Solid Strips

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Abstract

In this paper we introduce the idea of a Solid Strip which is the generalization to higher dimensions of 2-dimensional Untwisted and Mobius Strips.

Key Words: compact manifold, topology.

1 Introduction

For the purpose of this paper, we will call a strip is a 2-dimensional manifold with boundary obtained by identifying 2 opposite edges of the 4 edges of a square. It can be done without a twist (Untwisted Strip) or with a twist (Mobius Strip).

In 2-dimensions we have only two possible configurations. An untwisted strip which has a boundary composed by two circles and it is orientable and a Mobius Strip that has a boundary composed only by one circles and it is not orientable. We want to generalise the idea of a strip to the 3-dimensional case.

2 Solid Strips

The idea of a strip described above can be easily extended to 3-dimensional manifolds. In 2-dimensions we have 2D strips obtained by identifying one couple of opposite edges of the two couples of edges of a square. In 3-dimensions we will have "Solid Strips" which are 3D "strips" obtained by identifying two couples of opposite faces of the three couples of faces of a cube.

Solid trips are like "rings" in $\mathbb{R}^4$ meaning that they can be linked like the links of a common steel chain (the one you can buy in an hardware shop) in $\mathbb{R}^3$, and they cannot be separated without breaking them. There are 8 ways of identifying two opposite faces of a cube (group of symmetries of a square keeping the other one fixed) which, for two couples, gives a total of 64 different possible manifolds. However these manifolds form homeomorphic classes and therefore the number of actual spaces is much lower.

All the above 64 mentioned configurations are reported in the table in Appendix A.1 where vertices numbering of the cube used to build the configurations is defined in Fig. 1. The 2^{nd} and 3^{rd} columns of the two tables contain the way opposite faces of the cube are identified for that specific configuration and the

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4^{th} and 5^{th} columns contain the way edges and vertices of the two remaining faces (the up and bottom faces in Fig. 1) get identified to form the boundary (to better understand how to read these 2 columns, refer to Fig. 1).

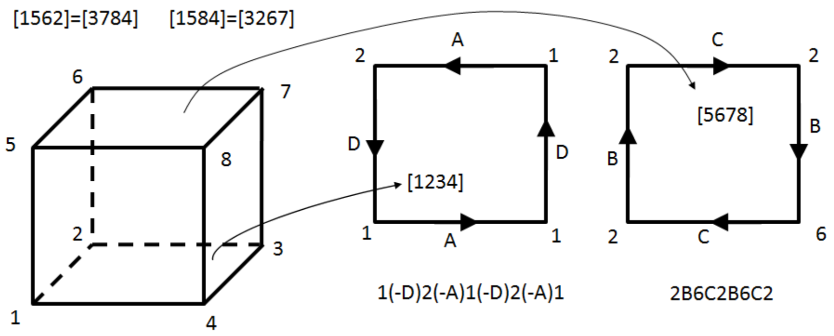

Figure 1: Example of a Solid Strip Configuration - Boundary $\mathbf{RP}^2 \vee \mathbf{RP}^2$

Given the above configurations, 2 out of 64 (configurations 63 and 64 in the table in Appendix A.1) lead to a not feasible space. To see this let us consider configuration 63. The identification requirements from columns 2 ask edge [15] to be identified with edge [37] while from column 3 we are asked to identify edge [15] to edge [73] and the two identification are obviously impossible to be done at the same time.

The 64 configurations in some cases are equivalent each other. For example, in configuration 40 in table A1.1 we are requested to rotate a face of a couple by 90deg and then identify it to the other face. In configuration 41 we are requested to do the same thing to the other couple of faces. The two configurations are obviously equivalent and we can go from one to another by rotating the cube by 90deg around the z axis. More analytically we have that configurations 40 and 41 are given by the following identification requirements:

$$\begin{array}{ll} C_{40}: & [1562]=[4873] \quad [1584]=[6732] \\ C_{41}: & [1562]=[3487] \quad [1584]=[2673] \end{array} \tag{1}$$

by applying a rotation of 90deg around the z axis, to configuration 40, given by the following permutation of cube vertices:

$$RotZ = \begin{pmatrix} 12345678 \\ 23416785 \end{pmatrix} \tag{2}$$

we get the following configuration:

$$C'_{40}: \quad [2673]=[1584] \quad [2651]=[7843] \tag{3}$$

which is, by permuting the order of the vertices, configuration 41. The two configuration are therefore equivalent and the two relevant spaces are homeomorphic.

We have written a simple code that applies all the 48 symmetries of a cube to each configuration in Appendix A.1 and compares the results with all other configurations looking for equivalences. By using this code, we have classified all 64 spaces in 22 equivalent classes (one of which composed of non feasible configurations) leading to a total of 21 equivalent homeomorphic classes of spaces reported in the table below:

For each class, we have finally evaluated the boundary of each space (see Appendix A.2) and we have found 13 different boundaries which are reported in the last column of the table below.

Cla.	Ident. 1	Ident. 2	Face 1 [1234]	Face 2 [5678]	Bound.
1	[1562]=[4873]	[1584]=[2673]	1(-C)1(-E)1C1E1	5A5D5(-A)5(-D)5	$\mathbf{T}^2\,;\mathbf{T}^2$
2	[1562]=[4873]	[1584]=[3762]	1(-C)1E1C1E1	5A5(-D)5(-A)5(-D)5	$\mathbf{K}\,;\mathbf{K}$
3	[1562]=[3784]	[1584]=[3762]	1(-D)2F1(-D)2F1	5B6(-E)5B6(-E)5	$\mathbf{RP}^2\,;\mathbf{RP}^2$
4	[1562]=[3784]	[1584]=[3267]	1(-D)2(-A)1(-D)2(-A)1	2B6C2B6C2	$\mathbf{RP}^2 \vee \mathbf{RP}^2$
5	[1562]=[3487]	[1584]=[3267]	1(-D)2(-A)1A2D1	2B6(-B)2(-C)6C2	$\mathbf{X}_1 \vee \mathbf{X}_1$
6	[1562]=[7348]	[1584]=[7326]	1(-D)2(-E)3B4F1	3B4F1(-D)2(-E)3	$\mathbf{S}^2$
7	[1562]=[7348]	[1584]=[2376]	1(-D)1A3B4C1	3B4C1(-D)1A3	$\mathbf{X}_1$
8	[1562]=[4873]	[1584]=[6237]	1(-C)2D2C1E1	2A1(-E)1(-A)2(-D)2	$\mathbf{T}^2$
9	[1562]=[4873]	[1584]=[7326]	1(-C)2(-D)2C1E1	2A1E1(-A)2(-D)2	$\mathbf{T}^2$
10	[1562]=[8437]	[1584]=[6237]	1(-C)2D1(-A)2E1	2A1(-E)2C1(-D)2	$\mathbf{T}^2$
11	[1562]=[8437]	[1584]=[3762]	1(-C)2E1(-A)2E1	2A1(-D)2C1(-D)2	$\mathbf{K}$
12	[1562]=[8437]	[1584]=[3267]	1(-D)2(-A)1(-B)2(-C)1	2B1A2D1C2	$\mathbf{X}_2$
13	[1562]=[8437]	[1584]=[7326]	1(-C)1(-D)3(-A)3E1	3A3E1C1(-D)3	$\mathbf{K}$
14	[1562]=[8437]	[1584]=[2376]	1(-D)1A3(-B)3C1	3B3(-A)1D1(-C)3	$\mathbf{X}_2$
15	[1562]=[3487]	[1584]=[6732]	1(-D)2B1A2(-C)1	2B1A2(-C)1(-D)2	$\mathbf{X}_2$
16	[1562]=[4378]	[1584]=[2376]	1(-D)1A3(-A)1C1	3B1(-D)1C1(-B)3	$\mathbf{X}_2$
17	[1562]=[4873]	[1584]=[3267]	1(-D)1(-A)1D1C1	1B1(-A)1(-B)1C1	$\mathbf{Y}_1$
18	[1562]=[4873]	[1584]=[2376]	1(-D)1A1D1C1	1B1A1(-B)1C1	$\mathbf{Y}_1$
19	[1562]=[3784]	[1584]=[2376]	1(-D)1A1(-D)1C1	1B1(-C)1B1(-A)1	$\mathbf{Y}_1$
20	[1562]=[3487]	[1584]=[7326]	1(-B)1(-C)1B1D1	1A1D1(-A)1(-C)1	$\mathbf{Y}_1$
21	[1562]=[3487]	[1584]=[2376]	1(-D)1A1A1C1	1B1B1(-C)1D1	$\mathbf{Z}_1$

Table 1 : Classes of Equivalent Solid Strip Configurations

where:

- Space $\mathbf{X}_1$: is a 2-sphere where two separate points of the sphere are identified (see also Appendix A.2). This space has a point where the space is not locally homomorphic to $\mathbb{R}^2$ and therefore it is not a manifold.
- Space $\mathbf{X}_1 \vee \mathbf{X}_1$: is a wedge sum of two $\mathbf{X}_1$ spaces (see also Appendix A.2). This space has three points where the space is not locally homomorphic to $\mathbb{R}^2$ and therefore it is not a manifold.
- Space $\mathbf{X}_2$: is a 2-sphere where two couple of separate points of the sphere are identified (see also Appendix A.2). This space has two points where the space is not locally homomorphic to $\mathbb{R}^2$ and therefore it is not a manifold.
- Space $\mathbf{Y}_1$: is a 2-torus where two separate points of the torus are identified (see also Appendix A.2). This space has a point where the space is not locally homomorphic to $\mathbb{R}^2$ and therefore it is not a manifold.
- Space $\mathbf{Z}_1$: is a Klein Bottle where two separate points of the Klein Bottle are identified (see also Appendix A.2). This space has a point where

the manifold is not locally homomorphic to $\mathbb{R}^2$ and therefore it is not a manifold.

Our hypothesis is that spaces with the same boundary are likely to be homeomorphic and therefore classes of solid trips may be further grouped in 13 classes corresponding to the 13 different type of boundary found. To prove that, we should decompose the cube in simplexes and permute them looking for equivalent configurations. This may be done in a further issue of this paper.

We note explicitly that each solid trip boundary is made up of two identical squares, where identical means that they edges and vertices are identified with the same rules (See Appendix A.2)

3 The ξ Notation

For sake of visualization, we imagine to stretch the cube along one directions so that the final configuration will really look like a ring. We identify the two "short" faces (actually square faces) and two opposite "long" faces. The two remaining long faces will form the boundary.

We said above that the way we identify faces are elements of the group of symmetries of a square and we can give a name to them. These element are reported in the tables below where vertices of the square used as a reference for the permutations are named from 1 to 4 clockwise:

Permutations	Symmetry	Face 1	Face 2	Group Element
1234	Identity	[1562]	[4873]	a_0
2143	SymX	[5126]	[8437]	a_1
2341	-RotZ	[5621]	[8734]	a_2
4123	RotZ	[2156]	[3487]	a_3
4321	SymY	[2651]	[3784]	a_4
3412	2*RotZ	[6215]	[7348]	a_5
1432	SymXY	[1265]	[4378]	a_6
3214	SymX(-Y)	[6512]	[7843]	a_7

Table 1 : Long Faces - Naming of Face Orientation Group

Permutations	Symmetry	Face 1	Face 2	Group Element
1234	Identity	[1584]	[2673]	g_0
2143	SymX	[5148]	[6237]	g_1
2341	-RotZ	[5841]	[6732]	g_2
4123	RotZ	[4158]	[3267]	g_3
4321	SymY	[4851]	[3762]	g_4
3412	2*RotZ	[8415]	[7326]	g_5
1432	SymXY	[1485]	[2376]	g_6
3214	SymX(-Y)	[8514]	[7623]	g_7

Table 2 : Short Faces - Naming of Face Orientation Group

With the above group elements for face orientation we will use the notation $\xi([g_i, g_j], [a_k, a_l])$ for a given solid strip, where the first element group of each identification is the orientation of the first face and the second is the orientation of the second face to be identified. Obviously each identification can be given with the orientation of face one being the identity. In this case we will use the notation $\xi(g_j, a_l)$ which imply that the two first faces are oriented as g_0 and a_0.

In the table below we present all the solid strips grouped by class with the ξ notation and the relevant boundary.

Cla.	Conf.	Bound.
1	$\xi(g_0, a_0)$	$\mathbf{T}^2\,;\mathbf{T}^2$
2	$\xi(g_4, a_0)$, $\xi(g_0, a_4)$	$\mathbf{K}\,;\mathbf{K}$
3	$\xi(g_4, a_4)$	$\mathbf{RP}^2\,;\mathbf{RP}^2$
4	$\xi(g_3, a_4)$, $\xi(g_2, a_4)$, $\xi(g_4, a_3)$, $\xi(g_4, a_2)$	$\mathbf{RP}^2 \vee \mathbf{RP}^2$
5	$\xi(g_3, a_3)$, $\xi(g_2, a_2)$	$\mathbf{X}_1 \vee \mathbf{X}_1$
6	$\xi(g_5, a_5)$	$\mathbf{S}^2$
7	$\xi(g_6, a_5)$, $\xi(g_7, a_5)$, $\xi(g_5, a_6)$, $\xi(g_5, a_7)$	$\mathbf{X}_1$
8	$\xi(g_1, a_0)$, $\xi(g_0, a_1)$	$\mathbf{T}^2$
9	$\xi(g_5, a_0)$, $\xi(g_0, a_5)$	$\mathbf{T}^2$
10	$\xi(g_1, a_1)$	$\mathbf{T}^2$
11	$\xi(g_4, a_1)$, $\xi(g_1, a_4)$	$\mathbf{K}$
12	$\xi(g_3, a_1)$, $\xi(g_2, a_1)$, $\xi(g_1, a_3)$, $\xi(g_1, a_2)$	$\mathbf{X}_2$
13	$\xi(g_5, a_1)$, $\xi(g_1, a_5)$	$\mathbf{K}$
14	$\xi(g_6, a_1)$, $\xi(g_7, a_1)$, $\xi(g_1, a_6)$, $\xi(g_1, a_7)$	$\mathbf{X}_2$
15	$\xi(g_2, a_3)$, $\xi(g_3, a_2)$	$\mathbf{X}_2$
16	$\xi(g_6, a_6)$, $\xi(g_7, a_6)$, $\xi(g_6, a_7)$, $\xi(g_7, a_7)$	$\mathbf{X}_2$
17	$\xi(g_3, a_0)$, $\xi(g_2, a_0)$, $\xi(g_0, a_3)$, $\xi(g_0, a_2)$	$\mathbf{Y}_1$
18	$\xi(g_6, a_0)$, $\xi(g_7, a_0)$, $\xi(g_0, a_6)$, $\xi(g_0, a_7)$	$\mathbf{Y}_1$
19	$\xi(g_6, a_4)$, $\xi(g_7, a_4)$, $\xi(g_4, a_6)$, $\xi(g_4, a_7)$	$\mathbf{Y}_1$
20	$\xi(g_5, a_3)$, $\xi(g_5, a_2)$, $\xi(g_3, a_5)$, $\xi(g_2, a_5)$	$\mathbf{Y}_1$
21	$\xi(g_6, a_3)$, $\xi(g_7, a_3)$, $\xi(g_6, a_2)$, $\xi(g_7, a_2)$, $\xi(g_3, a_6)$, $\xi(g_2, a_6)$, $\xi(g_3, a_7)$, $\xi(g_2, a_7)$	$\mathbf{Z}_1$
22	$\xi(g_5, a_4)$, $\xi(g_4, a_5)$	Not Feasible

Table 3 : Solid Strip Classes with ξ Notation and Boundaries.

Appendix

A.1 Full Set of Solid Strip Configurations

In this appendix the full set of 64 solid strip configurations, obtained by identifying in couples 4 of the 6 faces of a cube, is given. The configurations are grouped in 22 equivalent classes as specified in the right columns of the tables. Note that one class leads to non feasible configurations and therefore the number of possible different solid strips that can be obtained is only 21. However, it is not studied in this paper the possibility to further partition the above classes in classes of homeomorphic spaces.

n	Ident. 1	Ident. 2	Face 1 [1234]	Face 2 [5678]	Class
1	[1562]=[4873]	[1584]=[2673]	1(-C)1(-E)1C1E1	5A5D5(-A)5(-D)5	1
2	[1562]=[4873]	[1584]=[3762]	1(-C)1E1C1E1	5A5(-D)5(-A)5(-D)5	2
3	[1562]=[3784]	[1584]=[2673]	1(-C)1(-E)1(-C)1E1	5A5D5A5(-D)5	2
4	[1562]=[3784]	[1584]=[3762]	1(-D)2F1(-D)2F1	5B6(-E)5B6(-E)5	3
5	[1562]=[3784]	[1584]=[3267]	1(-D)2(-A)1(-D)2(-A)1	2B6C2B6C2	4
6	[1562]=[3784]	[1584]=[6732]	1(-D)2(-C)1(-D)2(-C)1	5B1A5B1A5	4
7	[1562]=[3487]	[1584]=[3762]	1(-B)2D1(-B)2D1	2(-A)6(-C)2(-A)6(-C)2	4
8	[1562]=[8734]	[1584]=[3762]	1(-B)2D1(-B)2D1	5A1(-C)5A1(-C)5	4
9	[1562]=[3487]	[1584]=[3267]	1(-D)2(-A)1A2D1	2B6(-B)2(-C)6C2	5
10	[1562]=[8734]	[1584]=[6732]	1(-D)2D1C2(-C)1	5B1A5(-A)1(-B)5	5
11	[1562]=[7348]	[1584]=[7326]	1(-D)2(-E)3B4F1	3B4F1(-D)2(-E)3	6
12	[1562]=[7348]	[1584]=[2376]	1(-D)1A3B4C1	3B4C1(-D)1A3	7
13	[1562]=[7348]	[1584]=[7623]	1(-D)2(-C)3B3(-A)1	3B3(-A)1(-D)2(-C)3	7
14	[1562]=[4378]	[1584]=[7326]	1(-B)2(-C)3A1D1	3A1D1(-B)2(-C)3	7
15	[1562]=[7843]	[1584]=[7326]	1(-B)2(-C)2(-A)4D1	2(-A)4D1(-B)2(-C)2	7
16	[1562]=[4873]	[1584]=[6237]	1(-C)2D2C1E1	2A1(-E)1(-A)2(-D)2	8
17	[1562]=[8437]	[1584]=[2673]	1(-C)1(-E)3(-A)3E1	3A3D1C1(-D)3	8
18	[1562]=[4873]	[1584]=[7326]	1(-C)2(-D)2C1E1	2A1E1(-A)2(-D)2	9
19	[1562]=[7348]	[1584]=[2673]	1(-C)1(-E)3A3E1	3A3D1(-C)1(-D)3	9
20	[1562]=[8437]	[1584]=[6237]	1(-C)2D1(-A)2E1	2A1(-E)2C1(-D)2	10
21	[1562]=[8437]	[1584]=[3762]	1(-C)2E1(-A)2E1	2A1(-D)2C1(-D)2	11
22	[1562]=[3784]	[1584]=[6237]	1(-C)2D1(-C)2E1	2A1(-E)2A1(-D)2	11
23	[1562]=[8437]	[1584]=[3267]	1(-D)2(-A)1(-B)2(-C)1	2B1A2D1C2	12
24	[1562]=[8437]	[1584]=[6732]	1(-D)2(-A)1(-B)2(-C)1	2B1A2D1C2	12
25	[1562]=[3487]	[1584]=[6237]	1(-B)2C1A2D1	2B1(-D)2(-A)1(-C)2	12
26	[1562]=[8734]	[1584]=[6237]	1(-B)2C1A2D1	2B1(-D)2(-A)1(-C)2	12
27	[1562]=[8437]	[1584]=[7326]	1(-C)1(-D)3(-A)3E1	3A3E1C1(-D)3	13
28	[1562]=[7348]	[1584]=[6237]	1(-C)2D2A1E1	2A1(-E)1(-C)2(-D)2	13
29	[1562]=[8437]	[1584]=[2376]	1(-D)1A3(-B)3C1	3B3(-A)1D1(-C)3	14
30	[1562]=[8437]	[1584]=[7623]	1(-D)1A3(-B)3C1	3B3(-A)1D1(-C)3	14
31	[1562]=[4378]	[1584]=[6237]	1(-B)2C2(-A)1D1	2B1(-D)1A2(-C)2	14
32	[1562]=[7843]	[1584]=[6237]	1(-B)2C2(-A)1D1	2B1(-D)1A2(-C)2	14

n	Ident. 1	Ident. 2	Face 1 [1234]	Face 2 [5678]	Class
33	[1562]=[3487]	[1584]=[6732]	1(-D)2B1A2(-C)1	2B1A2(-C)1(-D)2	15
34	[1562]=[8734]	[1584]=[3267]	1(-D)2(-A)1C2B1	2B1(-D)2(-A)1C2	15
35	[1562]=[4378]	[1584]=[2376]	1(-D)1A3(-A)1C1	3B1(-D)1C1(-B)3	16
36	[1562]=[4378]	[1584]=[7623]	1(-D)2D1(-A)1B1	1B1(-A)1C2(-C)1	16
37	[1562]=[7843]	[1584]=[2376]	1(-D)1A1(-C)4C1	1B4(-B)1A1(-D)1	16
38	[1562]=[7843]	[1584]=[7623]	1(-D)2B2(-C)2D1	2B2(-A)1A2(-C)2	16
39	[1562]=[4873]	[1584]=[3267]	1(-D)1(-A)1D1C1	1B1(-A)1(-B)1C1	17
40	[1562]=[4873]	[1584]=[6732]	1(-D)1A1D1(-C)1	1B1A1(-B)1(-C)1	17
41	[1562]=[3487]	[1584]=[2673]	1(-B)1(-D)1(-A)1D1	1(-B)1C1(-A)1(-C)1	17
42	[1562]=[8734]	[1584]=[2673]	1(-B)1(-D)1A1D1	1(-B)1C1A1(-C)1	17
43	[1562]=[4873]	[1584]=[2376]	1(-D)1A1D1C1	1B1A1(-B)1C1	18
44	[1562]=[4873]	[1584]=[7623]	1(-D)1(-A)1D1(-C)1	1B1(-A)1(-B)1(-C)1	18
45	[1562]=[4378]	[1584]=[2673]	1(-B)1(-D)1A1D1	1(-B)1C1A1(-C)1	18
46	[1562]=[7843]	[1584]=[2673]	1(-B)1(-D)1(-A)1D1	1(-B)1C1(-A)1(-C)1	18
47	[1562]=[3784]	[1584]=[2376]	1(-D)1A1(-D)1C1	1B1(-C)1B1(-A)1	19
48	[1562]=[3784]	[1584]=[7623]	1(-D)1C1(-D)1A1	1B1(-A)1B1(-C)1	19
49	[1562]=[4378]	[1584]=[3762]	1(-B)1D1(-A)1D1	1A1(-C)1B1(-C)1	19
50	[1562]=[7843]	[1584]=[3762]	1(-B)1D1(-A)1D1	1A1(-C)1B1(-C)1	19
51	[1562]=[3487]	[1584]=[7326]	1(-B)1(-C)1B1D1	1A1D1(-A)1(-C)1	20
52	[1562]=[8734]	[1584]=[7326]	1(-B)1(-C)1B1D1	1A1D1(-A)1(-C)1	20
53	[1562]=[7348]	[1584]=[3267]	1(-D)1(-A)1B1A1	1B1(-C)1(-D)1C1	20
54	[1562]=[7348]	[1584]=[6732]	1(-D)1C1B1(-C)1	1B1A1(-D)1(-A)1	20
55	[1562]=[3487]	[1584]=[2376]	1(-D)1A1A1C1	1B1B1(-C)1D1	21
56	[1562]=[3487]	[1584]=[7623]	1(-D)1(-B)1A1(-D)1	1B1(-A)1(-C)1(-C)1	21
57	[1562]=[8734]	[1584]=[2376]	1(-D)1A1C1C1	1B1D1(-A)1B1	21
58	[1562]=[8734]	[1584]=[7623]	1(-D)1(-D)1C1(-B)1	1B1(-A)1(-A)1(-C)1	21
59	[1562]=[4378]	[1584]=[3267]	1(-D)1(-A)1(-A)1(-B)1	1B1D1C1C1	21
60	[1562]=[4378]	[1584]=[6732]	1(-D)1(-D)1(-A)1(-C)1	1B1A1C1B1	21
61	[1562]=[7843]	[1584]=[3267]	1(-D)1(-A)1(-C)1(-D)1	1B1B1A1C1	21
62	[1562]=[7843]	[1584]=[6732]	1(-D)1(-B)1(-C)1(-C)1	1B1A1A1D1	21
63	[1562]=[3784]	[1584]=[7326]	Not Feasible	Not Feasible	22
64	[1562]=[7348]	[1584]=[3762]	Not Feasible	Not Feasible	22

Table A1.1 : Solid Strip Configurations with Relevant Class

A.2 Boundaries of Strip Configurations by Class

This appendix contains the boundaries of the 21 solid strip configurations in a pictorial form.

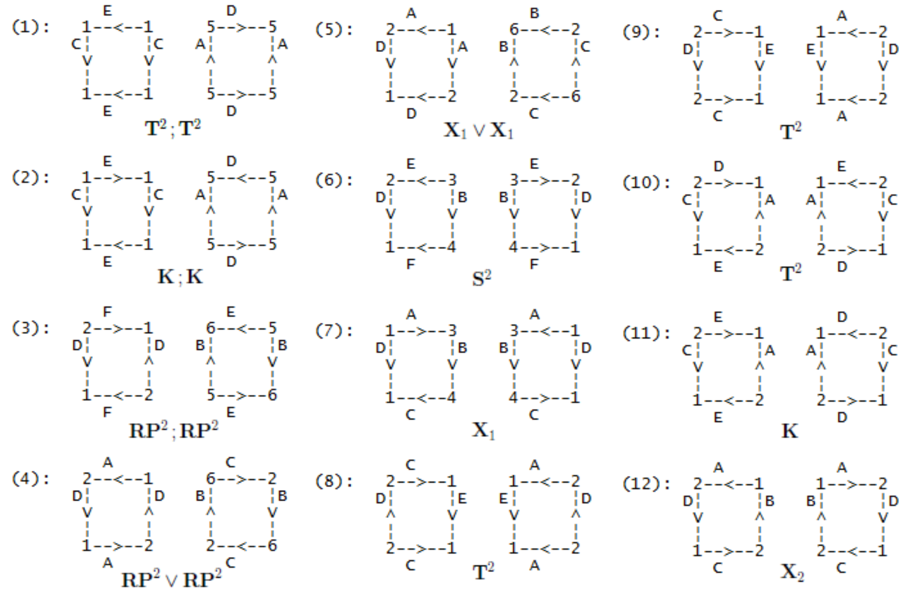

Figure 2: Solid Strip Configuration Boundaries for Classes 1-12

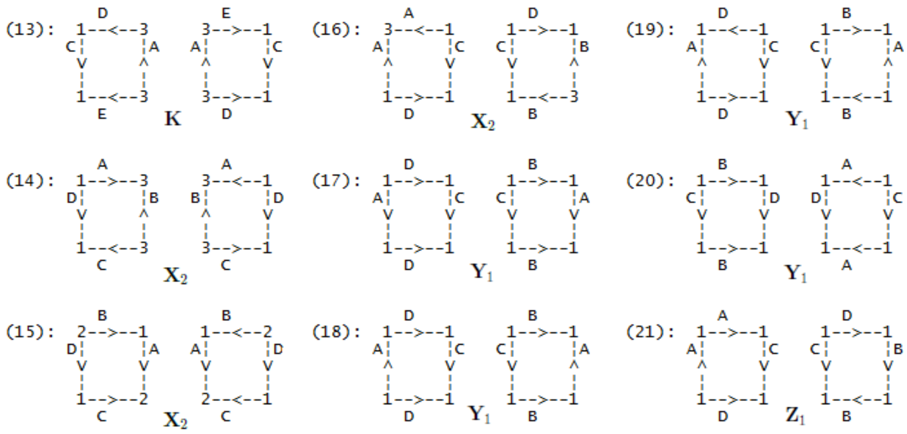

Figure 3: Solid Strip Configuration Boundaries for Classes 13-21