Numbers:Part 4,"Pi Representations"

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ABSTRACT. We give some representations of Pi.

keywords: number Pi , arctangents sums , continued fractions.

I. Master Formula

$$\pi = 2\,x + 2\tan^{-1}\left(\frac{1}{\tan(x)}\right) \tag{1}$$

Example:

$$\pi = 3 + 2\tan^{-1}\left(\frac{1}{\tan(3/2)}\right) \tag{2}$$

formula (2) is equivalent to

$$\pi = 3 + 2\tan^{-1}\left(\frac{2}{3} - \frac{3}{6-}\,\frac{3^2}{10-}\,\frac{3^2}{14-}\,\frac{3^2}{18-}\,...\right) \tag{3}$$

$$\pi = 3 + 2\tan^{-1}\left(\frac{1}{14+}\,\frac{1}{9+}\,\frac{1}{1+}\,\frac{1}{6+}\,\frac{1}{7+}\,\frac{1}{59+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2+}\,...\right) \tag{4}$$

This note presents some representations of Pi:

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2\,n+1} = 3.1415926535\;... \tag{5}$$

II. Some Representations of Pi

Entry 1.

$$\pi = \frac{22}{7} + 2\tan^{-1}\left(\cot\left(\frac{11}{7}\right)\right) \tag{6}$$

$$\pi = \frac{22}{7} - 2\tan^{-1}\left(\frac{1}{1581+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{176+}\,\frac{1}{1+}\,\frac{1}{19+}\,...\right) \tag{7}$$

$$\pi = \frac{333}{106} + 2\tan^{-1}\left(\cot\left(\frac{333}{212}\right)\right) \tag{8}$$

$$\pi = \frac{333}{106} + 2\tan^{-1}\left(\frac{1}{24\,032+}\;\frac{1}{1+}\;\frac{1}{3+}\;\frac{1}{1+}\;\frac{1}{4+}\;\frac{1}{4+}\;\frac{1}{1+}\;\ldots\right) \tag{9}$$

$$\pi = \frac{355}{113} + 2\tan^{-1}\left(\cot\left(\frac{355}{226}\right)\right) \tag{10}$$

$$\pi = \frac{355}{113} - 2\tan^{-1}\left(\frac{1}{7\,497\,258+}\;\frac{1}{5+}\;\frac{1}{2+}\;\frac{1}{1+}\;\frac{1}{1+}\;\frac{1}{9+}\;\frac{1}{5+}\;\ldots\right) \tag{11}$$

$$\pi = \frac{103\,993}{33\,102} + 2\tan^{-1}\left(\cot\left(\frac{103\,993}{66\,204}\right)\right) \tag{12}$$

$$\pi = \frac{103\,993}{33\,102} + 2\tan^{-1}\left(\frac{1}{3\,460\,862\,455+}\;\frac{1}{1+}\;\frac{1}{1+}\;\frac{1}{2+}\;\frac{1}{4+}\;\frac{1}{1+}\;\frac{1}{2+}\;\ldots\right) \tag{13}$$

Entry 2.

$$\pi = 3 + 2\tan^{-1}\left(\frac{1}{15}\right) + 2\tan^{-1}\left(\frac{15 - \tan(3/2)}{1 + 15\tan(3/2)}\right) \tag{14}$$

$$\pi = 3 + 2\tan^{-1}\left(\frac{1}{15}\right) + 2\tan^{-1}\left(\frac{1}{236+}\;\frac{1}{1+}\;\frac{1}{1+}\;\frac{1}{31+}\;\frac{1}{15+}\;\frac{1}{5+}\;\frac{1}{3+}\;\ldots\right) \tag{15}$$

$$\pi = 3 + 2\tan^{-1}\left(\frac{1}{14}\right) - 2\tan^{-1}\left(\frac{\tan(3/2) - 14}{1 + 14\tan(3/2)}\right) \tag{16}$$

$$\pi = 3 + 2\tan^{-1}\left(\frac{1}{14}\right) - 2\tan^{-1}\left(\frac{1}{1956+}\;\frac{1}{2+}\;\frac{1}{2+}\;\frac{1}{1+}\;\frac{1}{1+}\;\frac{1}{2+}\;\frac{1}{1+}\;\ldots\right) \tag{17}$$

Entry 3.

$$\pi = 3 + 2\tan^{-1}\left(\frac{1}{15}\right) + 2\tan^{-1}\left(\frac{1}{237}\right) + 2\tan^{-1}\left(\frac{3554 - 252\tan(3/2)}{252 + 3554\tan(3/2)}\right) \tag{18}$$

$$\pi = 3 + 2\tan^{-1}\left(\frac{1}{15}\right) + 2\tan^{-1}\left(\frac{1}{237}\right) + 2\tan^{-1}\left(\frac{1}{113\,911+}\;\frac{1}{10+}\;\frac{1}{2+}\;\frac{1}{6+}\;\frac{1}{72+}\;\ldots\right) \tag{19}$$

$$\pi = 3 + 2\tan^{-1}\left(\frac{1}{15}\right) + 2\tan^{-1}\left(\frac{1}{236}\right) - 2\tan^{-1}\left(\frac{251\tan(3/2) - 3539}{251 + 3539\tan(3/2)}\right) \tag{20}$$

$$\pi = 3 + 2\tan^{-1}\left(\frac{1}{15}\right) + 2\tan^{-1}\left(\frac{1}{236}\right) - 2\tan^{-1}\left(\frac{1}{109\,893+}\;\frac{1}{23+}\;\frac{1}{1+}\;\frac{1}{1+}\;\frac{1}{1+}\;\ldots\right) \tag{21}$$

Entry 4.

$$\pi = 3 + 2\tan^{-1}\left(\frac{1}{14}\right) - 2\tan^{-1}\left(\frac{1}{1956}\right) + 2\tan^{-1}\left(\frac{27\,385 - 1942\tan(3/2)}{1942 + 27\,385\tan(3/2)}\right) \tag{22}$$

$$\pi = 3 + 2\tan^{-1}\left(\frac{1}{14}\right) - 2\tan^{-1}\left(\frac{1}{1956}\right) + 2\tan^{-1}\left(\frac{1}{9\,140\,030+}\;\frac{1}{2+}\;\frac{1}{1+}\;\frac{1}{2+}\;\frac{1}{3+}\;\ldots\right) \tag{23}$$

$$\pi = 3 + 2\tan^{-1}\left(\frac{1}{14}\right) - 2\tan^{-1}\left(\frac{1}{1957}\right) - 2\tan^{-1}\left(\frac{1943\tan(3/2) - 27\,399}{1943 + 27\,399\tan(3/2)}\right) \tag{24}$$

$$\pi = 3 + 2\tan^{-1}\left(\frac{1}{14}\right) - 2\tan^{-1}\left(\frac{1}{1957}\right) - 2\tan^{-1}\left(\frac{1}{6\,586\,248+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{1+}\cdots\right) \tag{25}$$

Entry 5.

$$\pi = \sqrt{10} + 2\tan^{-1}\left(\cot\left(\sqrt{\frac{5}{2}}\right)\right) \tag{26}$$

$$\pi = \sqrt{10} - 2\tan^{-1}\left(\frac{1}{96+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{5+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\cdots\right) \tag{27}$$

Entry 6.

$$\pi = 3.14 + 2\tan^{-1}(\cot(1.57)) \tag{28}$$

$$\pi = 3.14 + 2\tan^{-1}\left(\frac{1}{1255+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{7+}\cdots\right) \tag{29}$$

$$\pi = 3.141592 + 2\tan^{-1}(\cot(1.570796)) \tag{30}$$

$$\pi = 3.141592 + 2\tan^{-1}\left(\frac{1}{3\,060\,023+}\,\frac{1}{3+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{14+}\,\frac{1}{1+}\,\frac{1}{2+}\cdots\right) \tag{31}$$

Entry 7.

$$\begin{aligned}\pi = 3 &+ 2\tan^{-1}\left(\frac{1}{15}\right) + 2\tan^{-1}\left(\frac{1}{237}\right) + 2\tan^{-1}\left(\frac{1}{113\,912}\right) + 2\tan^{-1}\left(\frac{1}{14\,347\,259\,163}\right) + \\ &2\tan^{-1}\left(\frac{1}{276\,584\,715\,711\,988\,129\,216}\right) + \ldots + 2\tan^{-1}\left(\frac{1}{n_k}\right) + \ldots\end{aligned} \tag{32}$$

where

$$n_1 = 15\,,\;\; n_{k+1} = 1 + \operatorname{floor}\left(\left(\tan^{-1}\left(\cot\left(\frac{3}{2}\right)\right) - \sum_{m=1}^{k}\tan^{-1}\left(\frac{1}{n_m}\right)\right)^{-1}\right) \tag{33}$$

remark: floor(x) is the floor function.

III. References

1. Andrews, G.E.: Number Theory. Dover, New York, 1994.
2. Arndt, J., and Haenel, C.: π unleashed. Springer-Verlag, 2001.
3. Beckmann, P.: A History of π. St Martin's Press, New York, 1971.