} \ge \dots$. Study of the situation in any given positive-length open interval around c will of course *show the presence of points* x' *at which* $f(x')$ *is below* $\sup_{x\in[a,b]} f(x)$ *by less than any specified amount*. Since $f(x)$ *is assumed to be continuous at every point in* $[a,b]$, *including at the point* c, $f(c)$ can therefore only have the value $\sup_{x\in[a,b]} f(x)$, *quashing* the initial supposition *that* $f(x)$ *is strictly below* $\sup_{x\in[a,b]} f(x)$ *at every* $x \in [a,b]$. The reader is invited *to develop the detailed arguments which establish the situation in every positive-length open interval around* c by using the previously developed similar arguments *as a guide*. To show that a *continuous* real function $f(x)$ on a closed interval $[a,b]$ *attains its greatest lower bound* $\inf_{x\in[a,b]} f(x)$ *on that interval obviously involves arguments very closely analogous to those made here*.

Given a *continuous* real function $f(x)$ on $[a,b]$ for which $f(a) < f(b)$, we now show that $f(x)$ *assumes in* $[a,b]$ *any real* r *such that* $f(a) < r < f(b)$. Denoting as $S^{\text{re}}_{f<r}$ *the set of right endpoints* y *of the closed sub-intervals* $[a,y]$ *of* $[a,b]$ *that satisfy* $x \in [a,y] \Rightarrow f(x) < r$, and denoting $\sup S^{\text{re}}_{f<r}$ as d, we note that $a < d < b$ because *continuity of* $f(x)$ *at* $x = a$ *and* $x = b$ *plus* $f(a) < r < f(b)$ implies the existence of x_1 and x_2 in $[a,b]$ such that $a < x_1 < x_2 < b$ and $f(x) < r$ for all $x \in [a, x_1]$ and also $f(x) > r$ for all $x \in [x_2, b]$. Since $d > a$, *a sufficiently short positive-length open interval around* d always has points $x' < d$ at which $f(x') < r$ *because* d *is the least upper bound of* $S^{\text{re}}_{f<r}$, and since $d < b$, such an interval *also* always has points $x'' \ge d$ at which $f(x'') \ge r$ *because* d *is an upper bound of* $S^{\text{re}}_{f<r}$. Therefore $f(d)$ *must be equal to* r since (1) if $f(d) > r$, such an interval *which doesn't have points* x' *where* $f(x') < r$ *always exists by continuity*, and (2) if $f(d) < r$ such an interval *which doesn't have points* x'' *where* $f(x'') \ge r$ *always exists by continuity*. Given $f(a) > f(b)$, this demonstration can obviously be readily *extended* to any real s such that $f(a) > s > f(b)$.

These lemmas concerning "filling in" the function values between the $[a,b]$ closed-interval endpoint values $f(a)$ and $f(b)$ of a continuous function $f(x)$ are readily extended to "filling in" the function values *between the minimum and maximum of* $f(x)$ *on the closed interval* $[a,b]$ because there is a point $c \in [a,b]$ where $f(x)$ attains its minimum value $\min_{x\in[a,b]} f(x)$ and a point $d \in [a,b]$ where $f(x)$ attains its maximum value $\max_{x\in[a,b]} f(x)$, so to demonstrate the "filling in" between the minimum and maximum values of $f(x)$ on the closed interval $[a,b]$ *entails simply shifting focus to the particular closed sub-interval* $[c,d]$ *of* $[a,b]$. The *upshot* is that a continuous function $f(x)$ on a closed interval $[a,b]$ *assumes every value between its minimum and maximum value on* $[a,b]$—it in fact maps the closed interval $[a,b]$ onto the closed interval $[\min_{x\in[a,b]} f(x),\ \max_{x\in[a,b]} f(x)]$, so *continuous functions map closed intervals into closed intervals*.

The mean-value theorems from continuous-function and calculus basics

A *continuous function* $f(x)$ *on* $[a,b]$ *attains its minimum* $\min_{x\in[a,b]} f(x)$ *and maximum* $\max_{x\in[a,b]} f(x)$. This and any of the usual definitions of the integral—e.g., the Riemann definition of the integral—yield that,

$$\begin{gathered}\int_a^b f(x)dx \ge \big(\min_{x\in[a,b]} f(x)\big)(b-a) \ \text{ and } \ \int_a^b f(x)dx \le \big(\max_{x\in[a,b]} f(x)\big)(b-a), \text{ which implies that,} \\ \min_{x\in[a,b]} f(x) \ \le \ (b-a)^{-1} \int_a^b f(x)dx \ \le \ \max_{x\in[a,b]} f(x).\end{gathered} \tag{1a}$$

Since $f(x)$ *assumes every value between its minimum and maximum on* $[a,b]$, there is a $c \in [a,b]$ such that,

$$f(c) = (b-a)^{-1} \int_a^b f(x)dx, \tag{1b}$$

a result which after being multiplied through by $(b-a)$ yields,

$$\int_a^b f(x)dx = f(c)(b-a) \text{ for } c \in [a,b]. \tag{1c}$$

This is *the integral form of the mean-value theorem; it assumes that* $f(x)$ *is continuous for* $x \in [a,b]$. If *we replace* $f(x)$ *in* Eq. (1c) *by a function's derivative* $F'(x)$ *which is continuous for* $x \in [a,b]$, we obtain,

$$\begin{gathered}\int_a^b F'(x)dx = F'(c)(b-a) \text{ for } c \in [a,b], \text{ which by the fundamental theorem of the calculus yields that,} \\ (F(b) - F(a)) = F'(c)(b-a) \text{ for } c \in [a,b].\end{gathered} \tag{1d}$$

This is *the differential form of the mean-value theorem; it assumes that* $F'(x)$ *is continuous for* $x \in [a,b]$.

Taylor expansion and estimation of the error entailed

The coefficient c_k, $k = 0, 1, 2, \ldots, n$, of the kth power of x in the polynomial $P_n(x)$,

$$P_n(x) \stackrel{\text{def}}{=} \textstyle\sum_{k=0}^{n} c_k\, x^k, \tag{2a}$$

is rather simply related to the kth-order derivative with respect to x of $P_n(x)$, evaluated at $x = 0$,

$$\begin{gathered} d^l(P_n(x))/dx^l = \theta(n - l + \tfrac{1}{2}) \textstyle\sum_{k=l}^{n} c_k\, k(k-1)\cdots(k-l+1)x^{k-l} \;\Rightarrow \\ \big(d^l(P_n(x))/dx^l\big)_{x=0} = \theta(n - l + \tfrac{1}{2})\, c_l\, l! \;\Rightarrow\; c_k = \big(d^k(P_n(x))/dx^k\big)_{x=0}/k!\,,\; k = 0, 1, 2 \ldots . \end{gathered} \tag{2b}$$

The *upshot* of the Eq. (2b) determination of the coefficient c_k of the kth power of x in the polynomial $P_n(x)$ in terms of the kth-order derivative of $P_n(x)$, evaluated at $x = 0$, *is a representation of $P_n(x)$ in terms of its derivatives to all orders, evaluated at* $x = 0$, namely,

$$P_n(x) = P_n(x = 0) + \textstyle\sum_{k=1}^{\infty} \big(d^k(P_n(x))/dx^k\big)_{x=0}\, x^k/k!. \tag{2c}$$

The Eq. (2c) and (2a) representations of $P_n(x)$ are in terms of powers of x, which is advantageous when x has a value close to zero. Since that isn't always the case, we now *extend* the Eq. (2a) and (2c) representations of $P_n(x)$ from powers of x to powers of $(x - x_0)$, where x_0 is an arbitrary constant,

$$\begin{gathered} P_n(x) = \textstyle\sum_{k=0}^{n} b_k\,(x - x_0)^k \;\Rightarrow\; \big(d^l(P_n(x))/dx^l\big)_{x=x_0} = \theta(n - l + \tfrac{1}{2})\, b_l\, l! \;\Rightarrow \\ \big(P_n(x) - P_n(x_0)\big) = \textstyle\sum_{k=1}^{\infty} \big(d^k(P_n(x))/dx^k\big)_{x=x_0}\,(x - x_0)^k/k!. \end{gathered} \tag{2d}$$

These Taylor representations of any polynomial $P_n(x)$ are exact in principle since the kth-order derivatives $d^k(P_n(x))/dx^k$ that occur in Eq. (2d) vanish identically for all values of k such that $k > n$. However, notwithstanding that we have *developed* the Eq. (2d) Taylor representations *by using polynomials* $P_n(x)$, for which these Taylor representations *are exact in principle*, we obviously *would like to apply the* Eq. (2d) *Taylor representations to any function $F(x)$ which possesses the requisite kth-order derivatives at $x = x_0$.* For such *general-function application* of the Eq. (2d) Taylor representations, we obviously *also* would wish to have a systematic means *of estimating the errors which are made*. To *begin thinking* about error estimation for Eq. (2d) applied to a general $F(x)$, *we first drastically downgrade* Eq. (2d) *by truncating its entire right side*, thus producing *the most rudimentary conceivable extension of Taylor expansion to $F(x)$, i.e.*,

$$\big(F(x) - F(x_0)\big) = 0. \tag{2e}$$

For this most rudimentary conceivable Taylor expansion of $F(x)$, the Eq. (1d) differential form of the mean-value theorem *supplies the desired error estimate*. Taking $b = x$ and $a = x_0$ in Eq. (1d) yields,

$$\big(F(x) - F(x_0)\big) = F'(c)(x - x_0) \text{ for } c \in [x_0,\, x], \tag{2f}$$

Of course this application of Eq. (1d) *assumes that the derivative $F'(x')$ of $F(x')$ is continuous for all values of x' in the closed interval* $[x_0, x]$. The *derivation* of the Eq. (1d) differential form of the mean-value theorem from Eq. (1c) *involved the fundamental theorem of the calculus*, and it turns out that the fundamental theorem *itself* can be made to generate error renditions *for less and less severely truncated versions of the* Eq. (2d) *Taylor expansion*. To *begin* to show this, we *slightly modify* that theorem's *presentation* as follows,

$$\textstyle\int_{x_0}^{x} F'(x')dx' = F(x) - F(x_0) \;\Rightarrow\; \big(F(x) - F(x_0)\big) = \int_{x_0}^{x} F^{(1)}(x')dx'. \tag{2g}$$

From Eq. (2g) we see *in particular* that for the Eq. (2e) *most rudimentary conceivable Taylor expansion* $(F(x) - F(x_0)) = 0$, the Eq. (2g) fundamental theorem of the calculus *yields the error as* $\int_{x_0}^{x} F^{(1)}(x')dx'$, *to which application of the* Eq. (1c) *integral form of the mean-value theorem yields the* Eq. (2f) *error estimate*.

But *vastly more interesting* is that *successive integrations by parts* of the integral term $\int_{x_0}^{x} F^{(1)}(x')dx'$ of the Eq. (2g) fundamental theorem *generates successive refinements* of the Eq. (2e) most rudimentary conceivable Taylor expansion $(F(x) - F(x_0)) = 0$ *in complete correspondence with successive terms of the* Eq. (2d) *Taylor sum* (which of course is exact for polynomials). In *addition* to generating the successive terms of the Eq. (2d) Taylor sum for $F(x)$, the successive integrations by parts of the fundamental theorem of the calculus *also feature a compact integral rendition of the error entailed by approximating $F(x)$ by the Taylor-sum terms which they have generated*. We now display the results *of the first three integrations by parts* of the integral $\int_{x_0}^{x} F^{(1)}(x')dx'$ of the fundamental theorem of the calculus,

$$\big(F(x) - F(x_0)\big) = \int_{x_0}^{x} F^{(1)}(x')dx' = F^{(1)}(x_0)(x - x_0) + \int_{x_0}^{x}(x - x')F^{(2)}(x')dx' =$$

$$F^{(1)}(x_0)(x - x_0) + F^{(2)}(x_0)(x - x_0)^2/2 + (1/2)\int_{x_0}^{x}(x - x')^2 F^{(3)}(x')dx' = \tag{2h}$$

$$F^{(1)}(x_0)(x - x_0) + F^{(2)}(x_0)(x - x_0)^2/2 + F^{(3)}(x_0)(x - x_0)^3/6 + (1/6)\int_{x_0}^{x}(x - x')^3 F^{(4)}(x')dx'.$$

It now seems apparent that nth integration by parts of the Eq. (2g) fundamental calculus theorem yields,

$$\big(F(x) - F(x_0)\big) = \theta(n - \tfrac{1}{2})\sum_{k=1}^{n} F^{(k)}(x_0)(x - x_0)^k/k! + (1/n!)\int_{x_0}^{x}(x - x')^n F^{(n+1)}(x')dx', \tag{2i}$$

which is readily *inductively verified* via integration by parts of its last term. Eq. (2i) presents only the first n terms of the Eq. (2d) *infinite* Taylor sum, *but its last term is an integral rendition of the error entailed by using those first n terms of the* Eq. (2d) *infinite Taylor sum.* The Eq. (2d) *infinite* Taylor sum is exact for polynomials and entire functions, *but it can also easily diverge*, e.g., for analytic functions which hem in its radius of convergence with complex-plane points of non-analyticity. A *sufficient* condition *for the overall validity of* Eq. (2i) *with its error integral included* is for $F^{(n+1)}(x')$ to be continuous for all $x' \in [x_0, x]$.

Direct estimation of *the integral rendition of the error in* Eq. (2i) may be inconvenient in practice. We therefore work out an integral-free mean-value-theorem style of presentation of that error term. To do so *we of course assume that $F^{(n+1)}(x')$ is continuous for all $x' \in [x_0, x]$.*

Direct application to the error-rendition integral $(1/n!)\int_{x_0}^{x}(x-x')^n F^{(n+1)}(x')dx'$ in Eq. (2i) of the integral mean-value theorem is inhibited by the presence of the factor $(x - x')^n$ in the integrand of that integral. Therefore we attempt to "flatten" that factor *so that it effectively becomes constant* by applying a suitable change of integration variable *before* we apply the integral mean-value theorem itself.

A technical issue which complicates finding a suitable change of integration variable is that although $(x - x') \geq 0$ when $x > x_0$, $(x - x') \leq 0$ when $x_0 > x$, *so it is best to make two different changes of integration variable in the two cases $x > x_0$ and $x_0 > x$.*

If $x > x_0$, we change the integration variable from x' to $U(x') = (x - x')^{n+1}$, so $dU(x')/dx' = -(n + 1)(x - x')^n$. Thus the factor $(x - x')^n$ we wish to "flatten" equals $-(1/(n + 1))dU(x')/dx'$, which implies that $(x - x')^n dx' \propto dU$. Additional relevant facts are that $U(x' = x) = 0$, $U(x' = x_0) = (x - x_0)^{n+1}$ and $x' = x - (U(x'))^{1/(n+1)}$. Thus for this $x > x_0$ case,

$$(1/n!)\int_{x_0}^{x}(x - x')^n F^{(n+1)}(x')\,dx' = (1/(n + 1)!)\int_{0}^{(x-x_0)^{n+1}} F^{(n+1)}(x - U^{1/(n+1)})\,dU. \tag{3a}$$

Eq. (3a) shows the $x' \to U$ variable change "flattened" the factor $(x - x')^n$ because $(x - x')^n dx' \propto dU$. Thus there exists a "mean value" $F^{(n+1)}(c)$ of $F^{(n+1)}(x - U^{1/(n+1)})$ which can be taken *outside* the integration over U between 0 and $(x - x_0)^{n+1}$—where $(x - U^{1/(n+1)}) \in [x_0, x]$ and $F^{(n+1)}(x - U^{1/(n+1)})$ *is continuous.* With this function *outside* the U integration, that integration becomes simple. Thus for $x > x_0$ we obtain,

$$(1/n!)\int_{x_0}^{x}(x - x')^n F^{(n+1)}(x')\,dx' = \big(F^{(n+1)}(c)/(n + 1)!\big)\int_{0}^{(x-x_0)^{n+1}} dU = F^{(n+1)}(c)(x - x_0)^{n+1}/(n + 1)!, \tag{3b}$$

for $c \in [x_0, x]$.

If $x_0 > x$, we change the integration variable from x' to $V(x') = (x' - x)^{n+1}$, so $dV(x')/dx' = (n + 1)(x' - x)^n$. Thus the factor $(x - x')^n$ we wish to "flatten" equals $(-1)^n(1/(n+1))dV(x')/dx'$, which implies that $(x - x')^n dx' \propto dV$. Additional relevant facts are that $V(x' = x) = 0$, $V(x' = x_0) = (x_0 - x)^{n+1}$ and $x' = x + (V(x'))^{1/(n+1)}$. Thus for this $x_0 > x$ case,

$$(1/n!)\int_{x_0}^{x}(x - x')^n F^{(n+1)}(x')\,dx' = (-1)^{n+1}(1/(n + 1)!)\int_{0}^{(x_0-x)^{n+1}} F^{(n+1)}(x + V^{1/(n+1)})\,dV. \tag{3c}$$

Eq. (3c) shows the $x' \to V$ variable change "flattened" the factor $(x - x')^n$ because $(x - x')^n dx' \propto dV$. Thus there exists a "mean value" $F^{(n+1)}(c)$ of $F^{(n+1)}(x + V^{1/(n+1)})$ which can be taken *outside* the integration over V between 0 and $(x_0 - x)^{n+1}$—where $(x + V^{1/(n+1)}) \in [x_0, x]$ and $F^{(n+1)}(x + V^{1/(n+1)})$ *is continuous.* With this function *outside* the V integration, that integration becomes simple. Thus for $x_0 > x$ we obtain,

$$\begin{aligned}(1/n!)\int_{x_0}^{x}(x - x')^n F^{(n+1)}(x')\,dx' &= (-1)^{n+1}\big(F^{(n+1)}(c)/(n + 1)!\big)\int_{0}^{(x_0-x)^{n+1}} dV = \\ (-1)^{n+1}\big(F^{(n+1)}(c)/(n + 1)!\big)(x_0 - x)^{n+1} &= F^{(n+1)}(c)(x - x_0)^{n+1}/(n + 1)!,\end{aligned} \tag{3d}$$

for $c \in [x_0, x]$. This result for the $x_0 > x$ case *has come out to be exactly the same as the* Eq. (3b) *result for the $x > x_0$ case.* Thus *we can replace the integral error term in the* Eq. (2i) *Taylor expression by the considerably simpler result obtained here*, which produces,

$$\big(F(x) - F(x_0)\big) = \theta(n - \tfrac{1}{2})\sum_{k=1}^{n} F^{(k)}(x_0)(x - x_0)^k/k! + F^{(n+1)}(c)(x - x_0)^{n+1}/(n + 1)!, \tag{3e}$$

for $c \in [x_0, x]$. Of course, $F^{(n+1)}(x')$ *is assumed to be continuous for all $x' \in [x_0, x]$.* The $n = 0$ case of the Eq. (3e) Taylor theorem is the Eq. (1d) differential form of the mean value theorem.