Stefan-Boltzmann constant incorrect by a factor of 2π

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Abstract- Due to wrong applications of Planck's radiation law a wrong Stefan-Boltzmann constant has been introduced. This article describes 2 of such applications and proves that the mentioned constant is a factor 2π too high. One of the consequences is that the alleged power density of the sun is also 2π too high. Or its temperature is 1.6 times higher!

1. Planck's original description of the two power density spectra

These descriptions are found in [1]. The relevant parameters are:

h	Planck's constant	$6.6 \cdot 10^{-34}$	Js
c	speed of light	$3.0 \cdot 10^{8}$	m/s
k	Boltzmann's constant	$1.4 \cdot 10^{-23}$	J/K

" Moreover the specific intensity K of a monochromatic plane polarized ray of frequency v is, according to equation (160),

$$K = hv^3c^{-2}/(\exp(hv/kT) - 1) \qquad (274)"$$

Respectively:

" If we refer the specific intensity of a monochromatic ray not to the frequency v but, as is usually done in experimental physics, to the wave length λ, by making use of (15) and (16) we obtain the expression

$$E_\lambda = (hc^2/\lambda^5) \;/\; (\exp(hc/k\lambda T) - 1) \qquad (276)$$

This is the specific intensity of a monochromatic plane polarized ray of the wave length λ which is emitted from a black body at the temperature T into vacuum in a direction perpendicular to the surface. "

N.B. The text doesn't show any remark about solid angles, neither are there numerical constants used in the expression for K and E_λ.

Attention has to be paid to the condition: ".......emitted .... in a direction perpendicular to the surface".

2. Application of Planck's law to stars

The stars, inclusive the sun, are assumed to have a spherical shape. Their radiation therefore fulfils Planck's condition: the emission has to be in a direction perpendicular to the surface under consideration. All points on the surface of stars emit in such a direction! So Planck's law can be applied to all places of the surface of stars, but one have to keep in mind that it is eventually restricted to a relatively very small part of the total surface.

The first step is to integrate the spectrum over de frequency. This integral can be written as:

$$C \cdot \int_0^\infty x^3/(e^x - 1)dx$$

with $x = hv/kT$, so $v = xkT/h$, thus $dv = dx \cdot kT/h$. As a result: $C = h/c^2 . (kT/h)^4$.

According to [2] the integral $\int_0^\infty x^3/(e^x - 1)dx$ "is a particular case of a Bose–Einstein integral, the polylogarithm, or the Riemann zeta function ζ(s). The value of the integral is $6 \cdot \zeta(4) = \pi^4/15$, ".

The final result is: $\int_0^\infty hv^3c^{-2}/(\exp(hv/kT) - 1)\, dv = h/c^2 . (kT/h)^4 \cdot \pi^4/15 = (\pi^4/15) \cdot k^4/h^3c^2 \cdot T^4 = \sigma_S T^4$.

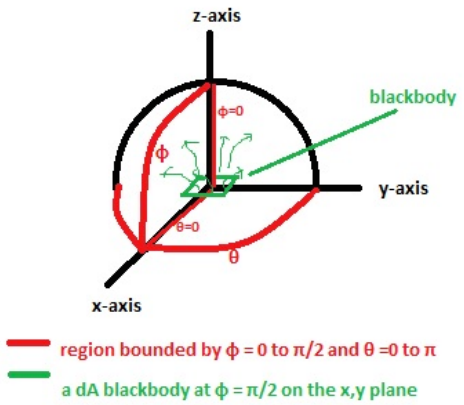
This constant σ_S is a factor 2π lower than the generally accepted $\sigma = 2\pi \cdot \pi^4 k^4/15h^3c^2$ W/m²/K⁴.

3. Application of Planck's law to a small flat black body

Reference [2] shows an application of the power density spectrum based on a factor 2 added to the original equation and an elaboration based on the application of the solid angle approach. Strange enough [2] doesn't show the application elaborated in section 2. The text is in Italics and intended.

> " *Derivation from Planck's law*
>
> *The law can be derived by considering a small flat black body surface radiating out into a half-sphere. This derivation uses spherical coordinates, with θ as the zenith angle and φ as the azimuthal angle; and the small flat blackbody surface lies on the xy-plane, where $\theta = \pi/2$.*"

Comment:
See figure below, copied from [2]. However in this figure the zenith angle is represented by φ, while θ is the azimuth angle. As a consequence the last expression has to be $\varphi = \pi/2$.



> " *The intensity of the light emitted from the blackbody surface is given by Planck's law:*
>
> $$I(v,T) = 2 \cdot hv^3 c^{-2} / (\exp(hv/kT) - 1)$$
>
> *$I(v,T)$ is the amount of power per unit surface area per unit solid angle per unit frequency emitted at a frequency v by a black body at temperature T.* "

Comment:
Planck's law is violated here two times: by the addition of the factor 2 and by the addition of: "per unit solid angle".

> " *The quantity $I(v,T)\ A\ dv\ d\Omega$ is the power radiated by a surface of area A through a solid angle $d\Omega$ in the frequency range between v and $v + dv$* "

Comment:
The addition "through a solid angle dΩ" is fundamentally wrong, because the quantity $I(v,T)\ A\ \mathrm{d}v \cdot \mathrm{d}\Omega$ has no physical meaning at all. The text should have sound: The quantity $I(v,T)\ A\ dv$ is the power radiated by a surface of area A in the frequency range between v and $v + \mathrm{d}v$.

> " *The Stefan–Boltzmann law (*power density equals Stefan–Boltzmann constant multiplied by T^4*) gives the power emitted per unit area of the emitting body,*
>
> $$P/A = \int_0^\infty I(v,T)\,\mathrm{d}v \int \cos\theta\,\mathrm{d}\Omega$$ "

Comment:
According to the figure above this expression has to be: $P/A = \int_0^\infty I(v,T)\,\mathrm{d}v \int \cos\varphi\,\mathrm{d}\Omega$

" *Note that the cosine appears because black bodies are Lambertian (i.e. they obey Lambert's cosine law), meaning that the intensity observed along the sphere will be the actual intensity times the cosine of the zenith angle.*
To derive the Stefan–Boltzmann law, we must integrate dΩ = sin(θ) dθ dφ over the half-sphere and integrate ν from 0 to ∞."

Comment:
According to the figure above the solid angle dΩ has to be expressed as sinφ dφ dθ. If for the moment a radius r, significant larger than the size of the black body, of the hemisphere is introduced, then the surface of the related infinite small area at this distance from the centre is represented by $r\mathrm{d}\theta \sin\varphi \cdot r\mathrm{d}\varphi$. It shows that the horizontal dimension of this area is zero around the z-axis, $r\mathrm{d}\theta$ in the x-y plane and that the 'height' is always $r\mathrm{d}\varphi$. "Obeing Lambert's cosine law" the correct integral to be calculated thus is:

$$P = r^2 \int_0^\infty I(\nu,T)\,\mathrm{d}\nu \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi/2} \cos\varphi \sin\varphi\, \mathrm{d}\varphi$$

instead of the presented integral:

$$P = A \int_0^\infty I(\nu,T)\,\mathrm{d}\nu \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi/2} \cos\theta \sin\theta\, \mathrm{d}\theta$$

The only relevant difference to be considered is the term r^2 versus A.
The outcome of $\int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi/2} \cos\varphi \sin\varphi\, \mathrm{d}\varphi$ is $2\pi \cdot 0.5 = \pi$.
So the correct calculation is: $P = \pi\, r^2 \int_0^\infty I(\nu,T)\,\mathrm{d}\nu$ versus the incorrect one: $P = \pi A \int_0^\infty I(\nu,T)\,\mathrm{d}\nu$.

The relevance of the question above: what is physically meant with the quantity $A\, I(\nu,T)\,\mathrm{d}\nu\, \mathrm{d}\Omega$, pops up. The two expressions for P show that $A = r^2$, which is not true. It would mean that the surface A is, for example, a square with size r. This proves that the approach shown in [2] has to be rejected.

Seemingly the integral $\int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi/2} \cos\varphi \sin\varphi\, \mathrm{d}\varphi$ has been introduced, in combination with the incorrectly addition of the factor 2, in order to obtain the generally accepted expression of the Stefan-Boltzmann constant. In order to measure the temperature of such a black body one has to aim the power measuring device along the z-axis, taking care that its beamwidth is narrow enough to cover the body only partly. The measured power divided by the area that is covered at the source then is: $\pi^4 k^4/15h^3c^2 \cdot T^4$. The result is a temperature in K that is 1.6 times higher than calculated with the present Stefan-Boltzmann constant.

4. Consequence for the alleged power density of the sun

Assumed that the temperature of the sun at its surface is indeed 5777 K and applying a 2π times smaller σ, the result is a 2π times lower power density of the sun at its surface.
This power density transported to the position of the earth, ignoring phenomena like the albedo effect, leads to the rounded value: 220 W/m^2, instead of the generally accepted value: 1360 W/m^2.

N.B. The area at the surface of the sun that just covers the earth is $2.8 \cdot 10^9$ m^2, being relatively an extremely small (~ $5 \cdot 10^{-10}$) part the total surface of the sun! That makes the application of Lambert's cosine law superfluous.

Conclusions

1 The Stefan-Boltzmann constant $\sigma = 2\pi \cdot \pi^4 k^4/15h^3c^2$ has to be replaced by $\sigma_S = \pi^4 k^4/15h^3c^2$ W/m^2/K^4.

2 One of the consequences is that the alleged power density of the radiation of the sun has to be reduced by this same factor 2π. Or its temperature has to be raised to 9146 K.

References

[1] http://www.gutenberg.org/files/40030/40030-pdf.pdf
[2] https://en.wikipedia.org/wiki/Stefan-Boltzmann_law