Somsikov A.I.

Historical problem of mathematics. Complex numbers

The description of the complex number which is not containing a concept of the complex plane is offered.

If the XOY coordinate system contains vectors $\vec{A}$ and $\vec{B}$, so that the difference in their arguments is equal to φ, and the origins coincide, vector $\vec{B}$ can be obtained from vector $\vec{A}$ by a simple summation:

$$\vec{B} = \vec{A} + \vec{C}$$

where $\vec{C}$ is the vector connecting the ends of vectors $\vec{A}$ and $\vec{B}$ (closing vector).

If modules of vectors $\vec{A}$ and $\vec{B}$ are changed with unchanged arguments, the module and the argument of vector $\vec{C}$ change.

However, it is possible to perform vector summation so that the closing vector argument does not depend on the modules of vectors $\vec{A}$ and $\vec{B}$. To do this, let's project vector $\vec{B}$ on vector $\vec{A}$ and perpendicular to vector $\vec{A}$ and write down the vector sum:

$$B\cos\varphi\frac{\vec{A}}{A} + B\sin\frac{\vec{C}'}{C'} = \vec{B} \qquad (1)$$

Vector $\vec{C}'$ is perpendicular to vector $\vec{A}$.

So, if we take the vector $\frac{B}{A}\cos\varphi\,\vec{A}$ instead of the vector $\vec{A}$ inside the vector sum, i.e. simply multiply the vector $\vec{A}$ by the coefficient, the closing vector acquires a certain argument, which does not depend on the modules of vectors $\vec{A}$ and $\vec{B}$, namely, it is perpendicular to the vector $\vec{A}$.

This circumstance can be symbolically written as follows: if vector $\vec{A}$ should be rotated $\frac{\pi}{2}$ counterclockwise to get equality of arguments of vectors $\vec{C}'$ and $\vec{A}$:

$$\frac{\vec{C}'}{C'} = i\frac{\vec{A}}{A}$$

if vector $\vec{A}$ should be rotated $\frac{\pi}{2}$ clockwise to get equality of arguments of vectors $\vec{C}'$ and $\vec{A}$:

$$\frac{\vec{C}'}{C'} = -i\frac{\vec{A}}{A}$$

So, the symbol i next to the vector means that this vector is rotated $\frac{\pi}{2}$ counterclockwise.

Considering the entered designations, vector equation (1) will be rewritten:

$$\frac{B}{A}\cos\varphi\,\vec{A} \pm i\frac{B}{A}\sin\varphi\,\vec{A} = \vec{B}$$

Let us assume that $\frac{B}{A}\cos\varphi = a$; $\frac{B}{A}\sin\varphi = b$.

$$a\vec{A} \pm ib\vec{A} = \vec{B}$$

The last equation means: to get vector $\vec{B}$, the vector $\vec{A}$ must be multiplied by the coefficient a, then the vector $\vec{A}$ must be multiplied by the coefficient b and rotated $\frac{\pi}{2}$ clockwise or counter-clockwise, and the vectors must be added together.

The operations that need to be made on vector $\vec{A}$ to get vector $\vec{B}$ can be written as follows:

$$(a \pm bi)\vec{A} = \vec{B}$$

The operator $(a \pm bi)$ is called a *complex number*.

So, the symbolic operator $(a \pm bi)$ serves for transition from one vector to another on the plane.

To show where such a transition may be needed, let's consider a system of vector equations:

$$\overrightarrow{X_1} + \overrightarrow{X_2} = \vec{A},$$
$$(a_1 + b_1 i)\overrightarrow{X_1} + (a_2+b_2 i)\overrightarrow{X_2} = \vec{B} = (a_3+b_3 i)\vec{A},$$

where $\overrightarrow{X_1}$ and $\overrightarrow{X_2}$ are vectors with unknown modules and arguments, and $\vec{A}$ and $\vec{B}$ are known vectors.

Note that since $\vec{A}$ and $\vec{B}$ are known, $\vec{B} = (a_3+b_3 i)\vec{A}$ can be found:

$$\overrightarrow{X_1} + \overrightarrow{X_2} = \vec{A}, \qquad (2)$$
$$(a_1 + b_1 i)\overrightarrow{X_1} + (a_2+b_2 i)\overrightarrow{X_2} = \vec{B} = (a_3+b_3 i)\vec{A}, \qquad (3)$$

If equation (2) had a vector $(a_1 + b_1 i)\overrightarrow{X_1}$ instead of $\overrightarrow{X_1}$, we would have subtracted equation (3) from equation (2) and the vector $(a_1 + b_1 i)\overrightarrow{X_1}$ would have disappeared. We would get the equation with one unknown $\overrightarrow{X_2}$. The systems of first degree algebraic equations with two unknowns are solved in a similar way.

So, it is necessary to be able to apply the operator $(a + bi)$ to vector equations. It is necessary to learn how to apply the operator $(a + bi)$ to the sum (difference) of vectors and to apply the operator $(a + bi)$ to the vector to which the operator of this kind is already applied.

Consider: $(a + bi)\vec{A}$.

Let us assume that $\vec{A} + \vec{B} = \vec{C}$.

By definition:

$(a + bi)\vec{A} = a\vec{A} + bi\vec{A} = a(\vec{B} + \vec{C}) + bi(\vec{B} + \vec{C})$.

And then $i\vec{B} + i\vec{C} = i\vec{A} = i(\vec{B} + \vec{C})$ (4)

(vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$, as well as i$\vec{A}$, i$\vec{B}$ and i$\vec{C}$ perpendicular to them form closed triangles).

Then: $bi(\vec{B} + \vec{C}) = i(b\vec{B} + b\vec{C}) = bi\vec{B} + bi\vec{C}$.

So: $(a + bi)\left(\vec{B} + \vec{C}\right) = a\vec{B} + a\vec{C} + bi\vec{B} + bi\vec{C} = (a + bi)\vec{B} + (a + bi)\vec{C}$.

Consequently, equation (2) is easily converted to the desired form:

$$(a_1 + b_1 i)(\overrightarrow{X_1} + \overrightarrow{X_2}) = (a_1+b_1 i)\vec{A};$$
$$(a_1 + b_1 i)\overrightarrow{X_1} + (a_1 + b_1 i)\overrightarrow{X_2} = (a_1+b_1 i)\vec{A}.$$

Consider:

$$\vec{A} \pm \vec{B} = (a_1 + b_1 i)\vec{C} \pm (a_2 + b_2 i)\vec{C} = a_1\vec{C} + b_1 i\vec{C} \pm a_2\vec{C} \pm b_2 i\vec{C} = (a_1 \pm a_2)\vec{C} + (b_1 \pm b_2)i\vec{C} = [(a_1 \pm a_2) + (b_1 \pm b_2)i]\vec{C}.$$

This result can be symbolically written down as follows:

$$(a_1 + b_1 i) \pm (a_2 + b_2 i) = (a_1 \pm a_2) + (b_1 \pm b_2)i.$$

This symbolic operation corresponding to summation (subtraction) of vectors is called summation (subtraction) of complex numbers.

Let's finally consider $(a_1 + b_1 i)\vec{A}$, if $A = (a_2 + b_2 i)\vec{B} = a_2\vec{B} + b_2 i\vec{B}$.

According to (4): $i\left(b_1 a_2 \vec{B} + b_1 b_2 i\vec{B}\right) = b_1 a_2 i\vec{B} + b_1 b_2 ii\vec{B}$.

Repeated application of i means that the vector $b_1 b_2 i\vec{B}$ already rotated $\frac{\pi}{2}$ counterclockwise with respect to vector $b_1 b_2 \vec{B}$ is additionally rotated $\frac{\pi}{2}$ in the same direction, which is equivalent to entering the vector opposite to $b_1 b_2 \vec{B}$.

Then: $b_1 b_2 ii\vec{B} = -b_1 b_2 \vec{B}$.

So: $(a_1 + b_1 i)\vec{A} = a_1 a_2 \vec{B} + a_1 b_2 i\vec{B} + b_1 a_2 i\vec{B} - b_1 b_2 \vec{B} = (a_1 a_2 - b_1 b_2)\vec{B} + (a_1 b_2 - b_1 a_2)i\vec{B} = [(a_1 a_2 - b_1 b_2) + (a_1 b_2 - b_1 a_2)i]\vec{B}$.

Or: $(a_1 + b_1 i)(a_2 + b_2 i)\vec{B} = [(a_1 a_2 - b_1 b_2) + (a_1 b_2 - b_1 a_2)i]\vec{B}$.

This result written symbolically: $(a_1 + b_1 i)(a_2 + b_2 i) = (a_1 a_2 - b_1 b_2) + (a_1 b_2 -$ $b1a2i$

is called multiplication of complex numbers.

The division of complex numbers is the search for a complex number, which, being multiplied by a given complex number, gives the second given complex number.

In the example above, this problem occurs when the coefficient at $\overrightarrow{X_1}$ is not equal to one, but is also a complex number, and still needs to be turned into $(a_1 + b_1 i)$.

So, given:

$$\alpha_1 = a + bi,$$
$$\alpha_2 = c + di.$$

Find a α_3, so that $\alpha_1 \alpha_3 = \alpha_2$.

Let us assume that $\alpha_3 = x + yi$.

Then, taking by definition: $\alpha_1 \alpha_3 = (a + bi)(x + yi) = (ax - by) + (ay + bx)i$,

we find: $(ax - by) + (ay + bx)i = c + di$.

The equality of complex numbers means the equality of the resulting vectors, if the initial vectors are equal.

Hence, the requirement to perform the equality of actual and imaginary parts of equal complex numbers is clear.

In fact, the real part gives the projection of the resulting vector on the initial one, and the imaginary part gives the projection on the perpendicular to the initial vector, and the projections of equal vectors on the same axis are equal.

So:

$$ax - by = c,$$
$$ay + bx = d.$$

Solution of this system:

$$x = \frac{ac + bd}{a^2 + b^2},$$
$$y = \frac{ad - bc}{a^2 + b^2}.$$

Formally, the same result can be obtained by representing the desired complex number as a fraction and producing actions similar to liberation from the irrationality of the denominator:

$$x + yi = \frac{c + di}{a + bi} = \frac{(c + di)(a - bi)}{(a + bi)(a - bi)} = \frac{(ac + bd) + (ad - bc)i}{a^2 + b^2}.$$

So, the system of vector equations is given:

$$\alpha_1\overrightarrow{X_1} + \alpha_2\overrightarrow{X_2} = \vec{A}, \qquad (5)$$
$$\alpha_3\overrightarrow{X_1} + \alpha_4\overrightarrow{X_2} = \vec{B}, \qquad (6)$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are complex numbers;
$\vec{A}, \vec{B}$ are vectors, modules and arguments of which are known.
Find vectors $\overrightarrow{X_1}$ and $\overrightarrow{X_2}$.
Let us assume that $\vec{B} = (a + bi)\vec{A} = \alpha\vec{A}$.

$$\alpha_1\overrightarrow{X_1} + \alpha_2\overrightarrow{X_2} = \vec{A},$$
$$\alpha_3\overrightarrow{X_1} + \alpha_4\overrightarrow{X_2} = \alpha\vec{A}.$$

Apply a complex number α_5 to one of the equations, for example (5), so that, for example, $\alpha_1\alpha_5 = \alpha_3$ $\quad (\alpha_5 = \frac{\alpha_3}{\alpha_1})$.

Then:

$$\alpha_3\overrightarrow{X_1} + \alpha_2\alpha_5\overrightarrow{X_2} = \alpha_5\vec{A},$$
$$\alpha_3\overrightarrow{X_1} + \alpha_4\overrightarrow{X_2} = \alpha\vec{A}.$$

Let's deduct equation (6) from equation (5):

$$\alpha_3\overrightarrow{X_1} + \alpha_2\alpha_5\overrightarrow{X_2} - \alpha_3\overrightarrow{X_1} - \alpha_4\overrightarrow{X_2} = \alpha_5\vec{A} - \alpha\vec{A}.$$

Then: $(\alpha_2\alpha_5 - \alpha_4)\overrightarrow{X_2} = (\alpha_5 - \alpha)\vec{A}$.

Or finally:

$$\overrightarrow{X_2} = \frac{\alpha_5 - \alpha}{\alpha_2\alpha_5 - \alpha_4}\vec{A}.$$

Similarly, systems of n vector equations with n unknown vectors are solved in the same way as systems of linear algebraic equations.

Complex numbers arise when solving systems of integro-differential equations from harmonic functions in the following order:

- the integro-differential equations are solved, after which the system turns into a system of trigonometric equations from harmonic functions,
- transition from the system of trigonometric functions to the vector diagram,
- a system of vector equations corresponding to the obtained vector diagram is compiled.

We have just described the solution of the vector equation system.