Derivation of field equations of f(R) gravity from Euler-Poisson equation

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Abstract
We derived the field equations of $f(R)$ gravity using Euler-Poisson equation, which allows the boundary term to vanish in a natural way from the principle of Calculus of variation in contrary to the original theory of H. A. Buchdahl 1970, in which the boundary term was not treated.

Keywords: Calculus of variation; f(R) gravity

Contents

1. Introduction

In his paper "Non-Linear Lagrangians and Cosmological Theory" [1] H. A. Buchdahl, proposed his generalization to the Einstein field equations by considering a generalization of the gravitational Lagrangian $\phi(R)$ to be a general function the Riemanian scalar tensor rather than just a linear function proportional to the Riemanian curvature tensor. Nowadays it is called $f(R)$ gravity. Most references that discuss $f(R)$ gravity rarely refer to H. A. Buchdahl as the first to propose such non-linear Lagrangian functional model.
He suggested a Lagrangian functional of the form

$$L = \phi(R) \quad (1.0)$$

Where $\phi(R)$ is unspecified.
He has given the generalization to the Einstein field equations as a tensor equation, which contains derivatives of $\phi(R)$ with respect to the Ricci

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scalar as well as derivatives of the Ricci scalar with respect to space-time coordinates, which reads

$$(-R_{;k}\ R_{;l}+g_{kl}\ R_{;m}\ R^{;m})\ \phi'''+(-R_{;kl}+g_{kl}\ \Box R)\ \phi''-R_{kl}\ \phi'+\frac{1}{2}\ g_{kl}\ \phi=T_{kl}\quad (1.1)$$

He has given the result of variation of $\sqrt{-g}\ \phi(R)$ as

$$\delta(\sqrt{-g}\ \phi)\approx\sqrt{-g}\ P_{kl}\ \delta g_{kl}\quad (1.2)$$

Where the sign $\approx$ denotes equality to within additive divergence, and P_{kl} is given by

$$(-R_{;k}\ R_{;l}+g_{kl}\ R_{;m}\ R^{;m})\ \phi'''+(-R_{;kl}+g_{kl}\ \Box R)\ \phi''-R_{kl}\ \phi'+\frac{1}{2}\ g_{kl}\ \phi\quad (1.3)$$

Without treating the boundary term.

2. Euler-Poisson equation of the calculus of Variation

The Euler-Poisson equation of a general Lagrangian functional L is given by the following expression [2], [3]

$$\frac{\partial L}{\partial q}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)+\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}}\right)=0\quad (2.1)$$

With the boundary term given by

$$\left[\left(\frac{\partial L}{\partial \dot{q}}-\frac{d}{dt}\frac{\partial L}{\partial \ddot{q}}\right)\delta q+\left(\frac{\partial L}{\partial \ddot{q}}\right)\delta \dot{q}\right]_{x_1}^{x_2}\quad (2.2)$$

Which vanishes if δq and $\delta \dot{q}$ vanish at the two end points x_1 and x_2.

To derive field equations for any functional of the fundamental metric tensor $g_{mp}(x^e)$ we make into the Euler-Poisson equation the following change of variables

$$
\begin{aligned}
& t \Rightarrow x^{a} \\
& q(t) \Rightarrow g_{mp}(x^{e}) \\
& \dot{q}(t) = \frac{dq(t)}{dt} \Rightarrow g_{mp,s}(x^{e}) = \frac{\partial g_{mp}(x^{e})}{\partial x^{s}} \\
& \ddot{q}(t) = \frac{d^{2}q(t)}{dt^{2}} \Rightarrow g_{mp,sr}(x^{e}) = \frac{\partial}{\partial x^{s}}\left(\frac{\partial g_{mp}(x^{e})}{\partial x^{r}}\right) = \frac{\partial^{2} g_{mp}(x^{e})}{\partial x^{s}\,\partial x^{r}} \\
& L\left(t,\ q(t),\ \dot{q}(t),\ \ddot{q}(t)\right) \Rightarrow L\left(x^{e},\ g_{mp}(x^{e}),\ g_{mp,s}(x^{e}),\ g_{mp,sr}(x^{e})\right)
\end{aligned}
\qquad (2.3)
$$

The Lagrangian of the $f(R)$ gravity is [4-6],

$$
L(g_{ab},\ g_{ab,c},\ g_{ab,cd}) = \sqrt{-g}\ L_{Gravity} \quad (2.4)
$$

Where

$$
L_{Gravity}(g_{ab},\ g_{ab,c},\ g_{ab,cd}) = f(R) \quad (2.5)
$$

Substituting Eq. (2.3), (2.4) and (2.5) in Eq. (2.1), the Euler-Poisson equation of the Lagrangian of $f(R)$ gravity may be written explicitly as

$$
\left(\frac{\partial[\sqrt{-g}\ f(R)]}{\partial g_{mp}}\right) - \frac{\partial}{\partial x^{s}}\left(\frac{\partial[\sqrt{-g}\ f(R)]}{\partial g_{mp,s}}\right) + \frac{\partial^{2}}{\partial x^{r}\,\partial x^{s}}\left(\frac{\partial[\sqrt{-g}\ f(R)]}{\partial g_{mp,sr}}\right) = 0 \quad (2.6)
$$

This is our Euler-Poisson equation, which produces the same equation of motion of $f(R)$ gravity as Buchdahl equation.
Recalling that the Ricci scalar R may written in terms of the metric tensor and its partial derivatives by

$$
\begin{aligned}
R = {} & \frac{1}{2}\ g^{ab}\ g^{ch}\ [(g_{ch,ab} - g_{ac,hb}) - (g_{bh,ac} - g_{ab,hc})] \\
& - (1/4)\ g^{ab}\ g^{ce}\ g^{hq}\ [(g_{ah,c} + g_{ch,a} - g_{ac,h})\ (g_{bq,e} + g_{eq,b} - g_{be,q}) \\
& - (g_{ah,b} + g_{bh,a} - g_{ab,h})\ (g_{cq,e} + g_{eq,c} - g_{ce,q})]
\end{aligned}
\quad (2.7)
$$

This in local coordinates like a geodesic coordinate system [4], a local inertial frame [5], or a Riemann Normal Coordinates system [7], which are characterized by

$$\frac{\partial g_{ab}}{\partial x^{c}}=0, \qquad \Gamma_{abc}=0, \qquad \Gamma_{ab}{}^{c}=0 \quad (2.8)$$

And

$$\frac{\partial}{\partial x^{c}}\left(\frac{\partial g_{ab}}{\partial x^{d}}\right)\neq 0, \qquad \frac{\partial \Gamma_{abc}}{\partial x^{d}}\neq 0, \qquad \frac{\partial \Gamma_{ab}{}^{c}}{\partial x^{d}}\neq 0 \quad (2.9)$$

Where Γ_{abc} is the Christoffel symbol of the first kind so, the Ricci Scalar in Eq. (2.7) may be rewritten as

$$R=\frac{1}{2}\, g^{ab}\, g^{ch}\, [(g_{ch,ab}-g_{ac,hb})-(g_{bh,ac}-g_{ab,hc})] \quad (2.10)$$

In which the first partial derivate of the metric tensor vanishes.
Since (a, b, c, h) are dummy indices ($\equiv$summed over), the Ricci Scalar R may be rewritten as

$$\begin{aligned}
R &=\frac{1}{2}\, g^{ab}\, g^{ch}\, [(g_{ch,ab}-g_{ac,hb})-(g_{bh,ac}-g_{ab,hc})] \\
&=\frac{1}{2}\, g^{ab}\, g^{ch}\, [(g_{ch,ab}-g_{ac,hb}-g_{bh,ac}+g_{ab,hc})] \\
&=\frac{1}{2}\, [(g^{ab}\, g^{ch}\, g_{ch,ab}-g^{ab}\, g^{ch}\, g_{ac,hb}-g^{ab}\, g^{ch}\, g_{bh,ac}+g^{ab}\, g^{ch}\, g_{ab,hc})] \\
&=\frac{1}{2}\, [(g^{ab}\, g^{ch}\, g_{ch,ab}-g^{ab}\, g^{ch}\, g_{ac,hb}-g^{ba}\, g^{hc}\, g_{ac,bh}+g^{ch}\, g^{ab}\, g_{ch,ab})] \quad (2.11) \\
&=\frac{1}{2}\, [(2\, g^{ab}\, g^{ch}\, g_{ab,ch}-2\, g^{ab}\, g^{ch}\, g_{ac,bh})] \\
&= [(g^{ab}\, g^{ch}\, g_{ab,ch}-g^{ab}\, g^{ch}\, g_{ac,bh})] \\
&= g^{ab}\, g^{ch}\, [(g_{ab,ch}-g_{ac,bh})]
\end{aligned}$$

This is resulting from making the indices changes $(a \to c,\ b \to h)$ in the first term and $(a \to b,\ c \to h)$ in the second term, respectively.
To determine the various differentiations in three terms in brackets in Eq. (2.6) in the Euler-Lagrange equation we make use of the derivation of Einstein field equation from Einstein-Hilbert [4-7] in which

$$\frac{\delta R}{\delta g^{ab}}=R_{ab} \quad (2.12)$$

When written as

$$\delta R = \frac{\partial R}{\partial g^{ab}} \delta g^{ab} \quad (2.13)$$

It implies

$$\frac{\partial R}{\partial g^{ab}} = R_{ab} \quad (2.14)$$

Since we are using the covariant metric tensor, we may transform the above equation to be rewritten in terms of the contra-variant metric tensor as

$$\frac{\partial R}{\partial g_{mp}} = \frac{\partial R}{\partial g^{ab}} \frac{\partial g^{ab}}{\partial g_{mp}} = R_{ab} \frac{\partial g^{ab}}{\partial g_{mp}} \quad (2.15)$$

Using the identity

$$\frac{\partial g^{ab}}{\partial g_{mp}} = - g^{am} g^{bp} \quad (2.16)$$

This is resulting from differentiating

$$g^{ab} g_{bc} = \delta^{a}{}_{c} \quad (2.17)$$

With respect to g_{mp}.

We get the differentiation of the Ricci scalar with respect to the covariant metric tensor as

$$\frac{\partial R}{\partial g_{mp}} = \frac{\partial R}{\partial g^{ab}} (- g^{am} g^{bp}) = R_{ab} (- g^{am} g^{bp}) = (- g^{am} g^{bp} R_{ab}) = - R^{mp} \quad (2.18)$$

Since we are considering local coordinates in which the Christoffel symbols of both kinds and the first derivative of the metric tensor vanish and do not appear in R expression in Eq. (2.7), we get

$$\frac{\partial R}{\partial g_{mp,s}} = 0 \quad (2.19)$$

The derivative of Ricci scalar with respect to second derivative of the metric tensor with respect to coordinates is given by

$$
\begin{aligned}
\frac{\partial R}{\partial g_{mp,sr}} &= \frac{\partial}{\partial g_{mp,sr}}\left[g^{ab}\, g^{ch}\,(g_{ab,ch} - g_{ac,bh})\right] = g^{ab}\, g^{ch}\,\frac{\partial}{\partial g_{mp,sr}}\left(g_{ab,ch} - g_{ac,bh}\right) \\
&= g^{ab}\, g^{ch}\left(\frac{\partial}{\partial g_{mp,sr}}\,(g_{ab,ch}) + \frac{\partial}{\partial g_{mp,sr}}\,(-\, g_{ac,bh})\right) \\
&= g^{ab}\, g^{ch}\left(\frac{\partial}{\partial g_{mp,sr}}\,(g_{ab,ch}) - \frac{\partial}{\partial g_{mp,sr}}\,(g_{ac,bh})\right) \\
&= g^{ab}\, g^{ch}\,[(\delta^{a}{}_{m}\,\delta^{b}{}_{p}\,\delta^{c}{}_{s}\,\delta^{h}{}_{r}) - \delta^{b}{}_{m}\,\delta^{h}{}_{p}\,\delta^{a}{}_{s}\,\delta^{c}{}_{r})] \\
&= g^{ab}\, g^{ch}\,(\delta^{c}{}_{m}\,\delta^{h}{}_{p}\,\delta^{a}{}_{s}\,\delta^{b}{}_{r} - \delta^{a}{}_{m}\,\delta^{c}{}_{p}\,\delta^{h}{}_{s}\,\delta^{b}{}_{r}) \\
&= (g^{ab}\, g^{ch}\,\delta^{c}{}_{m}\,\delta^{h}{}_{p}\,\delta^{a}{}_{s}\,\delta^{b}{}_{r}) - (g^{ab}\, g^{ch}\,\delta^{a}{}_{m}\,\delta^{c}{}_{p}\,\delta^{h}{}_{s}\,\delta^{b}{}_{r}) \\
&= (g^{mp}\, g^{sr}) - (g^{ms}\, g^{pr}) = (g^{mp}\, g^{sr} - g^{mr}\, g^{ps})
\end{aligned}
$$

(2.20)

Now we have at our disposal all the derivatives needed to derive our field equations of $f(R)$ gravity. We summarize these as

$$
\begin{aligned}
\frac{\partial R}{\partial g_{mp}} &= -\, R^{mp} \\
\frac{\partial\, R}{\partial g_{mp,s}} &= 0 \qquad\qquad (2.21) \\
\frac{\partial\, R}{\partial g_{mp,sr}} &= (g^{mp}\, g^{sr} - g^{mr}\, g^{ps})
\end{aligned}
$$

Having derived the main terms needed we are now ready to derive the field equations of the $f(R)$ gravity.

3. Derivation of the field equation of $f(R)$ gravity

We derive the field equations of $f(R)$ gravity in absence of external energy-momentum source by applying the Euler-Poisson equation to $f(R)$ gravity

$$\frac{\partial}{\partial g_{mp}}[(\sqrt{-g}\; f\,)] - \frac{\partial}{\partial x^{s}}[\frac{\partial}{\partial g_{mp,s}}(\sqrt{-g}\; f\,)] + \frac{\partial^{2}}{\partial x^{r}\,\partial x^{s}}[\frac{\partial}{\partial g_{mp,sr}}(\sqrt{-g}\; f)] = 0$$

$$[\sqrt{-g}\;\frac{\partial f}{\partial g_{mp}} + f\;\frac{\partial\sqrt{-g}}{\partial g_{mp}}] - \frac{\partial}{\partial x^{s}}[\sqrt{-g}\;\frac{\partial f}{\partial g_{mp,s}} + f\;\frac{\partial\sqrt{-g}}{\partial g_{mp,s}}] \qquad (3.1)$$

$$+\frac{\partial^{2}}{\partial x^{r}\,\partial x^{s}}[\sqrt{-g}\;\frac{\partial f}{\partial g_{mp,sr}} + f\;\frac{\partial\sqrt{-g}}{\partial g_{mp,sr}}] = 0$$

Make use of the following identities

$$\frac{\partial\sqrt{-g}}{\partial g_{mp,s}} = 0; \qquad \frac{\partial\sqrt{-g}}{\partial g_{mp,sr}} = 0 \quad (3.2)$$

Then Eq. (3.1) becomes

$$[\sqrt{-g}\;\frac{\partial f}{\partial g_{mp}} + f\;\frac{\partial\sqrt{-g}}{\partial g_{mp}}] - \frac{\partial}{\partial x^{s}}[\sqrt{-g}\;\frac{\partial f}{\partial g_{mp,s}}] + \frac{\partial^{2}}{\partial x^{r}\,\partial x^{s}}[\sqrt{-g}\;\frac{\partial f}{\partial g_{mp,sr}}] = 0 \quad (3.4)$$

Using the chain rule of differentiation

$$\frac{\partial f}{\partial g_{mp,s}} = \frac{df}{dR}\frac{\partial R}{\partial g_{mp,s}}, \qquad \frac{\partial f}{\partial g_{mp,sr}} = \frac{df}{dR}\frac{\partial R}{\partial g_{mp,sr}} \quad (3.5)$$

Substituting Eq. (3.5) into Eq. (3.4), we get

$$[\sqrt{-g}\;\frac{df}{dR}\frac{\partial R}{\partial g_{mp}} + f\;\frac{\partial\sqrt{-g}}{\partial g_{mp}}] - \frac{\partial}{\partial x^{s}}[\sqrt{-g}\,\frac{df}{dR}\frac{\partial R}{\partial g_{mp,s}}] + \frac{\partial^{2}}{\partial x^{r}\,\partial x^{s}}[\sqrt{-g}\,\frac{df}{dR}\frac{\partial R}{\partial g_{mp,sr}}] = 0$$

$$(3.6)$$

Since $\frac{\partial R}{\partial g_{mp,s}} = 0$ in a local coordinate system, the third term in Eq. (3.6) disappears from the equation of motion, so we are left with

$$\sqrt{-g}\;[\frac{df}{dR}(-R^{mp}) + \frac{1}{2}g^{mp}\,f\,] + \frac{\partial^{2}}{\partial x^{r}\,\partial x^{s}}[\sqrt{-g}\;\frac{df}{dR}(g^{mp}\,g^{sr} - g^{mr}\,g^{ps})] = 0$$

$$\sqrt{-g}\;[-R^{mp}\,\frac{df}{dR} + \frac{1}{2}g^{mp}\,f\,] + \frac{\partial}{\partial x^{r}}\{\frac{\partial}{\partial x^{s}}[\sqrt{-g}\;\frac{df}{dR}(g^{mp}\,g^{sr} - g^{mr}\,g^{ps})]\} = 0 \, (3.7)$$

$$\sqrt{-g}\;[-R^{mp}\,f_{R} + \frac{1}{2}g^{mp}\,f\,] + \frac{\partial}{\partial x^{r}}\{\frac{\partial}{\partial x^{s}}[\sqrt{-g}\;f_{R}\,(g^{mp}\,g^{sr} - g^{mr}\,g^{ps})]\} = 0$$

Where we have substituted

$$\frac{\partial R}{\partial g_{mp}} = -R^{mp}, \qquad \frac{\partial \sqrt{-g}}{\partial g_{mp}} = \frac{1}{2}\sqrt{-g}\, g^{mp}, \qquad f_R \equiv \frac{df}{dR} \quad (3.8)$$

Performing the first partial differentiation - the $s-$ differentiation - in the third term of Eq. (3.7) we get

$$\sqrt{-g}\ [-R^{mp} f_R + \frac{1}{2} g^{mp} f\,] + \frac{\partial}{\partial x^r}\{[\sqrt{-g}\ f_R\ (\frac{\partial}{\partial x^s}(g^{mp}\ g^{sr} - g^{mr}\ g^{ps})]$$
$$+ [\sqrt{-g}\ (\frac{\partial}{\partial x^s} f_R)\ (g^{sr}\ g^{mp} - g^{ms}\ g^{pr})] + [(\frac{\partial}{\partial x^s}\sqrt{-g})\, f_R\ (g^{mp}\ g^{sr} - g^{mr}\ g^{ps})]\} = 0$$
$$(3.9)$$

Making use of the following identities

$$\frac{\partial \sqrt{-g}}{\partial x^a} = \frac{\partial \sqrt{-g}}{\partial g_{mp}}\ \frac{\partial g_{mp}}{\partial x^a} = \frac{\partial \sqrt{-g}}{\partial g_{mp}}\ (0) = 0 \quad (3.10)$$

Since we are considering a local coordinate system in which $\frac{\partial g_{mp}}{\partial x^a} = 0$, Eq. (3.9) yields

$$\sqrt{-g}\ [-R^{mp} f_R + \frac{1}{2}\ g^{mp} f\,] + \frac{\partial}{\partial x^r}\{[\sqrt{-g}\ (\frac{\partial}{\partial x^s} f_R)\ (g^{mp}\ g^{sr} - g^{mr}\ g^{ps})]\} = 0$$
$$(3.11)$$

Writing partial derivative with respect to space-time coordinates of the derivative of $f\,(R)$ with respect to Ricci scalar explicitly as

$$\frac{\partial}{\partial x^s} f_R = \frac{d}{dR}(f_R)\frac{\partial}{\partial x^s}R = \frac{d}{dR}(\frac{df}{dR})\frac{\partial}{\partial x^s}R = (\frac{d^2 f}{dR^2})\ R_{,s} = f_{RR}\ R_{,s} \quad (3.12)$$

Substituting Eq. (3.12) into Eq. (3.11) yields

$$\sqrt{-g}\ [-R^{mp} f_R + \frac{1}{2}\ g^{mp} f\,] + \frac{\partial}{\partial x^r}\{[\sqrt{-g}\ (f_{RR}\ R_{,s})\ (g^{mp}\ g^{sr} - g^{mr}\ g^{ps})]\} = 0$$
$$(3.13)$$

Performing the $r-$partial derivative in Eq. (3.13) becomes

$$\sqrt{-g}\ [-R^{mp} f_R + \frac{1}{2} g^{mp} f\,] + \{\sqrt{-g}\ (f_{RR}\ R_{,s})\ \frac{\partial}{\partial x^r}(g^{mp}\ g^{sr} - g^{mr}\ g^{ps})]$$
$$+\{[\sqrt{-g}\ \frac{\partial}{\partial x^r}(f_{RR}\ R_{,s})\ (g^{sr}\ g^{mp} - g^{ms}\ g^{pr})] \qquad (3.14)$$
$$+[(\frac{\partial}{\partial x^r}\sqrt{-g})\ (f_{RR}\ R_{,s})\ (g^{mp}\ g^{sr} - g^{mr}\ g^{ps})] = 0$$

Substituting the identities in Eq. (3.10) into Eq. (3.14), we get

$$\sqrt{-g}\ [-R^{mp} f_R + \frac{1}{2} g^{mp} f\,] + [\sqrt{-g}\ (g^{mp}\ g^{sr} - g^{mr}\ g^{ps})\ \frac{\partial}{\partial x^r}(f_{RR}\ R_{,s})] = 0 \quad (3.15)$$

Using the Leibniz rule of differentiation

$$\begin{aligned}\frac{\partial}{\partial x^r}\ (f_{RR}\ R_{,s}) &= f_{RR}\ \frac{\partial}{\partial x^r}(R_{,s}) + R_{,s}\ \frac{\partial}{\partial x^r}(f_{RR})\\ &= f_{RR}\ R_{,s,r} + R_{,s}\ \frac{d}{dR}(f_{RR})\ \frac{\partial R}{\partial x^r}\\ &= f_{RR}\ R_{,s,r} + R_{,s}\ \frac{d}{dR}(\frac{d^2 f}{dR^2})\ \frac{\partial R}{\partial x^r} \quad (3.16)\\ &= f_{RR}\ R_{,s,r} + R_{,s}\ (\frac{d^3 f}{dR^3})\ R_{,r}\\ &= f_{RR}\ R_{,s,r} + R_{,s}\ R_{,r}\, f_{RRR}\end{aligned}$$

Substituting Eq. (3.16) into Eq. (3.15), we get

$$\sqrt{-g}\ [-R^{mp} f_R + \frac{1}{2}\ g^{mp} f\,] + [\sqrt{-g}\ (g^{mp}\ g^{sr} - g^{mr}\ g^{ps})\ (f_{RR}\ R_{,s,r} + R_{,s}\ R_{,r}\, f_{RRR})] = 0$$
$$(3.17)$$

Multiplying the two brackets in the second term in Eq. (3.17), we arrive at

$$\sqrt{-g}\ \{[-R^{mp} f_R + \frac{1}{2} g^{mp} f] + [(f_{RR} R_{,s,r} g^{mp} g^{sr} - f_{RR} R_{,s,r} g^{mr} g^{ps}) + (R_{,s} R_{,r} f_{RRR} g^{mp} g^{sr} - R_{,s} R_{,r} f_{RRR} g^{mr} g^{ps})]\} = 0$$

(3.18)

$$\sqrt{-g}\ \{[-R^{mp} f_R + \frac{1}{2} g^{mp} f] + [(f_{RR} R^{,r}{}_{,r} g^{mp} - f_{RR} R^{,m,p}) + (R^{,r} R_{,r} f_{RRR} g^{mp} - R^{,m} R^{,p} f_{RRR})]\} = 0$$

Since $\sqrt{-g} \neq 0$, the form of the field equations of $f(R)$ gravity in a local coordinate system in absence of external energy-momentum source would be

$$[-R^{mp} f_R + \frac{1}{2} g^{mp} f] + [(f_{RR} R^{,r}{}_{,r} g^{mp} - f_{RR} R^{,m,p}) + (R^{,r} R_{,r} f_{RRR} g^{mp} - R^{,m} R^{,p} f_{RRR})] = 0$$

(3.19)

Rearranging in the descending order of the derivative of $f(R)$ with respect to Ricci scalar R yields our completed derivation of the field equations of the $f(R)$ absence of external energy-momentum source as

$$(g^{mp} R^{,r} R_{,r} - R^{,m} R^{,p}) f_{RRR} + (g^{mp} R^{,r}{}_{,r} - R^{,m,p}) f_{RR} - R^{mp} f_R + \frac{1}{2} g^{mp} f = 0$$

(3.20)

Using the principle of general covariance Eq. (3.20) can be made to be valid in any general coordinate system by replacing the partial derivatives with covariant derivatives; in simple language the commas $(,)$ in the Eq. (3.20) will be replaced by semicolon $(;)$, i.e. $\left((,) \Rightarrow (;)\right)$.

So, our equation of motion of the $f(R)$ in the covariant form in absence of external energy-momentum source may be written as

$$(g^{mp} R^{;r} R_{;r} - R^{;m} R^{;p}) f_{RRR} + (g^{mp} R^{;r}{}_{;r} - R^{;m;p}) f_{RR} - R^{mp} f_R + \frac{1}{2} g^{mp} f = 0$$

(3.21)

Which is exactly the same as the Eq. (1.1) derived by H. A. Buchdahl.

It may clearly be seen that it requires both the metric tensor components and the form of the function $f(R)$ as an explicit function of the Ricci scalar R as its basic ingredients for it applications.

4. Trace of the field equation of the $f(R)$ gravity

Contracting Eq. (3.21) (i.e. multiplying by g_{mp} and sum) and notice that in a $4-$dimensions space-time and for the diagonal metric tensor g_{mp}

$$g^{mp}\, g_{mp} \equiv \sum_{m=0}^{3}\ \sum_{p=0}^{3} g^{mp}\, g_{mp} = 4 \quad (4.1)$$

Yields

$$g_{mp}\ \{(g^{mp}\, R^{;r}\, R_{;r} - R^{;m}\, R^{;p})\, f_{RRR} + (g^{mp}\, R^{;r}{}_{;r} - R^{;m;p})\, f_{RR} - R^{mp}\, f_R + \frac{1}{2}\ g^{mp}\, f\} = 0$$

$$(g_{mp}\, g^{mp}\, R^{;r}\, R_{;r} - g_{mp}\, R^{;m}\, R^{;p})\, f_{RRR} + (g_{mp}\, g^{mp}\, R^{;r}{}_{;r} - g_{mp}\, R^{;m;p})\, f_{RR} - g_{mp}\, R^{mp}\, f_R$$

$$+\frac{1}{2}\ g_{mp}\, g^{mp}\, f = 0$$

$$(4\ R^{;r}\, R_{;r} - R_{;p}\, R^{;p})\, f_{RRR} + (4\ R^{;r}{}_{;r} - R_{;p}{}^{;p})\, f_{RR} - R^{m}{}_{m}\, f_R + \frac{1}{2}\ (4)\, f = 0$$

$$(4.2)$$

Where we have made use of the identities

$$g^{mp}\, g_{mp} = 4, \qquad g_{mp}\, R^{;m} = R_{;p}, \qquad g_{mp}\, R^{mp} = R^{m}{}_{m} = R \quad (4.3)$$

Performing the arithmetic operation having in mind that $r,\ p$ and m are dummy indices (each ranges from 0 to 3), we get

$$\begin{aligned}
&(4\ R^{;r}\, R_{;r} - R_{;p}\, R^{;p})\, f_{RRR} + (4\ R^{;r}{}_{;r} - R_{;p}{}^{;p})\, f_{RR} - R^{m}{}_{m}\, f_R + \frac{1}{2}\ (4)\, f = 0 \\
&(4\ R^{;p}\, R_{;p} - R_{;p}\, R^{;p})\, f_{RRR} + (4\ R^{;p}{}_{;p} - R_{;p}{}^{;p})\, f_{RR} - R\, f_R + \frac{1}{2}\ (4)\, f = 0 \qquad (4.4) \\
&(3\ R^{;p}\, R_{;p})\, f_{RRR} + (3\ R^{;p}{}_{;p})\, f_{RR} - R\, f_R + 2\, f = 0 \\
&3\ (R^{;p}\, R_{;p})\, f_{RRR} + 3\ (R^{;p}{}_{;p})\, f_{RR} - R\, f_R + 2\, f = 0
\end{aligned}$$

Then the contracted field equations of $f(R)$ in absence of external energy-momentum source is

$$3\ (R^{;p}\, R_{;p})\, f_{RRR} + 3\ (R^{;p}{}_{;p})\, f_{RR} - R\, f_R + 2\, f = 0 \quad (4.5)$$

It is important to notice that in our case in which $T_{ab} = 0$ (i.e. in absence of external energy-momentum source) no longer implies $R = 0$ or even a constant Ricci curvature denoted by ($R = R_0$) as in contracted Einstein field equations.
The tensor form of the field equations of $f(R)$ gravity in Eq. (3.21) is a set of four equations -since we are working in space-time of four dimensions - we may write these equations explicitly as

$$[g^{00}(R^{;0}R_{;0} + R^{;1}R_{;1} + R^{;2}R_{;2} + R^{;3}R_{;3}) - R^{;0}R^{;0}]f_{RRR}$$
$$+[g^{00}(R^{;0}{}_{;0} + R^{;1}{}_{;1} + R^{;2}{}_{;2} + R^{;3}{}_{;3}) - R^{;0;0}]f_{RR} - R^{00}f_R + \frac{1}{2}g^{00}f = 0$$

$$[g^{11}(R^{;0}R_{;0} + R^{;1}R_{;1} + R^{;2}R_{;2} + R^{;3}R_{;3}) - R^{;1}R^{;1}]f_{RRR}$$
$$+[g^{11}(R^{;0}{}_{;0} + R^{;1}{}_{;1} + R^{;2}{}_{;2} + R^{;3}{}_{;3}) - R^{;1;1}]f_{RR} - R^{11}f_R + \frac{1}{2}g^{11}f = 0$$

$$[g^{22}(R^{;0}R_{;0} + R^{;1}R_{;1} + R^{;2}R_{;2} + R^{;3}R_{;3}) - R^{;2}R^{;2}]f_{RRR}$$
$$+[g^{22}(R^{;0}{}_{;0} + R^{;1}{}_{;1} + R^{;2}{}_{;2} + R^{;3}{}_{;3}) - R^{;2;2}]f_{RR} - R^{22}f_R + \frac{1}{2}g^{22}f = 0$$

$$[g^{33}(R^{;0}R_{;0} + R^{;1}R_{;1} + R^{;2}R_{;2} + R^{;3}R_{;3}) - R^{;3}R^{;3}]f_{RRR}$$
$$+[g^{33}(R^{;0}{}_{;0} + R^{;1}{}_{;1} + R^{;2}{}_{;2} + R^{;3}{}_{;3}) - R^{;3;3}]f_{RR} - R^{33}f_R + \frac{1}{2}g^{33}f = 0 \quad (4.6)$$

Since

$$\begin{aligned} g^{00}(R^{;0}R_{;0}) &= R^{;0}g^{00}R_{;0} = R^{;0}R^{;0} \\ g^{11}(R^{;1}R_{;1}) &= R^{;1}g^{11}R_{;1} = R^{;1}R^{;1} \\ g^{22}(R^{;2}R_{;2}) &= R^{;2}g^{22}R_{;2} = R^{;2}R^{;2} \\ g^{33}(R^{;3}R_{;3}) &= R^{;3}g^{33}R_{;3} = R^{;3}R^{;3} \end{aligned} \quad (4.7)$$

And

$$\begin{aligned} g^{00}(R^{;0}{}_{;0}) &= R^{;0;0} \\ g^{11}(R^{;1}{}_{;1}) &= R^{;1;1} \\ g^{22}(R^{;2}{}_{;2}) &= R^{;2;2} \\ g^{33}(R^{;3}{}_{;3}) &= R^{;3;3} \end{aligned} \quad (4.8)$$

The Eq. (4.6) becomes

$$
\begin{aligned}
&[g^{00}\,(R^{;1}R_{;1}+R^{;2}R_{;2}+R^{;3}R_{;3})]\,f_{RRR}+[g^{00}\,(R^{;1}{}_{;1}+R^{;2}{}_{;2}+R^{;3}{}_{;3})]\,f_{RR}-R^{00}f_R\\
&+\frac{1}{2}\,g^{00}f=0\\
&[g^{11}\,(R^{;0}R_{;0}+R^{;2}R_{;2}+R^{;3}R_{;3})]\,f_{RRR}+[g^{11}\,(R^{;0}{}_{;0}+R^{;2}{}_{;2}+R^{;3}{}_{;3})]\,f_{RR}-R^{11}f_R\\
&+\frac{1}{2}\,g^{11}f=0\\
&[g^{22}\,(R^{;0}R_{;0}+R^{;1}R_{;1}+R^{;3}R_{;3})]\,f_{RRR}+[g^{22}\,(R^{;0}{}_{;0}+R^{;1}{}_{;1}+R^{;3}{}_{;3})]\,f_{RR}-R^{22}f_R\\
&+\frac{1}{2}\,g^{22}f=0\\
&[g^{33}\,(R^{;0}R_{;0}+R^{;1}R_{;1}+R^{;2}R_{;2})]\,f_{RRR}+[g^{33}\,(R^{;0}{}_{;0}+R^{;1}{}_{;1}+R^{;2}{}_{;2})]\,f_{RR}-R^{33}f_R\\
&+\frac{1}{2}\,g^{33}f=0
\end{aligned}
\tag{4.9}
$$

In addition, the contacted form in Eq. (4.5) becomes

$$
\begin{aligned}
&3\,(R^{;0}R_{;0}+R^{;1}R_{;1}+R^{;2}R_{;2}+R^{;3}R_{;3})\,f_{RRR}+3\,(R^{;0}{}_{;0}+R^{;1}{}_{;1}+R^{;2}{}_{;2}+R^{;3}{}_{;3})\,f_{RR}\\
&-R\,f_R+2\,f=0
\end{aligned}
\tag{4.10}
$$

In some gravitational models, the metric tensor is a function of one space-time coordinates such that the Ricci scalar is a function only of the derivatives of the metric tensor with respect to that space-time coordinate. As examples, the time independent spherically symmetric metric [4-7], and the FLWR metric in Cosmology [4-7], are functions of one space-time coordinate only - $r-$coordinate in the former and $t-$ coordinate in the latter- so that there is only one surviving term in the brackets containing the derivatives of the Ricci scalar with respect to the space-time coordinate in Eq. (4.9) and (4.10).

5. Conclusion

Derivation of the field equations of the $f(R)$ gravity in absence of energy-momentum directly from Euler-Poisson equation by assuming the vanishing of the metric tensor and its first derivative at the boundary is straightforward and enlightening. The resulting field equations are the same with all its predictions as those derived by H. A. Buchdahl in absence of energy-momentum tensor.

6. References

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