Grimm's Conjecture

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Abstract: The collection of the consecutive composite integers is the composite connected, and no pair of its distinct integers may be generated by a single prime number. Composite connectedness implies the two-primes rule and the singularity propagation/breaking rule. Failure of the singularity propagation proves the Gremm's Conjecture.

Key words: Fundamental theorem of arithmetic, Bertrand's postulate, two-primes rule, singularity propagation rule.

Introduction

Carl Albert Grimm, April 1, 1926 - January 2, 2018, states that to each element of a set of consecutive composite numbers, one can assign a distinct prime that divides it. The problem is one of the important but still not solved problems in the Number Theory.

We define the set $\mathcal{S}$ to be a collection of the consecutive composed integers $x_1, x_2, x_3, \cdots, x_n$, perhaps placed between two consecutive primes α and β. All primes are outside of the set $\mathcal{S}$, and consequently the set $\mathcal{S}$ is the composite connected. The size of the set $\mathcal{S}$ cannot exceed α. Else, the size of the set would be greater than 2α, by Bertrand's postulate the set would contain at least one prime and would not be composite connected.

The collection of all prime divisors of an integer $x \in \mathcal{S}$ is the set $\mathrm{D}(x)$. According to the fundamental theorem of arithmetic each integer x is generated by its prime divisor set P, $x = \Pi p$, and will be identified by the pair (x, P). The set $\mathrm{D}(x,y) = \{d\colon\, d|x,\, d|y\}$ is the collection of all prime divisors common to both x and y integers. Further, $x = \mathrm{D}_x \Pi p'$ and $y = \mathrm{D}_y \Pi q'$, where D_x and D_y are products of the primes from $\mathrm{D}(x,y)$ particular to the integers x and y, and $\Pi p'$ and $\Pi q'$ are the products of their additional composing primes.

Example The following table shows the prime factorization and some collections of distinct prime factors of the composed integers between primes $\alpha = 89$ and $\beta = 97$. The set $\mathcal{S}$ has 7 and the collection $2, 3, 7, 5, 13, 19, 31, 43, 47$ of its prime divisors has 9 distinct members.

Table 1. The set $\mathcal{S}_{89}^{97}$ and Some Prime Divisors Selections

$\lvert i\rangle$	**89**	90	91	92	93	94	95	96	**97**
$\Pi\xi$	**89**	$2 \cdot 3^2 \cdot 5$	$7 \cdot 13$	$2^2 \cdot 43$	$3 \cdot 31$	$2 \cdot 47$	$5 \cdot 19$	$2^3 \cdot 3$	**97**
P_1		3	13	43	3	2	5	3	
$\mathrm{P}_\star$		2	7	43	31	47	19	3	
P_3		2	13	43	3	47	5	3	

Clearly, it is always possible to select a collection of 7 prime divisors. The primes $2, 3, 5$ are divisors of a few distinct integers from the set $\mathcal{S}$. However, Grimm's conjecture requires the set P of 7 distinct prime divisors selected from the 7 distinct integers of the set $\mathcal{S}$. Notice that such selection is not unique.

Description

Relations between two distinct integers from the set $\mathcal{S}$, are characterized by the set intersection of their prime divisor sets and by their greatest common divisor. For integers x and y in $\mathcal{S}$

1. $\mathrm{D}(x,y) = \emptyset$,
2. $\mathrm{D}(x,y) = \mathrm{D}(x) = \mathrm{D}(y)$,
3. $\mathrm{D}(x,y) = \mathrm{D}(x) \subset \mathrm{D}(y)$,
4. $\mathrm{D}(x,y) = \mathrm{D}(y) \cap \mathrm{D}(x) \neq \emptyset \,\&\, \mathrm{D}(x)$.

The greatest common divisor and $\mathrm{D}(x,y)$ are related. For, if $x = \Pi p^m$ and $y = \Pi p^n$, $n > m$ then

$$g = \Pi p^k,\ k = \max\{m,n\}\ \therefore\ p^k | x \,\&\, p^k\, | y \quad \Rightarrow \quad x = g\mathrm{G}_x,\ y = g\mathrm{G}_y,$$

where G_x and G_y are the g cofactor integers in the x and y. Two important cases are when integer x is the greatest common divisor of y, and when both x and y are generated by a single prime.

Corollary 1. *There are no distinct integers x and y in $\mathcal{S}$, such that either x divides y or that a single prime generates both x and y.*

If $x|y$ the integer x is the greatest common divisor of x and y and its cofactor yx^{-1} in the y is an integer greater or equal to 2. By the Bertrand's postulate there is a prime between x and y and the set $\mathcal{S}$ is not the composition connected, contradiction.

If both x and y are generated by a single prime p then $x = p^m$ and $y = p^n$, $n > m$. Hence $y = xp^{n-m}$ and x is the greatest common divisor of x and y with cofactor p^{n-m} in y greater or equal to 2. Again, by Bertrand's postulate there must be a prime between x and y, contradiction. There is no integer pair in the set $\mathcal{S}$ generated by the single prime. ■

Corollary 2. *Each pair (x,y) of the distinct composed integers from the set $\mathcal{S}$ lets a pair $|p,q\rangle$. of distinct prime divisors $p|x$ and $q|y$.*

We will consider one by one the set intersection cases of the prime divisor sets of a pair (x,y) of distinct composed integers $x \sim (p, \mathrm{P})$ and $y \sim (q, \mathrm{Q})$ from the set $\mathcal{S}$. We may assume $y > x$ when it is necessary.

1. When $\mathrm{D}(x,y) = \emptyset$ the integers x and y are composed on two distinct sets of the primes. It is sufficient to take a prime divisor p of x and a prime divisor q of y to make the pair $|p,q\rangle$ of the distinct primes.

2. When $\mathrm{D}(x) = \mathrm{D}(y)$, the set $\mathrm{D}(x,y)$ cannot be a single prime set, see Corollary 1. Consequently, the set $\mathrm{D}(x,y)$ must have at least two distinct prime divisors and the vector $|p,q\rangle$ of the distinct prime divisors of x and y cannot be empty.

3. When $\mathrm{D}(x) \subset \mathrm{D}(y)$ all primes in the set $\mathrm{D}(x,y)$ are the divisors of the x, and there is at least one prime q divisor of y not in the $\mathrm{D}(x,y)$. The vector $|p,q\rangle$, $p \in \mathrm{D}(x,y)$, is a pair of distinct prime divisors of x and y respectively.

4. Finally, when $\mathrm{D}(x,y)$ is nonempty intersection $\mathrm{D}(y) \cap \mathrm{D}(x)$ not identical to $\mathrm{D}(x)$ there are at least two distinct primes $p \in \mathrm{D}(x) \setminus \mathrm{D}(y)$ and $q \in \mathrm{D}(y) \setminus \mathrm{D}(x)$ such that $x = p\Pi d_x$ and $y = q\Pi d_y$, where $d_x, d_y \in \mathrm{D}(x,y)$. Definitely, at least, $|p,q\rangle$ is the vector of two distinct prime divisors of the pair (x,y). ■

Remark: Further, we are selecting a collection $\mathrm{P} = \{p_1, p_2, p_3, \cdots, p_n\}$ of the prime divisors from a set of the consecutive composed integers $\mathcal{S} = \{x_1, x_2, x_3, \cdots, x_n\}$, and require that the following, either affirmative or negative, statements of the Grimm's Conjecture hold

$$\forall\mathcal{S},\, \exists\mathrm{P} = |p_1, p_2, \cdots, p_i, \cdots, p_n\rangle,\, p_i|x_i \,\therefore \forall i \neq j \Rightarrow p_i \neq p_j,$$
$$\exists\mathcal{S},\, \forall\mathrm{P} = |p_1, p_2, \cdots, p_i, \cdots, p_n\rangle,\, p_i|x_i \,\therefore \exists i \neq j \Rightarrow p_i = p_j.$$

The working hypothesis is that Grimm's Conjecture is false, which is that each prime divisor selection "must" choose at least a single prime divisor ξ for at least a pair of the integers X and Y from $\mathcal{S}$. Such a couple of the integers we call singular, else regular couple.

The "must" statement of the hypotheses is equivalent to the saying *that there is no reconstruction of a chosen prime sequence, which would remove the singularity. The last suggests the proof strategy to be the reconstruction of the prime divisors set $\mathcal{P}$, created on the singular ground pair* (X, Y). *If the reconstruction removes the ground couple singularity, the Grimm's Conjecture is true. Otherwise, it is false.*

Remark: The essence of Grimm's Conjecture is in the natural relations between the integers of the set $\mathcal{S}$, All distinct prime constituents of $\mathcal{S}$ are Π, and all distinct prime divisors of an integers $\mathrm{X} \in \mathcal{S}$ are Π_{X}.

For convenience, we represent each integer from the set $\mathcal{S}$ by three primes, $\mathrm{X} = (p, q, q^{\mathrm{O}})$. The primes p and q may be identical and the additional prime divisors q^{O} may or may not be there. The prime p is the integer division representative or the first prime. The second prime q, the linking prime divisor or the second prime, relates the integer X to other integers. Recall that the integers of the ground pair (X, Y) are mutually related by the set of the common prime divisors $\mathrm{D}(\mathrm{X}, \mathrm{Y})$,

Direct family F_p of the prime p is the collection of all integers $\mathrm{Z} \in \mathcal{S}$ p divides. We will say that p is self-coupled or reproduced in the integers from the direct family. The actual family F'_p of the prime p is broader, it is the collection of all integers $\mathrm{Z} \in \mathcal{S}$ divisible by the all primes in $\mathrm{D}(\mathrm{X})$. The actual family F'_p is the union of direct families of the primes $q \neq p$, and $F_p \subset F'_p$. A prime is an isolated/lonely prime If its direct family is itself. All other primes in $\mathrm{D}(\mathrm{X})$ self-couple to all integers of their direct families F_q.

The prime pointer $\hat{q}$ is the second prime in X pointing to an integer $\mathrm{Y} \in \mathcal{S}$ it divides to make the pair (X, Y). Further, the pointer $\hat{q}$ is the selection operator, choosing an integer $\mathrm{Y} \in F_q$ from its direct family to self-couple as its first or the prime representative p'. The operation $\hat{q} : F_q \to \mathrm{Y} \sim (p', \hat{q}') = (q, \hat{q}')$ prescribes the selection procedure to reconstruct the prime divisors set $\mathcal{P}$.

Further, we will present any integer X by the triplet $\mathrm{X} \equiv \langle \mathrm{X},\, p,\, \hat{q}|$, the first prime p is its division representative, the second prime is the prime pointer or the selection operator. After selection is done the pair (X, Z) is the quadruplet $\langle \mathrm{X},\, p; q,\, \mathrm{Z}\rangle$, and the integer $\mathrm{Z} \equiv \langle \mathrm{Z},\, p',\, \hat{q}'|,\ p' = q$.

Singularity Propagation

The prime pointer self-coupling operation $\hat{q}q = p'$ at $\text{X} \Leftrightarrow (\text{X}, p; \hat{q})$ induces the mapping

$$\hat{q}: \quad \mathcal{S} \to \mathcal{P} \quad \Leftrightarrow \quad \hat{q}(\text{X}, p, q) = (\text{Y}, p', q').$$

<u>Definition</u>: *The consecutive application of the pointer operator to reconstruct a "must" prime sequence is the singular prime or just prime propagation.*

Under the working hypotheses, the reselection operator $\hat{q}$ starts at the singular prime $p = \xi \in \text{X}$, at the ground pair (X, Y), and, throughout the consecutive self-application recreates the singular chain $\gamma(\text{X}, p\,; m)$ of the prime divisors up to an integer N_m, and a pointer $\hat{q}_m$. Together with it the chain of the integers $\Gamma(\text{X}, p; i)$ they divide is created. We assume that such chain extensions are regular up to the next singular point, if such exists or to an isolated prime or until all integers of the set $\mathcal{S}$ are exhausted.

The smallest chain is an isolated prime chain $\gamma(\text{X}, \hat{q}\,; 1)$, the first next is the single link chain $\gamma(\text{X}, p, q\,; 2)$, and further come the linear chains $\gamma(\text{X}, p, q\,; m) = |p, q, \cdots; m\rangle$, the chain loops and so on. The prime divisor selection may have a few singular points and selection centers.

The singularity propagates throughout by the prime pointers linking the sequence of the quadruplets,

$$\langle \text{N}_1 \;\; p_1;\, q_1 \;\; \text{N}_2\rangle \;\xrightarrow{\hat{q}_1}\; \langle \text{N}_2 \;\; q_1;\, q_2 \;\; \text{N}_3\rangle \;\xrightarrow{\hat{q}_2}\; \langle \text{N}_3 \;\; q_2;\, q_3 \;\; \text{N}_4\rangle \;\xrightarrow{\hat{q}_3}\; \cdots .$$

The extension of the singular link $\Gamma(\text{X}, p; 1)$ are the integer and the prime regular chains

$$\begin{aligned} \Gamma(\text{X}, p; m) \quad &= |\text{X}, \text{Y}, \text{N}_1, \text{N}_2, \cdots, \text{N}_m\rangle \\ \gamma(\text{X}, p; m) \quad &= |p; q, q_1, q_2, \cdots, q_j\rangle, \quad m \in \mathbb{N}. \end{aligned}$$

<u>Definition</u>: *The collection of all consecutive images* $\hat{q} : \langle \text{X}_i, p_i; q_i \,| \to \hat{q}\, q_i = p_{i+1} \in \text{X}_{i+1}$ *is the singularity propagation function of the singular prime representative* p. *Its propagation chain of the order* n *is the discrete function*

$$\gamma(\text{X}, p; n) \quad = |p, p_1, p_2, p_3, \cdots, p_{n-1}\rangle.$$

A conditional statement of Grimm's Conjecture in terms of the propagation function is: "If the Grimm's Conjecture is true, the propagation function is $1:1$ *correspondence. Else, there is at least one pair* $(\text{X}_i, \text{X}_j) \sim \langle \text{X}_j \;\; p_j; q_j \;\; \text{Z}'\rangle$, *such that"*

$$\hat{q}q_i = p_i = q_j, \quad j \neq i.$$

Propagation of the chain, and therefore of the singularity, is carried out by the instant pointer. A free prime pointer connects to an integer outside of the chain to continue the chain, or the propagation breaks by the isolated pointer, or the last pointer self-coupling to the chain or by another singularity.

<u>Corollary</u> 3. *<u>Three Primes Rule</u>:*

If ξ *is the singular prime divisor of an integer pair* (X, Y), $\text{X} \prec \text{Y}$, *each of the integers* X *and* Y *must have an additional prime divisor.*

<u>The Singularity Propagation/ Breaking Rule:</u>

To propagate the singularity, each current pointer of the chain must self-couple in an additional integer in $\mathcal{S}$. Else, the pointer is the propagation breaking pointer, the prime singularity propagation breaks, the chain $\mathbf{P}$ *of the prime carrier divisors destructs all the way down to the ground level, and the regular chain* $\mathbf{Q}$ *of the distinct pointer primes in* $1:1$ *correspondence with integers of the set* $\mathcal{S}$ *is exhibited. The singularity propagation breaking pointer removes the singularity.*

$\square$ If the integers (X, Y), $\mathrm{X} \prec \mathrm{Y}$ do not have the second prime divisors, the single prime $\xi \in \mathrm{D}(\mathrm{X}, \mathrm{Y})$ generates both integers, the set $\mathcal{S}$ is not composite connected, and the Grimm's Conjecture does not satisfy the defining conditions. Hence both X and Y must have an additional prime divisor, and $\mathrm{X} \equiv \langle \mathrm{X}, \xi; \hat{q}_* |$ and $\mathrm{Y} \equiv \langle \mathrm{Y}, \xi; \hat{q} |$. The same holds for any pair $(\mathrm{N}_i, \mathrm{N}_{i+1})$ coupled by the pointer $\hat{q}_i = \eta$.

Further, in addition to the pair $\langle \mathrm{X},\, p\,; q,\, \mathrm{Y} \rangle$ we look at the pairs $\langle \mathrm{Y},\, \xi\,; q',\, \mathrm{Z} \rangle$ and $\langle \mathrm{Y}_*,\, q_*\,; q'',\, \mathrm{Z}_* \rangle$.
The pointer $\hat{\xi}$, couples X to Y in the integer pair $\mathrm{Y} \equiv \langle \mathrm{Y}, \xi; \hat{q} |$ of the chain $\mathbf{P} = |\xi, \xi\rangle$. If $\hat{q} \in \mathrm{Y}$ is the last chain pointer, it would be an isolated prime and could be chosen to be the prime divisor, a distinct representative of the integer Y, leaving $\xi = p$ to be the distinct representative of the integer X only. The chain of the distinct regular pointers $\mathbf{Q} = \langle q, p = \xi |$ in $1:1$ correspondence with the integers X, Y exhibits, and the the singularity of the ground pair cancels. However, Grim's conjecture is not true, and the pointer $\hat{q}$ must couple to an integer $\mathrm{Z} \equiv \langle \mathrm{Z}, q; \hat{q}' | \in \mathcal{S}$.

If the pointer $\hat{q}' \in \mathrm{Z}$ is the breaking pointer, the prime q' could be chosen to be the prime divisor of the Z, leaving the pointers q and ξ to be the prime divisor of the integers X and Y. Consequently, the sequence of the regular distinct prime pointers $\mathbf{Q} = \langle q', q, \xi |$ would be exhibited instead of the singular chain $\mathbf{P} = |xi, \xi, q\rangle$. The singularity at the ground pair is removed, and ....

It is obvious that the same holds at an arbitrary level m. If the pointer is the breaking pointer the regular chain $\mathbf{Q} = \langle q_m, \cdots, q', q, \xi |$ of the distinct primes in $1:1$ correspondence with the integers $\mathrm{N}_m, \mathrm{N}_{m-1} \cdots \mathrm{Z}, \mathrm{X}, \mathrm{Y}$ exists. Else the chain is the singular chain $\mathbf{P} = |\xi, \xi, q, q', \cdots, q_{m-1}\rangle$. $\blacksquare$

An obvious conclusion is that the three primes rule, the singularity propagation/breaking rule, and the finiteness of the set $\mathcal{S}$, governs and determine the singularity propagation.

Conclusion

The two primes rule and the singularity propagation/breaking rules govern the singularity development. If the current pointer is free, it will self-couple in an integer outside of the chain to continue the singularity propagation. Otherwise, the following singularity propagation outcomes are possible:

1. the isolated/lonely pointer case
2. the chain loop closure pointer case
3. the loop-appendix pointer realization .

Remark: In the case of any of three singularity propagation outcomes, the singularity propagation/break rule applies, the chain breaks, and Grimm's Conjecture is true.

To see this, we rely on the Corollary 3. We will go one by one through the cases. Tables 2 and 3 present the current self-coupling chains. The row **P** is the singular chain of the first prime p divisor representatives, and the row **Q** is the regular chain of the distinct prime divisors in the $1:1$ correspondence with the integers of the chain.

1. **Isolated Pointer**

   The singularity propagation breaks at the integer $\mathrm{N}_i = \mathrm{N}_m$ of the chain and, according to Corollary 3, the regular chain of the distinct second prime divisors $\mathbf{Q} = |q_m,\, q_{m-1},\, \cdots,\, q_2,\, q_1,\, q,\, \xi\,\rangle$ in the $1:1$ correspondence with the collection of the integers from the set $\mathcal{S}$ forms instead. The row translation $\hat{1}: \mathbf{P} \to \mathbf{P} \ominus \hat{1} = \mathbf{Q}$ is the chain regularization. We chose the prime divisors collection $\gamma(\xi, q, m+2)$, the singularity cancels, and the Grimm's Conjecture is true.

Table 2.Isolated Pointer and the Chain Loop

$\langle\,\mathrm{N}_i\vert$	N_{m+1}	N_m	N_{m-1}	$\cdots$	N_3	N_2	N_1	Y	X	Y_*
P		q_{m-1}	q_{m-2}	$\cdots$	q_2	q_1	q	ξ	ξ	q_*
Q		q_m	q_{m-1}	$\cdots$	q_3	q_2	q_1	q	ξ	q_*
$\gamma(\xi, q, m+2)$		$\mathbf{q}_m$	$\mathbf{q}_{m-1}$	$\cdots$	$\mathbf{q}_3$	$\mathbf{q}_2$	$\mathbf{q}_1$	$\mathbf{q}$	ξ	
$\Lambda(\mathrm{X}, \xi; \mathrm{N}_m)$		$\mathbf{q}_m \equiv q_*$	$\mathbf{q}_{m-1}$	$\cdots$	$\mathbf{q}_3$	$\mathbf{q}_2$	$\mathbf{q}_1$	$\mathbf{q}$	ξ	

2. **The Chain Loop**

   In this case the pointer $\hat{q}_m \in \mathrm{D}(\mathrm{X}_m \cap \mathrm{X})$ self-couples in the integer $\mathrm{X} \equiv \langle \mathrm{X},\, \hat{q}_*,\, \xi|$ The identification transformation $\hat{\mathrm{I}}: \xi \leftrightarrow q_m,\;\; \mathbf{P} \ominus \{q_*\} \to \mathbf{P} \ominus \{q_*\}$, see the last line of Table 2, closes the chain in the regular loop $\Lambda(\mathrm{X}, \xi; \mathrm{N}_m) = |q_m = q_*,\, q_{m-1},\, \cdots,\; q_2,\, q_1,\, q,\, \xi\,\rangle$, the singularity cancels, and the Grimm's Conjecture is true.

   If the prime divisors chain exhausts all integers of the set $\mathcal{S}$, the last integer $\mathrm{N}_n \equiv \langle \mathrm{N}_n,\, p_n,\, \hat{q}_n|$ must couple to the ground integer $\mathrm{X} \equiv \langle \mathrm{X},\, \hat{q}_*,\, \xi|$. The quadruplet $\langle \mathrm{N}_n,\, \hat{q}_n; \hat{q}_*,\, \mathrm{X}\rangle$, closes the chain in the loop, the singularity cancels, and the Grimm's Conjecture is true.

3 **The Loop with Appendix**

   In the first part of Table 3, the row **Q** of the second primes $\mathbf{Q} = \mathbf{P} \ominus \hat{1}$ is the back transformation of the first primes row. Without limitations on the pointer $\hat{q}_m$, the chain, row γ is the regular chain. However, the singularity propagation chain $\gamma(\mathrm{X}, \xi,\, m) \equiv \mathbf{P}$ attaches by its end point $\mathrm{N}_m \equiv \langle \mathrm{N}_m,\, q_{m-1}; \hat{q}\,|$ to an integer $\mathrm{N}_k \equiv \langle \mathrm{N}_k,\, p_k,\, \hat{q}_k| \in \Gamma(\mathrm{X}, p\,; m)$, $k < m$. Clearly, $q_m \in \mathrm{D}(\mathrm{N}_k \cap \mathrm{N}_m)$. If the pointer $\hat{q}_m$ contacts $q'' \in \mathrm{N}_k$ the self-coupling $q_m = q'' \neq p_k$ is just the chain external extension to an integer, which happens to be in N_k.

   Otherwise, $\hat{q}_m$ self-recreates as the first prime $p_k \equiv q_{k-1} = \xi'$, and the new singular pair $(\mathrm{N}_m, \mathrm{N}_k) \sim \langle \mathrm{X}', \xi'; \xi', \mathrm{Y}'_k\rangle$ creates, see the second part of Table 3. At this place we rename $m' = m - k$,

$$
\begin{aligned}
& (\mathrm{N}_k, \mathrm{N}_{k+1}, \mathrm{N}_{k+2}, \cdots, \mathrm{N}_{m-1}, \mathrm{N}_m) \to (\mathrm{Y}', \mathrm{N}'_1, \mathrm{N}''_2, \cdots, \mathrm{N}'_{m'-1}, \mathrm{X}'),\\
\mathbf{p}' = \; & (q_{k-1}, q_k, q_{k+1}, \cdots, q_{m-2}, q_{m-1}) \to (\xi', q', q'_1, \cdots, q'_{m'-2}, q'_{m'-1}),\\
\mathbf{q} = \; & (q_k, q_{k+1}, q_{k+2}, \cdots, q_{m-1}, q_m) \quad\; \to (q', q'_1, q'_2, \cdots, q'_{m'-1}, \xi')
\end{aligned}
$$

The row $\mathbf{P}' = |p\rangle \oplus \mathbf{p}$ of the first primes is the singular chain, such is also its second part $\mathbf{p}'$, and the row of the second primes $\mathbf{Q}'$ restriction to $\mathbf{q}'$ are all regular distinct primes. The backward translation $\hat{1} : |p\,\rangle \to |p\,\rangle \ominus \hat{1}$ creates the linear chain $\gamma(\mathrm{Y}_*, \xi; k+3) = |q'_*, q_*, \xi, q, q_1, \cdots, q_{k-2}, \rangle$ and the identification transformation $\hat{\mathrm{I}} : q'_{m'} \leftrightarrow q'_k = \xi'$, $\mathbf{q}\ominus q'_{m'} \to \mathbf{q}\ominus q'_{m'}$ create a loop $\Lambda(\mathrm{Y}', \xi'; m+1) = |\xi', q', q'_1, \cdots, q'_{m'-2}, q'_{m'-1}\rangle$. Consequently

$$\begin{aligned} &\gamma(\mathrm{Y}_*, \xi; k+3) &&= |q'_*, q_*, \xi, q, q_1, \cdots, q_{k-2}\rangle = \quad |q'_*\rangle + \gamma(\mathrm{X}, \xi; k+2) \\ \Rightarrow \quad &\gamma(\mathrm{X}_*, \xi; m+2) &&= \gamma(\mathrm{Y}_*, \xi; k+2) + \Lambda(\mathrm{X}', \xi'; m'+1), \end{aligned}$$

and the chain is a union of the regular loop and regular appendix. The Grimm's Conjecture is true.

Table 3. The Loop with Appendix

$\langle\,\mathrm{N}_i\vert$	Y^*	X	Y	N_1	N_2	$\cdots$	N_{k-1}	N_k	N_{k+1}	N_{k+2}	$\cdots$	N_{m-1}	N_m
$\mathbf{P}$	q'_*	q_*	ξ	q	q_1	$\cdots$	q_{k-2}	$p_k = q_{k-1}$	q_k	q_{k+1}	$\cdots$	q_{m-2}	q_{m-1}
$\mathbf{Q}$	q_*	ξ	q	q_1	q_2	$\cdots$	q_{k-1}	q_k	q_{k+1}	q_{k+2}	$\cdots$	q_{m-1}	q_m
γ		ξ	q	$\mathbf{q}_1$	$\mathbf{q}_2$	$\cdots$	$\mathbf{q}_{k-1}$	$\mathbf{q}_k$	$\mathbf{q}_{k+1}$	$\mathbf{q}_{k+2}$	$\cdots$	$\mathbf{q}_{m-1}$	$\mathbf{q}_m$
								Y'	N'_1	N'_2	$\cdots$	$\mathrm{N}'_{m'-1}$	X'
$\mathbf{P}'$	q'_*	q_*	ξ	q	q_1	$\cdots$	q_{k-2}	ξ'	q'	q'_1	$\cdots$	$q'_{m'-2}$	$q'_{m'-1}$
$\mathbf{Q}'$								$\mathbf{q}'$	$\mathbf{q}'_1$	$\mathbf{q}'_2$	$\cdots$	$\mathbf{q}'_{m'-1}$	ξ'
	Appendix $\gamma(\mathrm{Y}^*, \xi; k+3)$							**Loop** $\Lambda(\mathrm{Y}', \xi'; m+1)$					
γ^*		ξ	q	$\mathbf{q}_1$	$\mathbf{q}_2$	$\cdots$	$\mathbf{q}_{k-1}$	$\mathbf{q}'$	$\mathbf{q}'_1$	$\mathbf{q}'_2$	$\cdots$	$\mathbf{q}'_{m'-1}$	ξ'

Corollary: *The Grimm's Conjecture is true.*

□ If Grimm's Conjecture is not true, there is no prime divisor re-selection which would remove the singularity. Consequently, the singular chain $\gamma(\mathrm{X}, \xi; m)$ must couple with an integer outside of the chain at each evolution level $m \leq n$. The set $\mathcal{S}$ is finite, and the chain must couple to itself at $m = n$, the singularity propagation breaks, and the regular chain in $1 : 1$ correspondence with integers of $\mathcal{S}$ constructs. Hence, Grimm's Conjecture is true. ■

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