Fictitious forces, inertial forces and Mach's principle (I)

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Abstract. We study the application of the laws of mechanics in inertial and non-inertial reference frames. We verify that in the usual formulation of mechanics, there are no centrifugal forces. We understand the force of inertia as a force that acts on everybody that is accelerated with respect to an inertial reference frame. The so-called fictitious forces are not considered as forces since other bodies do not produce them. We indicate the superiority of the principle of dynamic equilibrium or D'Alembert's law over the usual way of expressing the second law of mechanics. In the second part of this research, Mach's principle develops and it is shown that inertia is produced by the action of the gravitational induction forces of the whole Universe.

1. Laws of the mechanics

Classical mechanics is based on the three laws of Newton: the law of inertia, the law of proportionality between force and acceleration and the law of action and reaction.

The law of inertia must be understood as the definition of good reference frames or inertial frames, which are those in which the laws of mechanics, in particular the second law, are valid. The reference frames that are accelerated with respect to an inertial frame are called non-inertial.

The realization of the inertial reference frame is the International Reference Celestial System ICRS, which has its origin in the barycentre of the solar system and its axes are aligned with a set of distant radio sources. Then, Barycentric Celestial Reference System BCRS is defined, which establishes the metric tensor of the reference frame [1, 2].

If $\mathbf{a}$ is the acceleration of a body of mass m with respect to an inertial frame K, $\mathbf{a}'$ the acceleration of the same body with respect to a non-inertial frame K', $\mathbf{R}$ is the position vector of the origin of K' with respect to K, $\mathbf{v}'$ and $\mathbf{r}'$ the velocity and position of the particle with respect to K' and $\boldsymbol{\omega}$ the intrinsic rotation velocity of K' with respect to K, then

$$\mathbf{a} = \mathbf{a}' + \ddot{\mathbf{R}} + 2\boldsymbol{\omega} \wedge \mathbf{v}' + \boldsymbol{\omega} \wedge \left(\boldsymbol{\omega} \wedge \mathbf{r}'\right) + \dot{\boldsymbol{\omega}} \wedge \mathbf{r}'. \qquad (1)$$

We assume that the force $\mathbf{F}$ acting on the body only depends the relative positions and on the relative velocities, therefore, the value of $\mathbf{F}$ is the same regardless of the reference system.

The second law of dynamics in an inertial reference system is

$$\mathbf{F} = m\mathbf{a}$$

using (1)

$$\mathbf{F} - m\ddot{\mathbf{R}} - m2\boldsymbol{\omega} \wedge \mathbf{v}' - m\boldsymbol{\omega} \wedge \left(\boldsymbol{\omega} \wedge \mathbf{r}'\right) - m\dot{\boldsymbol{\omega}} \wedge \mathbf{r}' = m\mathbf{a}' \quad \Rightarrow \quad \mathbf{F} + \mathbf{F}_f = m\mathbf{a}' \qquad (2)$$

1.- In the conception of Mach, the Universe defines the good reference frame, that is, the inertial frame. And also the Universe generates a force on accelerated bodies, which we will identify with the force of inertia.

2.- The reference frame derived from the ICRS is not strictly an inertial system, due to the geodesic precession. A gyroscope isolated in the barycenter of the solar system has a precession with respect to the ICRS, something that would not happen if the system were inertial. As the geodesic precession arises from the galactic movement of the solar system, it turns out to be very small so that we can identify for practical purposes the ICRS with an inertial frame.

which is the second law of dynamics in a non-inertial frame. $\mathbf{F}_f$ is erroneously called the force of inertia or fictitious force. As we will see, the force of inertia is another different concept. It is also wrong to call $\mathbf{F}_f$ fictitious force, because it is not a force, in the sense that it does not express the interaction between two bodies. $\mathbf{F}_f$ is the product of the mass by various types of accelerations, but it is only a term that appears in the second law of mechanics and not a real force. Equation (2) is valid in an inertial reference frame, and in this case $\mathbf{F}_f$ is null.

To $-2m\boldsymbol{\omega}\wedge\mathbf{v}'$ we will call it, following the custom, fictitious force of Coriolis, $-m\boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{r}')$ fictitious centrifugal force and $-m\dot{\boldsymbol{\omega}}\wedge\mathbf{r}'$ fictitious force of Euler.

The physical results should not depend on the reference system in which the observation is made. It is indifferent that we do the calculations in an inertial system or a non-inertial system; the only difference is that in the first there are no fictitious forces and if there are in the non-inertial systems.

2 The centrifugal force is not a real force

One of the phenomena that seem most apparent is the action of the centrifugal force, which seems to act on a body that is in rotation. It is a radial force, directed towards the convex part of the curved path and which is detected by an observer in an inertial reference system. However, the centrifugal force is not a real force, as we check in the following examples. However, it is a widespread mistake to consider the centrifugal force as real.

a) *A rope that rotates with a body located at its end*. In this simple experience, we notice that on our hand, which holds the end of the spinning rope, there is a force in the outward direction. It is understood that this force is the centrifuge that acts on the body that is at the other end of the rope, and that is transmitted to our hand through the rope.

b) *Two bodies tied to a rope and rotating around an intermediate point.* As the two bodies rotate, the rope is in tension, which seems to be a manifestation of the centrifugal forces acting on the two bodies that are spinning.

c) *A ring that moves freely on an axis that rotates around one of its ends*. When the axis starts to rotate, the ring moves until it reaches the other end. Everything seems as if a centrifugal force acts on the ring that rotates with the axis.

d) *The sensation that a passenger has in a vehicle when it takes a curve*. When taking the curve, the traveler feels the sensation that a force takes him out of the curve, a force that identifies as the centrifugal.

e) *Crane that rotates having at its end a rope from which a body hangs*. When the crane rotates the rope with the weight stops being vertical and takes an inclination; the body moves away from the axis of rotation. It seems that on the body when rotating, a force (the centrifuge) acts that tends to move it away from the axis of rotation.

We analyze these five examples as are observed from an inertial reference frame, and we verify that they are explained without the centrifugal force. Although if the observation is made from a non-inertial reference frame we must consider the fictitious centrifugal force, which as we have said is not a force but a term that appears in the equation of motion.

a) *Body tied to the end of a rope and that is turning from the other end.* The hand applies a force T to the rope, which transmits it to the body of mass m, it is a centripetal force, directed toward the concave part of the curve, which produces a normal or centripetal acceleration v^2/l in the body, being v the linear velocity of m and l the length of the rope. By the second law of mechanics applicable to an inertial frame

$$F_n = ma_n \quad \Rightarrow \quad T = m\frac{v^2}{l} = m\omega^2 l$$

When the hand applies a force to the rope, there is a reaction force, that is, a force of the rope on the hand, which gives the feeling that the body that turns drags us. There is no centrifugal force directed outward from the body's trajectory.

Now we analyze the experiment from a non-inertial frame K' that rotates with angular velocity $\boldsymbol{\omega}$ and with respect to which the body is at rest. Then we must apply the equation (2) valid in non-inertial systems and taking into account that the only fictitious force that exists is the centrifuge

$$\mathbf{F} + \mathbf{F}_f = \mathbf{a}' \quad \Rightarrow \quad \mathbf{F} - m\boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \mathbf{l}) = 0 \quad \Rightarrow \quad T = m\omega^2 l$$

recovering the same result as before.

b) *Two bodies tied to a rope and rotating around an intermediate point.* Bodies of masses m_1 and m_2 linked by a rope rotate freely. For the bodies to have the rotation motion, it is necessary to apply a centripetal force, which is the tension *T* of the rope. Applying the second law of mechanics with respect to the inertial frame *K* to each of the bodies

$$T = m_1 a_1 = m_1 \omega^2 l_1; \quad T = m_2 a_2 = m_2 \omega^2 l_2$$

l_1 and l_2 are the distances of the masses to the center of mass which, is where the axis of rotation passes. Note that the rope exerts a force on the masses (the centripetal force that causes the rotation of the masses) and these, in reaction, apply a force on the rope, equal and opposite, and is therefore, the cause of that the rope is in tension. This force of tension is what makes us think that there is a centrifugal force.

Now we analyze the experiment from a non-inertial frame K' that rotates with the two masses and that are therefore at rest with respect to K'. By application of (2) and knowing that the only fictitious force is the centrifuge

$$\mathbf{T} - m_1 \boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \mathbf{l}_1) = 0; \quad \mathbf{T} - m_2 \boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \mathbf{l}_2) = 0$$

we get the same result as before. **T** is the force that the rope exerts on each of the two masses.

c) *A ring that moves freely on an axis that rotates around one of its ends.* In this experiment, a ring moves freely and without friction on an axis that rotates with an angular velocity ω. The Cartesian coordinates of the ring with respect to an inertial reference system *K* with its origin in the axis of rotation are

$$x = r\cos\omega t; \quad y = r\sin\omega t \quad \Rightarrow \quad \mathbf{r} = x\mathbf{i} + y\mathbf{j}$$

r is the distance from the ring to the axis of rotation, **r** the vector of position of the ring, and we assume that the motion of the ring is the *x-y* plane. The velocity and acceleration of the ring are

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (\dot{r}\cos\omega t - r\omega\sin\omega t)\mathbf{i} + (\dot{r}\sin\omega t + r\omega\cos\omega t)\mathbf{j}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = (\ddot{r}\cos\omega t - 2\omega\dot{r}\sin\omega t - \omega^2 r\cos\omega t)\mathbf{i} + (\ddot{r}\sin\omega t + 2\dot{r}\omega\cos\omega t - \omega^2 r\sin\omega t)\mathbf{j}.$$

The force that the axis exerts on the ring is always normal to the axis, that is, perpendicular to **r**

$$\mathbf{F} = -F\sin\omega t\,\mathbf{i} + F\cos\omega t\,\mathbf{j}.$$

The components of the second law of dynamics $\mathbf{F} = m\mathbf{a}$ are

$$\begin{aligned} -F\sin\omega t &= m(\ddot{r}\cos\omega t - 2\omega\dot{r}\sin\omega t - \omega^2 r\cos\omega t) \\ F\cos\omega t &= m(\ddot{r}\sin\omega t + 2\dot{r}\omega\cos\omega t - \omega^2 r\sin\omega t), \end{aligned} \tag{3}$$

multiplying the first equation by $\cos\omega t$, the second equation by $\sin\omega t$ and adding we get

$$\ddot{r} - \omega^2 r = 0$$

integrating

$$r = r_0 \cosh(\omega t)$$

r_0 is the initial position of the ring. Therefore the movement of the ring along the axis is explained without considering the centrifugal force.

We calculate the value of the force on the ring by one of the two equations (3)

$$F = 2m\dot{r}\omega = 2mr_0\omega\sinh(\omega t)$$

the force direction is always perpendicular to the axis.

We analyze the motion of the ring with respect to a non-inertial frame K' that rotates together with the axis, therefore with an angular velocity ω with respect to an inertial system. Then we must apply equation (2), where there is the fictitious centrifugal force and the fictitious force of Coriolis. The acceleration with respect to K' is $\mathbf{a}'$ and the velocity $\mathbf{v}'$

$$\mathbf{F} - m\boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \mathbf{r}) - 2m\boldsymbol{\omega} \wedge \mathbf{v}' = m\mathbf{a}'$$

F is the force that the axis applies to the ring that is perpendicular to the axis. Using polar coordinates

$$\mathbf{v}' = \dot{r}\,\mathbf{u}_r; \quad \mathbf{a}' = \ddot{r}\,\mathbf{u}_r; \quad -m\boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{r}) = m\omega^2 r\,\mathbf{u}_r; \quad -2m\boldsymbol{\omega}\wedge\mathbf{v}' = -2m\omega\dot{r}\,\mathbf{u}_n$$

$\mathbf{u}_r$ is the radial unit vector and $\mathbf{u}_n$ is the normal unit vector.and the equation of motion (2) is

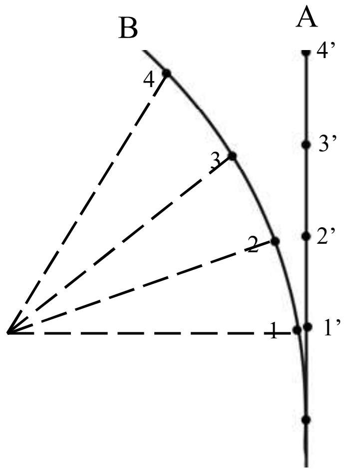
$$F\,\mathbf{u}_n + m\omega^2 r\,\mathbf{u}_r - 2m\omega\dot{r}\,\mathbf{u}_n = m\ddot{r}\,\mathbf{u}_r$$

doing the decomposition into radial and normal components

$$F = 2m\omega\dot{r}; \quad \ddot{r} - \omega^2 r = 0$$

they are the same results as those found in the frame *K*.

d) The sensation that a passenger has in a vehicle when it takes a curve. We consider a vehicle that initially has a constant velocity and a rectilinear motion. Inside there is a passenger that is on the floor of the vehicle, not having friction with the floor. At a precise moment, the vehicle begins to turn when a centripetal force acts on it. No force acts on the occupant due to not having friction with the floor of the vehicle, so that he remain in uniform and rectilinear motion. The motion of the vehicle and the passenger are those of the drawing 1.



Drawing 1.- *The rectilinear trajectory A with uniform motion is the one followed by the passenger. The vehicle follows a curved path B. The relative motion of the passenger with respect to the vehicle is a displacement towards the convex part of the curve, but it is not a radial movement.*

The numbers represent the positions of the vehicle and the occupant at the same time. Then with respect to the vehicle, the passenger moves out of the curve (but not exactly in the radial direction), until the moment he reaches the door of the vehicle, then the door transmits to the passenger the centripetal force acting on the vehicle, following the vehicle in its curved path. The mistaken feeling of the passenger is that a centrifugal force (directed out of the curve) acts on him, which he transmits to the vehicle's door and therefore the passenger receives the reaction force of the door.

e) *Crane that rotates having at its end a rope from which a body hangs*. At the end of a crane there is a rope of length *l*, which has at its end a body of mass *m*. At the beginning the rope is vertical, but when the crane starts to rotate with angular velocity ω, the rope moves away from the vertical an angle α (drawing 2).

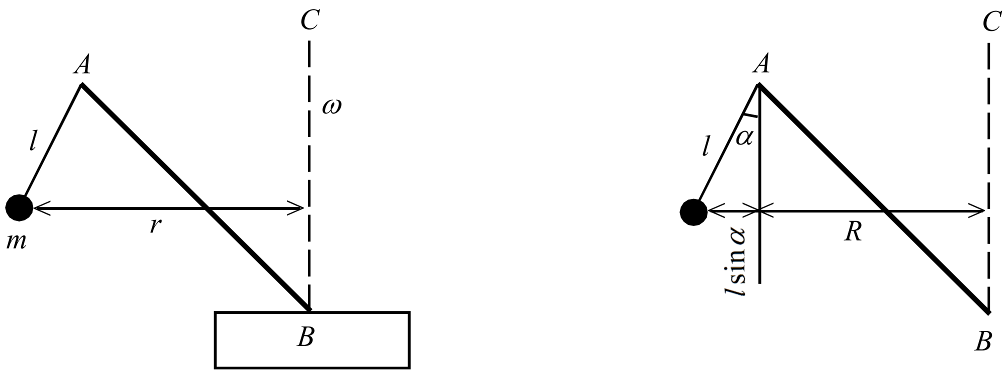
Two forces act on the body, the tension *T* of the rope and the weight *mg*. By applying the second law of mechanics and taking into account that the motion of the body *m* is in the horizontal plane, we have for an inertial reference frame

$$\mathbf{F} = m\mathbf{a} \quad \Rightarrow \quad T\cos\alpha = mg; \quad T\sin\alpha = m\omega^2 r$$

The rotational motion of the body has a centripetal acceleration $\omega^2 r$. *r* is the distance of the mass *m* to the axis of rotation and depends on the velocity ω, such that

$$r = R + l\sin\alpha$$

R is the distance from the body *m* to the axis of rotation before it began to rotate. So



Drawing 2.- *A crane has at its end A a rope of length l that holds a mass m. The crane rotates with angular velocity w around the BC axis. As a result of this rotation, the mass m moves away from the vertical.*

$$\begin{aligned} T\cos\alpha &= mg \\ T\sin\alpha &= m\omega^2 r = m\omega^2 R + m\omega^2 l\sin\alpha \end{aligned} \tag{4}$$

system of two equations with two unknowns T and α. If the angular velocity is small, then the angle α also is small and $R \gg l\sin\alpha$ which allows us to solve the equations by successive approximations.

Initially, we neglect the term $m\omega^2 l\sin\alpha$ and find as the first approximation for the angle α

$$\tan\alpha_1 = \frac{\omega^2 R}{g} \quad\Rightarrow\quad \sin\alpha_1 = \frac{\omega^2 R/g}{\sqrt{1+\omega^4 R^2/g^2}} \approx \omega^2 R/g \quad\Rightarrow\quad T_1 \approx mg + m\omega^2 l,$$

for the next approximation, we return to (4), but now in the second member of the second equation instead of α we put the value α_1 that we just found

$$\begin{aligned} T\cos\alpha &= mg \\ T\sin\alpha &= m\omega^2 R + m\omega^4 l\, R/g \end{aligned}$$

therefore the following approximation for α

$$\tan\alpha_2 = \frac{\omega^2 R}{g}\left(1 + \frac{\omega^2 l}{g}\right)$$

from here we find the approximation for T, and so it continues until the approximation we want.

In the previous examples we have shown that force centrifugal does not exist, although centrifugal fictitious force does exist, but as we have said this is not a force, but a term that appears in the equation of motion (2), that has the form mathematical of a force (ie, a mass multiplied by an acceleration) and has been mistakenly understood to be a force, but it is not, because a force is always the result of the interaction of two bodies, either by a contact action or by intermediation of a field.

3 Effects observed on the surface of the Earth as a result of its rotation

a) ***The decrease in the acceleration of gravity***

* Non-inertial reference frame

We examine this question in the reference frame K' which has its origin in the center of the Earth and rotates with it with its same angular velocity ω; its z'-axis has the direction and sense of the axis of rotation of the Earth, and the y'-axis is directed towards the meridian origin of the longitudes. A force towards the center of the Earth of value $m\mathbf{g}$ acts on a body located at a point of latitude λ. By applying the equation of motion (2) to a body that falls freely to Earth, we only have to consider the fictitious centrifugal force and the fictitious force of Coriolis

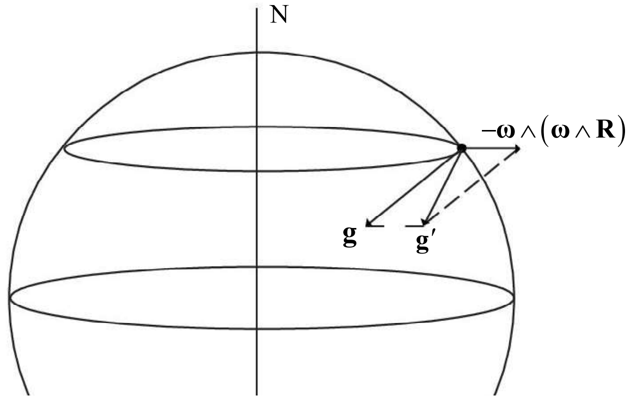
$$m\mathbf{g} - m\boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \mathbf{r}') - 2m\boldsymbol{\omega} \wedge \mathbf{v}' = m\mathbf{a}' \tag{5}$$

$\mathbf{g}$ is the acceleration gravity, $\mathbf{r}'$ is the position vector of the body that falls freely at latitude λ, $\mathbf{a}'$ is the acceleration with which the body falls to Earth according to the observer K' and $\mathbf{v}'$ is the velocity of fall also with respect to K'. We assume that the height from which the body falls is minimal with respect to the radius of the Earth, therefore r' coincides with the radius of the Earth R.

The fictitious force of Coriolis has the sense towards the east and therefore does not contribute to the acceleration of fall, at least in first approximation; that is, we are neglecting the displacement to the east that a body suffers when it falls freely. Therefore the equation of motion is

$$\mathbf{g} - \boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \mathbf{R}) = \mathbf{g}' \tag{6}$$

$\mathbf{g}'$ is the resulting acceleration on the Earth in rotation. The fictitious centrifugal force has the direction indicated in the drawing 3.



Drawing 3.- *The direction of centrifugal acceleration is out of the curve that describes the body. The acceleration* **g** *has radial direction, which does not happen with the acceleration* $\mathbf{g}'$.

The centrifugal acceleration is

$$-\boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \mathbf{R}) = R\omega^2 \cos\lambda \sin L \mathbf{i}' + R\omega^2 \cos\lambda \cos L \mathbf{j}' \tag{7}$$

L is the geographical longitude of the observation site. The radial component of (7) is

$$-\boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \mathbf{R}) \frac{\mathbf{R}}{R} = R\omega^2 \cos^2 \lambda$$

have the same sense as **R**. We call to $\mathbf{g}_{ef}$ the acceleration of effective gravity. Since the accelerations **g** and $\mathbf{g}_{ef}$ are radial and opposite to **R**, (5) is

$$g_{ef} = g - R\omega^2 \cos^2 \lambda \tag{8}$$

which means that the acceleration of gravity observed on the rotating Earth is smaller than the acceleration of the Earth caused by gravity **g**.

Fictitious centrifugal acceleration (7) not only has a radial component that decreases the acceleration of gravity, but it has a tangential component to the surface of the Earth, and whose direction is south, which produces an effect that we will analyze.

* Inertial reference frame

For an inertial reference frame, the acceleration of gravity is still **g**. However, the time of fall of a body is the same result as if it is analyzed from the frame K' with the formula (8). The reason is that in the frame K' the body falls vertically, but in the frame K, fixed in the center of the Earth, an elliptical motion of the body is observed when falling, to which we must add the curvature of the terrestrial surface.

* Fall time for the non-inertial frame

For the no-inertial frame K' the body falls vertically and with uniformly accelerated motion of acceleration g_{ef}, therefore the time it takes the body to reach the ground is

$$t = \sqrt{\frac{2h}{g_{ef}}} = \sqrt{\frac{2h}{g - \omega^2 R \cos\lambda}}. \tag{9}$$

* Fall time for the inertial frame

For the inertial frame K the motion of the body is the combination of a uniform and tangential movement to the Earth's surface of value $\omega R\cos\lambda$ and a uniformly accelerated vertical movement of acceleration g. The body falls from a height h above the surface of the Earth. If we choose a frame K whose axis z is vertical and the x axis is directed towards the east, then the equations of motion are

$$\begin{aligned} x &= \omega R\cos\lambda \\ z &= \left(R+h\right)-\frac{1}{2}gt^2 . \end{aligned} \tag{10}$$

The motion of the body is in the z-y plane. The cut of this plane with the terrestrial sphere is a circumference with the equation

$$x^2+z^2=R^2 . \tag{11}$$

The body reaches the ground when its trajectory cuts to the previous circumference, a condition that allows us to determine the time t that the body takes to fall. Therefore the time t is calculated by including equation (9) in (10). Since h is much smaller than R, we neglect the quantities of order h^2. Since gt^2 is very close to h, we also neglect the terms g^2t^4 and hgt^2. With these simplifications

$$\omega^2R^2\cos^2\lambda t^2+R^2+2hR-gt^2R=R^2 \quad\Rightarrow\quad t=\sqrt{\frac{2h}{g-\omega^2R\cos\lambda}} .$$

we obtain the equation (9) again, verifying that although the acceleration of the gravity is different in K than in K' the time it takes the body to reach the ground is the same for the two systems.

The time calculated in (9) is approximate. Among other things, because it considers that the motion of the body when falling is parabolic and not elliptical and does not consider the displacement towards the east and the south of the body in its fall motion.

<u>* The decrease in the weight of a body</u>

The acceleration of gravity in the frame K is not modified, but the same does not happen with the weight of a body. In the frame K' the weight is mg_{ef} and therefore, it is less than if the Earth did not turn. However, the same must happen in the inertial frame K. For the frame K, the body is rotating and therefore has a centripetal acceleration $\boldsymbol{\omega}\wedge\left(\boldsymbol{\omega}\wedge\mathbf{R}\right)$, which means that on this body must act a centripetal force $m\boldsymbol{\omega}\wedge\left(\boldsymbol{\omega}\wedge\mathbf{R}\right)$. The only force acting on the body is the gravitational force $m\mathbf{g}$, which decomposes in a force $\mathbf{F}$ (which is the cause of the weight) and in the centripetal force

$$m\mathbf{g}=\mathbf{F}+m\boldsymbol{\omega}\wedge\left(\boldsymbol{\omega}\wedge\mathbf{R}\right)$$

the radial component is

$$mg=P+m\omega^2R\cos\lambda \quad\Rightarrow\quad P=mg-m\omega^2R\cos\lambda=m\left(g-\omega^2R\cos\lambda\right) \tag{12}$$

P is the radial component of $\mathbf{F}$, that is, the apparent weight of the body. (12) is the same result as in K'. In addition to the radial component of $\mathbf{F}$, there is a component directed towards the south, for the observation made from the northern hemisphere.

b) ***Displacement to the east of a body that falls freely***

<u>* Non-inertial reference frame</u>

We have already said that if the observation is made from a rotating system there is a fictitious force of Coriolis that displaces the body towards the east, regardless of whether it is in the northern or southern hemisphere. We analyze this effect from a system K'' that has its origin in the point of the Earth's surface of latitude λ where the body falls. Its axis z'' is perpendicular to the Earth's surface, y'' is directed towards the north and the axis x'' towards the east and therefore K'' has an intrinsic rotation of angular velocity ω.

As we calculate the displacement to the east, we do not consider the fictitious centrifugal force, but we must say that there are two of these forces. One of them is the first term of (2), which is to say $-m\ddot{\mathbf{R}}$ and caused by the rotation of origin of the frame K''. The other centrifugal fictitious force is $-\boldsymbol{\omega}\wedge\left(\boldsymbol{\omega}\wedge\mathbf{r}''\right)$ where $\mathbf{r}''$ is the position vector with respect to K'', which in previous calculations we have considered $r''\ll R$.

The equation of motion (3) in the system K'' is

$$m\mathbf{g} - 2m\boldsymbol{\omega} \wedge \mathbf{v}' = m\mathbf{a}' \tag{13}$$

where we do not consider the effect of the fictitious centrifugal forces and **g** is the acceleration of gravity. The angular velocity **ω** and the linear velocity $\mathbf{v}'$ with respect to K'' are

$$\boldsymbol{\omega} = \omega\cos\lambda\,\mathbf{j}'' + \omega\sin\lambda\,\mathbf{k}''; \quad \mathbf{v}'' = -v''\mathbf{k}'' = -gt\mathbf{k}''$$

From equation (13) we deduce

$$a''_x = 2\omega g t\cos\lambda; \quad a''_y = 0; \quad a''_z = -g$$

doing integration

$$\begin{aligned} x'' &= \frac{1}{3}\omega g t^3 \cos\lambda \\ z'' &= -\frac{1}{2} g t^2 + h \end{aligned} \tag{14}$$

h is the height from which the body falls. The time it takes the body to reach the ground is derived from the last equation

$$t = \sqrt{\frac{2h}{g'}}$$

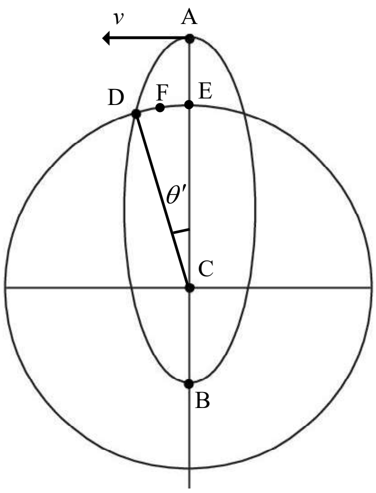
therefore of the first equation (14) we determine the displacement of the body towards the east

$$\delta = \frac{1}{3}\omega g'\left(\sqrt{\frac{2h}{g}}\right)^3 \cos\lambda. \tag{15}$$

This displacement to the east occurs in the northern hemisphere and the southern hemisphere. If the latitude is negative, its cosine is positive and the displacement has the same sense as when the body falls in the northern hemisphere.

* Inertial reference frame

The problem is considerably complicated when the calculations are made with respect to an observer in an inertial reference frame K that has its origin in the center of the Earth. In the system K it is observed that the body initially has a tangential velocity to the surface of the Earth of value $v = \omega r$, where r is the distance to the axis of rotation of the Earth. If the body is at latitude λ, then



Drawing 4.- *The elliptical orbit that would follow the body dropped from point A, observed from an inertial frame K. The point B is the perigee of the orbit. C is the center of the Earth. The point D is where the body crosses with the Earth's surface. The circle is the cut of the Earth's sphere with the plane where the path of the falling body lies, which contains the center of the Earth. The drawing is observed from the north pole, in such a way that the Earth rotates against the hands of the clock. The initial velocity of the body v at point A is to the left of the drawing and therefore the body moves to the east. Point E is located on the perpendicular of A. As a consequence of the Earth's rotation movement, point E moves to F during the time it takes the body to fall, describing an angle α.*

$$r=(R+h)\cos\lambda \quad \Rightarrow \quad v=\omega(R+h)\cos\lambda\ .$$

If the mass of the Earth were concentrated in its center, then the body would describe an ellipse since the kinetic energy is less than the potential and the total energy of the body is negative; therefore its trajectory must be closed. The apogee of this orbit would be the point where the body is initially.

The apogee and the perigee are the only points of the elliptical orbit in which the velocity of the orbiting body is perpendicular to the radius vector that joins it with the focus of the ellipse (that is, the center of the Earth). The initial point A of the body must be the apogee because from that point the body begins to descend and the distance to the center of the Earth is decreasing, something that would not happen if the initial point were the perigee.

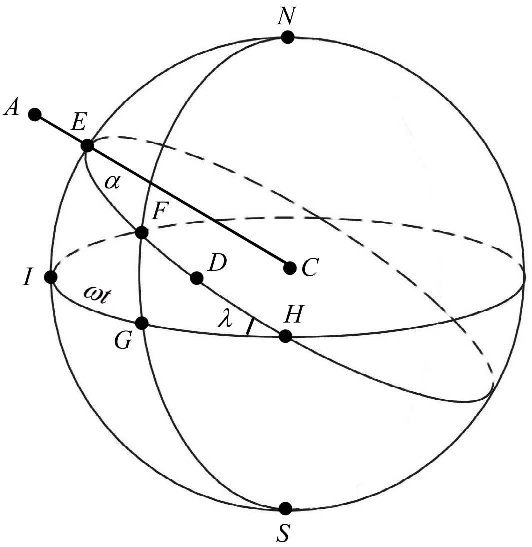
For an elliptical orbit, we have

$$r=\frac{p}{1+e\cos\theta};\quad L^2=GMm^2p;\quad e=\sqrt{1+\frac{2EL^2}{G^2M^2m^3}} \tag{16}$$

the first is the equation of the ellipse in polar coordinates, e is the eccentricity, θ is the true anomaly (angle measured from the perigee of the orbit to the point of the ellipse and in the sense of movement) and $p=a\left(1-e^2\right)$ (a is the semimajor axis of the ellipse). L is the angular moment of the body calculated with respect to the center of the Earth (which is the focus of the ellipse), G the gravitational constant, M the mass of the Earth, m the mass of the body that orbits, E the total energy of the body. The last two equations allow us to determine the coefficients p and e. L and E are constants of movement, so to find their value we calculate it for the initial moment

$$L=mv(R+h);\quad E=\frac{1}{2}mv^2-G\frac{Mm}{r}. \tag{17}$$

From the equations (16) and (17) we calculate



Drawing 5.- *The drawing represents the earth sphere, and HDFE is the circumference resulting from the cut of the sphere with the plane of the orbit that the body follows. The plane of the orbit is inclined at an angle λ to the equator. N and S are the north and south poles, and C is the center of the Earth. The turning motion of the Earth is counter-clockwise viewed from the north pole. The body is dropped at point A and collides with the surface of the Earth at point D. Meanwhile, the terrestrial sphere has moved an angle ωt, that is from I to G. At the same time point E has moved to F, traveling an angle α. The spherical triangle to be resolved is the HFG, from which we determine the relationship between the angles α and ωt. Note that point D is farther east than point F, hence the body moves eastward. Point D, which is where the body falls, is farther south than point A, which tells us that in addition to a displacement to the east there is another displacement to the south (if the movement it would have been in the southern hemisphere the displacement would have been towards the east and towards the north).*

$$p=\frac{\omega^2(R+h)^4\cos^2\lambda}{GM};\quad e=\sqrt{1+\frac{2p}{GM}\left(\frac{v^2}{2}-G\frac{M}{r}\right)}.$$

To find the displacement to the east, it is necessary to determine the point at which the ellipse cuts to the surface of the Earth (of equation $r=R$), which is the point on Earth where the body reaches the ground

$$R=\frac{p}{1+e\cos\theta}\quad\Rightarrow\quad\cos\theta=\frac{1}{e}\left(\frac{p}{R}-1\right),$$

but the angle that interests us is $\theta'=\pi-\theta$ (see drawing 4)

$$\cos\theta'=\frac{1}{e}\left(1-\frac{p}{R}\right)\quad\Rightarrow\quad\theta'=\cos^{-1}\left[\frac{1}{e}\left(1-\frac{p}{R}\right)\right].$$

While the body falls, the Earth is spinning. If t is the time it takes the body to reach the ground, then the arc described at the equator by the Earth is ωt, but the orbit that describes the body is inclined at an angle λ with respect to the equator, and therefore the circumference of drawing 4 has that same inclination. What we are interested in calculating is angle α of drawing 5.

The relationship between the angles of drawing 4 emerges from the spherical triangle *HFG*

$$\tan\alpha=\tan\omega t\cos\lambda.$$

The length of the arc of angle α is $R\alpha$, and the arc that has moved the body is $R\theta'$, so the displacement to the east is

$$\delta=R(\theta'-\alpha)=R\left\{\cos^{-1}\left[\frac{1}{e}\left(1-\frac{p}{R}\right)\right]-\alpha\right\}.\tag{18}$$

The numerical calculation with equation (18) have to do it with the maximum precision, therefore we can not use the time t calculated previously in (9), but it is necessary to calculate it rigorously.

* Calculation of the time of fall

To calculate the time it takes for the body to reach the ground we use the law of the areas applied to the elliptical path that the body follows when it falls, which tells us

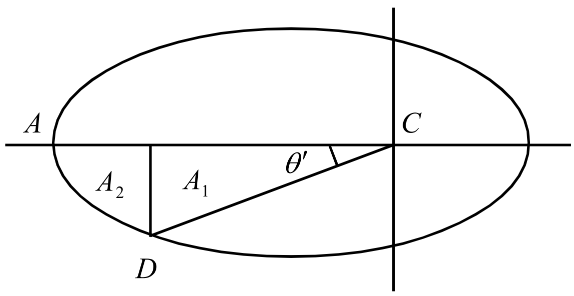
$$\frac{dS}{dt}=\frac{1}{2}r^2\frac{d\theta}{dt}=cte$$

where dS is the area swept by the radius vector when the body describes an angle $d\theta$. By definition, the angular momentum in polar coordinates is

$$L=mr^2\dot{\theta}\quad\Rightarrow\quad\frac{L}{2m}=\frac{1}{2}r^2\dot{\theta}=\frac{dS}{dt}\tag{19}$$

$L/2m$ it is the constant of the areas. If we calculate the area dS described by the radius vector from the initial position (point *A* of drawing 6) to the position characterized by θ' (point *D*) we can by (19) calculate the time t.

In drawing 6 the swept area is formed by points *A*, *D* and *C*, which is divided into two portions A_1 and A_2. A_1 is the area of a right triangle



Drawing 6.- *Elliptical orbit that would follow the body. Point A is the apogee, where it was initially the body. C is the center of the Earth, and the focus of the ellipse and D is the cut-off point of the ellipse with the surface of the Earth. The area swept by the body in its fall motion is ADC, which is divided into two portions A1 and A2.*

$$A_1 = \frac{1}{2} R\sin\theta' R\cos\theta' = \frac{1}{4} R^2 \sin 2\theta'.$$

To find A_2 we start from the equation of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \Rightarrow \quad y = b\sqrt{1 - \frac{x^2}{a^2}}$$

b is the semi-minor axis. A_2 is the integral of the previous function between points x_1 and x_2

$$x_1 = -a; \quad x_2 = -R\cos\theta' + ea$$

e is the eccentricity of the orbit,

$$A_2 = \int_{x_1}^{x_2} b\sqrt{1 - \frac{x^2}{a^2}}\, dx = \frac{ab}{2}\left[\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{a}\sqrt{1 - \frac{x^2}{a^2}}\right]_{x_1}^{x_2} = \frac{ab}{2}\left[\frac{\pi}{2} + \sin^{-1}\left(\frac{x_2}{a}\right) + \frac{x_2}{a}\sqrt{1 - \frac{x_2^2}{a^2}}\right].$$

Therefore the area swept at time t is

$$S = A_1 + A_2 = \frac{1}{4} R^2 \sin 2\theta' + \frac{ab}{2}\left[\frac{\pi}{2} + \sin^{-1}\left(\frac{x_2}{a}\right) + \frac{x_2}{a}\sqrt{1 - \frac{x_2^2}{a^2}}\right]$$

and the time t is

$$t = \frac{S}{L/2m}.$$

<u>* Example</u>

We apply the previous results to a body that drops from a height of 200 meters at a latitude of 30º north. The necessary data are

$$\begin{aligned}
&G = 6.67\cdot 10^{-11}\, N\cdot m^2/kg^2\\
&M = 5.974\cdot 10^{24} kg\\
&R = 6.378\cdot 10^{6} m\\
&h = 200 m\\
&\omega = 7.272205\cdot 10^{-5}\, rad/s\\
&\lambda = 30^{\circ}.
\end{aligned}$$

The initial velocity of the body is

$$v = (R+h)\omega\cos\lambda = 401.693\ 5681 m/s$$

the parameter p and the eccentricity e are

$$p = \frac{\omega^2 (R+h)^4 \cos^2\lambda}{GM} = 16\ 473.84479\ m; \quad e = \sqrt{1 + \frac{2p}{GM}\left(\frac{v^2}{2} - G\frac{M}{r}\right)} = 0.997\ 417\ 164$$

from which we deduced that the angle that the body moves is

$$\theta' = 0^{\circ}.023\ 0946108$$

finally, we calculate the dimensions of the ellipse

$$a = 3\,193\,223.828 m; \quad b = 229\,357.0814 m; \quad c = 3\,184\,976.255 m.$$

To apply (17) it is necessary to calculate the time that the body takes to fall, using (19). The limits of the integral are

$$x_1 = 3\,193\,223.828 m;\ x_2 = -3\,193\,023.227 m \ \Rightarrow\ A_1 = 343\,808.608 m^2;\ A_2 = 8\,198\,360\,049 m^2$$

and the time of fall is

$$t = 6.40003 s.$$

therefore

$$\omega t = 0^{\circ}.026\,666\,79087; \quad \alpha = 0^{\circ}.023\,094\,11875$$

with this result applies (17) obtaining

$$\delta = 0.0547 m$$

if we do the calculation with the formula (14)

$$\delta = 0.0536 m.$$

which means an error of 2%.

c) **Displacement to the south (or to the north) of a body that falls freely**

* Non-inertial reference frame

We refer the movtion of free fall of a body to the frame K'' that has its origin in the surface of the Earth where the body falls, the axis z'' is vertical to the surface of the Earth, the axis y'' has the north direction and the axis x'' the east direction. The equation of fall free of a body is (5). Since we are not interested in the motion to the east, we do not take into account the fictitious force of Coriolis, so the equation of motion is

$$\mathbf{a}' = \mathbf{g} - \boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \mathbf{R})$$

we have neglect the height from which the body falls, that is $r'' \ll R$. With respect to the frame K''

$$\mathbf{R} = R\mathbf{k}''; \quad \boldsymbol{\omega} = \omega\cos\lambda\,\mathbf{j}'' + \omega\sin\lambda\,\mathbf{k}''$$

therefore

$$\boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \mathbf{R}) = R\omega^2 \cos\lambda\sin\lambda\,\mathbf{j}'' - R\omega^2\cos^2\lambda\,\mathbf{k}'',$$

the first term of the second member is the one that produces the deviation towards the south or toward the north and is the one that we consider

$$\ddot{y}'' = -R\omega^2\cos\lambda\sin\lambda \quad \Rightarrow \quad y'' = -\frac{1}{2}R\omega^2\cos\lambda\sin\lambda\, t^2$$

t is the time it takes the body to reach the ground

$$t \approx \sqrt{\frac{2h}{g}}$$

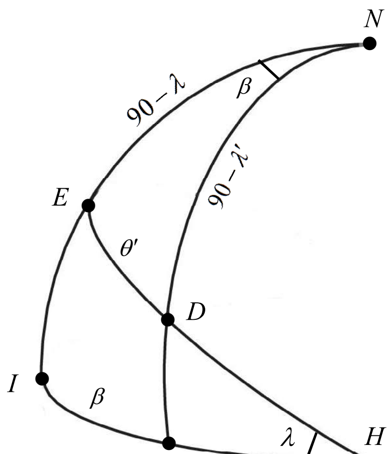
therefore the displacement is

$$y'' = \delta_S = -\frac{R\omega^2 h}{g}\cos\lambda\sin\lambda, \tag{20}$$

if the body falls in the northern hemisphere, where the latitude is positive, y'' is negative, that is, the body moves in the opposite direction to the axis y'', therefore towards the south. If the body falls in the southern hemisphere, then y'' is negative and y'' is positive, that is, the body moves to the north.

* Inertial reference frame

To determine the displacement to the south (or north) of a body that falls freely relative to an inertial frame *K*, we consider drawing 7 based on drawing 5. Point *D* is where the body reaches the



Drawing 7.- *It is a detail of drawing 4. Point D is the point where the body falls. The HJI arc is the equator. E is the point of the Earth on which the body was when it was dropped and has a latitude λ.*

surface of the Earth. Point *J* is the projection of the point *D* on the terrestrial equator. λ' is the latitude of point *D* and it is the unknown that must be determined.

The angle at the vertex *N* coincides with the angle β of the arc *IJ* which is determined by

$$\tan\theta' = \tan\beta\cos\lambda \quad \Rightarrow \quad \tan\beta = \frac{\tan\theta'}{\cos\lambda}.$$

Now we have to solve the spherical triangle *NED* and calculate λ', knowing that the angle in *E* is right. Using the sine theorem

$$\frac{\sin\theta'}{\sin\beta} = \frac{\sin(90-\lambda')}{\sin 90} \quad \Rightarrow \quad \cos\lambda' = \frac{\sin\theta'}{\sin\beta}.$$

Finally we must determine the length of the arc formed by the latitude points λ and λ' located on the same meridian

$$\delta_S = R(\lambda - \lambda').$$

* Example

Calculation of the displacement towards the south of a body that falls freely from a height of 200 meters at a point of the north latitude of 30 degrees.

Applying the formula (20) valid for the non-inertial frame K'' is found

$$\delta_S = -0.298m.$$

If we calculate in relation to an inertial reference frame *K*

$$\beta = 0^{\circ}.026\ 667\ 359; \quad \lambda' = 29^{\circ}.999\ 997\ 31,$$

and we find

$$\delta_S = -0.299m$$

practically the same result as in the frame K'', the small difference is due to the simplifications that have been made.

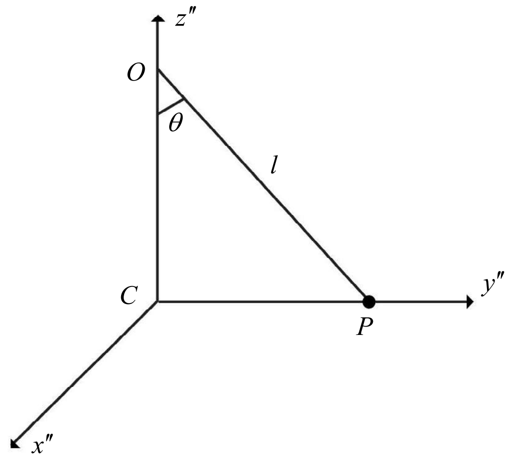
d) **Foucault's Pendulum**

* Non-inertial reference frame

It is called Foucault pendulum to a pendulum that can oscillate freely in any plane perpendicular to the Earth's surface and can remain in movement during a prolonged time. We study the movement of the weight of the pendulum from the non-inertial reference frame K'', which we have previously defined. On the weight of the pendulum act the tension of the rope *T* and the weight *mg*. If we neglect the centrifugal effect we only have to consider the fictitious force of Coriolis; therefore the equation of motion with respect to the non-inertial frame K'' is

$$\mathbf{T} + m\mathbf{g} - 2m\boldsymbol{\omega}\wedge\mathbf{v}'' = m\mathbf{a}'' \tag{21}$$

$\mathbf{v}''$ and $\mathbf{a}''$ are the velocity and acceleration with respect to K''.



Drawing 8.- *The pendulum has its oscillation point in O, and its weight in P. θ is the angle of oscillation. Initially, the pendulum is oscillating in the* $z''-y''$ *plane.*

We assume that the weight *P* of the pendulum (drawing 8) is on the y''-axis and that the pendulum oscillates in the $z'' - y''$ plane. If the oscillation amplitude of the pendulum is small, then its oscillation angle is

$$\theta = \theta_0 \cos \omega_0 t$$

θ_0 is the position of the pendulum in $t = 0$ and ω_0 is the angular frequency of oscillation. Since the oscillations are small, we assume that the weight of the pendulum moves along the y''-axis, being its distance to the equilibrium point C

$$y'' = l\theta_0 \cos \omega_0 t$$

l is the length of the pendulum, so the modulus of its velocity is

$$v'' = \theta_0 l \omega_0 \sin \omega_0 t,$$

since the movement is on the y''-axis, the velocity vector is

$$\mathbf{v}'' = -\theta_0 l \omega_0 \sin \omega_0 t \cos\varphi \mathbf{j}'',$$

we calculate the Coriolis acceleration

$$-2\boldsymbol{\omega} \wedge \mathbf{v}'' = \begin{vmatrix} \mathbf{i}'' & \mathbf{j}'' & \mathbf{k}'' \\ 0 & \omega\cos\lambda & \omega\sin\lambda \\ 0 & -\theta_0 l\omega_0 \sin\omega_0 t & 0 \end{vmatrix} = -2\theta_0 l\omega_0 \omega \sin\lambda \sin\omega_0 t\, \mathbf{i}''.$$

We are interested in the rotation of the pendulum oscillation plane due to the fictitious force of Coriolis, that is, we are interested in the motion of the coordinate x'' of the pendulum weight, which by (21) has the equation of motion

$$ma''_x = -2m(\boldsymbol{\omega} \wedge \mathbf{v}'')_x = -2m\theta_0 l\omega_0 \omega \sin\lambda \sin\omega_0 t$$

differential equation that has the solution

$$x'' = 2\omega l\theta_0 \sin\lambda \left(\frac{\sin\omega_0 t}{\omega_0} - t \right).$$

we assume that at $t = 0$, $x'' = 0$ and $\dot{x}'' = 0$. Initially the pendulum is oscillating in the plane $x'' = 0$, but when half a period $T/2$ has elapsed the pendulum weight is in the position

$$\begin{aligned} &x''(0) = 0; \quad x''(T/2) = 2\omega l\theta_0 \sin\lambda \left(\frac{\sin\pi}{\omega_0} - \frac{\pi}{\omega_0} \right) = -2\pi \frac{\omega}{\omega_0} l\theta_0 \sin\lambda \\ &y''(0) = l\theta_0; \quad y''(T/2) = -l\theta_0 \end{aligned} \qquad (22)$$

therefore, in half cycle the variation of the pendulum weight coordinates is

$$\Delta x'' = -2\pi \frac{\omega}{\omega_0} l\theta_0 \sin\lambda; \quad \Delta y'' = -2l\theta_0$$

the oscillation plane of the pendulum has rotated an angle

$$\Delta\varphi \simeq \frac{\Delta x''}{\Delta y''} = \pi \frac{\omega}{\omega_0} \sin\lambda$$

and the angular velocity of rotation of the oscillation plane of the pendulum is

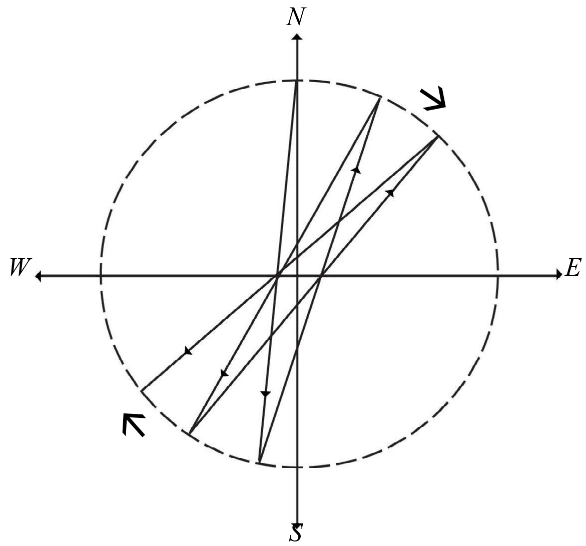
$$\Omega = \frac{\Delta\varphi}{T/2} = \omega \sin\lambda. \qquad (23)$$

In the northern hemisphere x'' in (22) has a negative value, which means that the oscillation plane of the pendulum has turned to the west, since the positive sense of x'' is towards the east. On the return path, that is, in the second part of the oscillation, the value of x'' is positive; therefore it rotates towards the east and so on.

The result is seen in drawing 9, which corresponds to a displacement of the oscillation plane in a clockwise direction. In the southern hemisphere (23) it has a negative sign because the latitude is negative, so the rotation of the oscillation plane of the pendulum is counter-clockwise.

* Inertial reference frame

With respect to the inertial reference frame *K*, the oscillation plane of the pendulum is



Drawing 8.- *The trajectory of the weight of the pendulum has been hugely exaggerated. The displacements of the plane of oscillation are alternate to the east and to the west. The result is that the oscillation plane of the pendulum rotates in a clockwise direction, as long as the pendulum is in the northern hemisphere.*

invariable, due to the conservation of the angular momentum caused by the fact that no mechanical moment acts on the pendulum. However, the horizontal plane, which is vertical to the plane of oscillation, rotates with the Earth. The result is that with respect to the horizontal plane, the plane of oscillation is rotating. This rotational velocity is the vertical component of ω, that is $\omega\sin\lambda$, equal to the result obtained in the non-inertial frame K''.

e) **Deviation of a projectile launched near the Earth's surface**

<u>* Non-inertial reference frame</u>

We consider a projectile that is launched from the point of latitude λ, with an initial velocity that has components $\dot{x}''_0, \dot{y}''_0, \dot{z}''_0$ and launched from the position $x''_0 = y''_0 = z''_0 = 0$. We want to find the effect that the rotation of the Earth has on its motion. The motion equation for the non-inertial reference frame K'' is

$$m\mathbf{a}'' = m\mathbf{g} - 2m\boldsymbol{\omega}\wedge\mathbf{v}'' - m\boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{R}) - m\boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{r}'')$$

R is the vector that joins the center of the Earth with the launch point located at the origin of K'' and $\mathbf{r}''$ is the position of the projectile with respect to the frame K''. As $r'' \ll R$ we neglect the last term of the equation of motion.

The components of Coriolis acceleration and centrifugal acceleration are

$$-2\boldsymbol{\omega}\wedge\mathbf{v}'' = -2(\dot{z}''\omega\cos\lambda - \dot{y}''\omega\sin\lambda)\mathbf{i}'' - 2\dot{x}''\omega\sin\lambda\mathbf{j}'' + 2\dot{x}''\omega\cos\lambda\mathbf{k}''$$
$$-\boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{R}) = -\omega^2R\cos\lambda\sin\lambda\mathbf{j}'' + \omega^2R\cos^2\lambda\mathbf{k}''$$

and the components of the motion equation are

$$\begin{aligned}\ddot{x}'' &= -2\dot{z}''\omega\cos\lambda + 2\dot{y}''\omega\sin\lambda \\ \ddot{y}'' &= -2\dot{x}''\omega\sin\lambda - \omega^2R\cos\lambda\sin\lambda \\ \ddot{z}'' &= -g + 2\dot{x}''\omega\cos\lambda + \omega^2R\cos^2\lambda.\end{aligned} \tag{24}$$

From the second equation (24)

$$d\dot{y}'' = -2\omega\sin\lambda\, dx'' - \omega^2R\cos\lambda\sin\lambda\, dt \quad\Rightarrow\quad \dot{y}'' = \dot{y}''_0 - 2\omega\sin\lambda x'' - \omega^2R\cos\lambda\sin\lambda t,$$

from the third equation (22)

$$d\dot{z}'' = -(g - \omega^2R\cos^2\lambda)dt + 2\omega\cos\lambda\, dx'' \quad\Rightarrow\quad \dot{z}'' = \dot{z}''_0 - (g - \omega^2R\cos^2\lambda)t + 2\omega\cos\lambda x''$$

these last two equations are inserted into the first equation (24)

$$\ddot{x}'' \approx -2\omega(\dot{z}''_0\cos\lambda - gt - \dot{y}''_0\sin\lambda)$$

As a first approximation we do not consider the terms of order ω^2

$$x'' = \dot{x}_0''t + \frac{1}{3}\omega g t^3 \cos\lambda - \omega t^2 \left(\dot{z}_0'' \cos\lambda - \dot{y}_0'' \sin\lambda\right) \tag{25}$$

the derivative is

$$\dot{x}'' = \dot{x}_0'' + \omega g t^2 \cos\lambda - 2\omega t \left(\dot{z}_0'' \cos\lambda - \dot{y}_0'' \sin\lambda\right) \tag{26}$$

including (26) in the second equation (24) and integrating

$$y'' = \dot{y}_0''t - \omega \dot{x}_0'' \sin\lambda t^2 \tag{27}$$

again we have neglected terms of order ω^2 or greater. Using the same procedure

$$z'' = \dot{z}_0''t - \frac{1}{2} g t^2 + \omega x_0'' \cos\lambda . \tag{28}$$

To make a new approximation, we take the solutions (25), (27) and (28) and insert them into the equations (24). Note that we have assumed that the geographical latitude remains the same, which means that the projectile has little travel.Otherwise, it is necessary to take the dependence of the latitude with respect to the coordinates of the projectile.

Now we assume that the projectile is fired with a motion at low altitude towards the north, therefore

$$\dot{y}_0'' = v_0 ; \quad \dot{x}_0'' = \dot{z}_0'' = 0$$

then equations (25) and (27) are simplified

$$x'' = \frac{1}{3}\omega g t^3 \cos\lambda + \omega t^2 v_0 \sin\lambda$$
$$y'' = v_0 t ,$$

the projectile velocity is fast enough therefore

$$v_0 \gg gt$$

then the deflection angle of the projectile is

$$\delta \approx \frac{x''}{y''} = \frac{\omega y \sin\lambda}{v_0}$$

if *d* is the distance traveled by the projectile, then

$$\delta = \frac{\omega d}{v_0} \sin\lambda \tag{29}$$

and the distance that the projectile has moved from its initial direction is

$$\Delta \approx d\delta = \frac{\omega d^2}{v_0} \sin\lambda .$$

<u>* Inertial reference frame</u>

With respect to an inertial reference frame *K*, the plane where the projectile is traveling has an angular velocity rotation $\omega \sin\lambda$. Then the angle rotated at a time *t* by this plane is

$$\omega t \sin\lambda = \omega \frac{d}{v_0} \sin\lambda$$

which is the angle rotated with respect to the initial direction of the projectile motion, finding that it is the same result as (29).

6 Isolated particle

<u>* Non-inertial reference frame</u>

We assume a body that is at rest with respect to an inertial reference frame, not being subjected to any force. We do the analysis from a non-inertial reference frame K' that is rotating with an intrinsic angular velocity $\boldsymbol{\omega}$, that is, the axis of rotation passes through the origin of K'. Regarding K', the body is in motion with an angular velocity $-\boldsymbol{\omega}$. If the body is in the position $\mathbf{r}'$ with respect to K' the equation of motion is

$$-2m\boldsymbol{\omega} \wedge \mathbf{v}' - m\boldsymbol{\omega} \wedge \left(\boldsymbol{\omega} \wedge \mathbf{r}'\right) = m\mathbf{a}'$$

$\mathbf{v}'$ is the velocity and $\mathbf{a}'$ the acceleration with respect to K'. As the body is rotating with angular velocity $-\boldsymbol{\omega}$ then

$$\mathbf{v}' = -\boldsymbol{\omega} \wedge \mathbf{r}'$$

therefore

$$2m\boldsymbol{\omega} \wedge \left(\boldsymbol{\omega} \wedge \mathbf{r}'\right) - m\boldsymbol{\omega} \wedge \left(\boldsymbol{\omega} \wedge \mathbf{r}'\right) = m\mathbf{a}' \quad \Rightarrow \quad \mathbf{a}' = \boldsymbol{\omega} \wedge \left(\boldsymbol{\omega} \wedge \mathbf{r}'\right) \tag{30}$$

which is the centripetal acceleration observed in the frame K'.

<u>* Inertial reference frame</u>

Equation (30) is also obtained if the analysis is made from an inertial frame K. Suppose that the z axis of K coincides with the axis of rotation of K'. So the coordinate transformation equation is

$$\begin{aligned} x' &= x\cos\omega t + y\sin\omega t \\ y' &= -x\sin\omega t + y\cos\omega t \\ z' &= z \end{aligned}$$

making the derivative with respect to the time of the previous equations, the components of the acceleration with respect to a rotating system are obtained. Keep in mind that the coordinates x, y, z do not change over time.

$$\ddot{x}' = -\omega^2 x'; \quad \ddot{y}' = -\omega^2 y'; \quad \ddot{z}'' = 0 \quad \Rightarrow \quad \mathbf{a}' = \boldsymbol{\omega} \wedge \left(\boldsymbol{\omega} \wedge \mathbf{r}'\right)$$

and we find the result (27).

5 Conservation of mechanical energy

For any inertial reference frame, the conservation of mechanical energy is valid

$$\frac{1}{2}mv_1^2 + E_{p1} = \frac{1}{2}mv_2^2 + E_{p2}$$

E_p is the potential energy of the body of mass m, which only depends on the relative positions. It is easy to verify that if the conservation theorem of energy is valid in an inertial frame, it is valid in any other inertial frame.

Now suppose a non-inertial reference frame K'. The second law of mechanics is by (2)

$$\mathbf{F} + \mathbf{F}_f = m\mathbf{a}',$$

the work done by all the forces to move the body from point 1 to 2 is

$$W_{1-2} = \int_1^2 \left(\mathbf{F} + \mathbf{F}_f\right) \cdot d\mathbf{r}' = \int_1^2 m\mathbf{a}' d\mathbf{r}' = \frac{1}{2}mv_2'^2 - \frac{1}{2}mv_1'^2,$$

if the force F derives from a potential energy, then

$$W_{1-2} = \int_1^2 \left(-\nabla E_p + \mathbf{F}_f\right) \cdot d\mathbf{r}' = E_{p1} - E_{p2} + \int_1^2 \mathbf{F}_f \cdot d\mathbf{r}',$$

the conservation theorem of energy in a non-inertial frame is

$$\frac{1}{2}mv_2'^2 + E_{p2} = \frac{1}{2}mv_1'^2 + E_{p1} + \int_1^2 \mathbf{F}_f \cdot d\mathbf{r}'$$

that is, the mechanical energy of the particle is not conserved. The work of the fictitious forces (which is negative) appears as a dissipated energy.

6 Principle dynamic equilibrium

Why maintain a body with uniform and rectilinear movement is not necessary to apply any force, however it is necessary to apply a force to maintain an accelerated movement? Why is the acceleration proportional to the force acting on the body? Why is the inertia of a body appreciated only when the velocity changes and not when the position changes? Why does force produce acceleration and not velocity? Why are there fictitious forces in non-inertial reference systems? Why are there differences between a non-inertial system and an inertial system? Why is the angular velocity $\boldsymbol{\omega}$ that appears in (2) identical to the angular velocity of the non-inertial system with respect to the Universe? Why can we determine the velocity of rotation of the Earth relative to the Universe by Foucault pendulum experiment? Why does the Universe not rotate? Why is inertia a constant in time? Why is the inertia the same in all directions? Why is inertia the same everywhere? Why is the inertia given by a scalar and not by a tensor? Why is the best reference system the one that does not rotate with respect to the Universe?

Newtonian mechanics does not explain all these questions. Now we reformulate the Newtonian mechanics and we answer the previous questions.

We start from the concept of inertia, understood as the opposition offered by bodies to change in their state of motion, that is, to change the velocity. Then we must understand that inertia is a force, which we call the force of inertia, which is opposed to the applied force, ie

$$\mathbf{F} + \mathbf{F}_i = 0$$

$\mathbf{F}$ is the applied force and $\mathbf{F}_i$ the inertial force. The above equation is called the principle of dynamic equilibrium or D'Alembert's law.

The force of inertia is

$$\mathbf{F}_i = -m\mathbf{a} \quad \Rightarrow \quad \mathbf{F} - m\mathbf{a} = 0 \tag{31}$$

where $\mathbf{a}$ is the acceleration with respect to an inertial reference system. In the development of the theory we demonstrate (31), what we leave for later.

It seems that (31) is the same as Newton's second law, but it is not like that. $\mathbf{F}_i$ is now a real force, acting on any body that is accelerated respect to an inertial frame. Since $\mathbf{F}_i$ is a real force, it must have its origin in another body, as we will see. On the other hand, (31) applies equally in an inertial or a non-inertial frame. Note that the force of inertia, as we now understand it, acts on every accelerated body and does not depend on the observer's reference frame, only on the acceleration of the body with respect to an inertial reference frame.

The infinite and uniform Universe of Newton, where the forces are static and action at a distance, produces a null net action on local bodies, so only the action of nearby bodies can influence the movement of bodies and the rest of the Universe does not affect local movements at all.

However, in field theory, in addition to the static forces, induction forces appear, which are those that are produced by the motion of the field source, or by the motion of the body on which the force acts. The gravitational theory has to be a field theory, only then can it be compatible with Relativity. Moreover, this theory, whatever it may be, has inductive effects.

Then the local bodies are no longer isolated from the rest of the cosmos because the inductive action of the Universe as a whole is not canceled, as we shall see. It is, therefore, necessary to accept that there is an inductive action of gravitational origin from the whole of the Universe on the local bodies, which we call the force of inertia.

To fully understand the law (31) it is necessary to verify that the inductive force of the whole of the Universe on an accelerated body is proportional to the acceleration and does not depend on the velocity. Then the law of inertia comes to us to say that the inductive action of the whole of the Universe does not depend on the velocity of the body. We must realize that this fact is circumstantial, not a law of Nature. The gravitational inductive force depends on the velocity, but when the total inductive force of the Universe is evaluated, the terms containing the body velocity are canceled in the classical approach.

These conceptions lead us to the conclusion that an inertial reference frame is one that has a uniform and rectilinear motion (that is, without acceleration) with respect to the whole of the Universe, so any body that in relation to an inertial frame is without acceleration does not will be subjected to the force of inertia and therefore will not act on the body any force according to (31), remaining accordingly with the same motion.

The opposite affirmation is also real. If a force does not act on a body, then by (31) there is no force of inertia and therefore no acceleration, so that the body would carry the uniform and rectilinear motion.

This situation changes when the observation is made from a non-inertial sframe. This system is accelerated with respect to the whole of the Universe, so any body that with respect to this non-inertial frame finds itself with a uniform and rectilinear movement, will be accelerated with respect to the whole of the Universe and on it the force of inertia will act. That is, according to (2) $\mathbf{a}'$ is zero and $\mathbf{F}_f$ is different from zero, so for the body to maintain the state of uniform and rectilinear motion with respect to the non-inertial frame it is necessety to apply a force $\mathbf{F}$ given by

(31), the necessary to balance $\mathbf{F}_f$.

In short, when a body acquires an acceleration with respect to an inertial system, it acquires it also with respect to the mass average of the Universe, which we consider as an inertial system. Then a gravitational inductive force appears on the accelerated body. This force is the force of inertia, which is proportional and opposite to the acceleration and which does not depend on the velocity.

7 Summary of the second part

In the second part of this investigation, the principle of dynamic equilibrium that we have stated above will be developed. We study the mechanism by which fields produce induction forces. In the field of classical physics, we obtain the techniques to determine the induction forces from the gravitational potentials. Finally, we apply the theory to some cosmological models; moreover we show that the force of inertia is produced by the gravitational forces of induction.