The Equation: $\psi(q) = 2$

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abstract

In this note we give $q = 0.645 \ldots$ such that: $\psi(q) = 2$, where $\psi(q)$ is the Ramanujan's theta function.

keywords: number Pi, Ramanujan's theta function, integral of Ramanujan's.

I. Introduction

The Ramanujan's $\psi(q)$ theta function is defined by

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty} \tag{1}$$

where

$$(a; q)_\infty = \prod_{n=0}^{\infty} (1 - a q^n) , \; |q| < 1 \tag{2}$$

II. The Equation $\psi(q) = 2$

$$\psi(q) = 2 \Longrightarrow q = 0.645222703236020979134251 66 \ldots \tag{3}$$

Iteration 1:

$$r_{k+1} = 1 - \frac{1 - r_k^2}{2} \prod_{n=1}^{\infty} \frac{1 - r_k^{2n+2}}{1 - r_k^{2n+1}} = 1 - \frac{1 - r_k}{2} \psi(r_k) , \; k = 1, 2, 3, \ldots \tag{4}$$

$$r_1 = \frac{1}{2} , \; r_k \longrightarrow q = 0.64522 \ldots \tag{5}$$

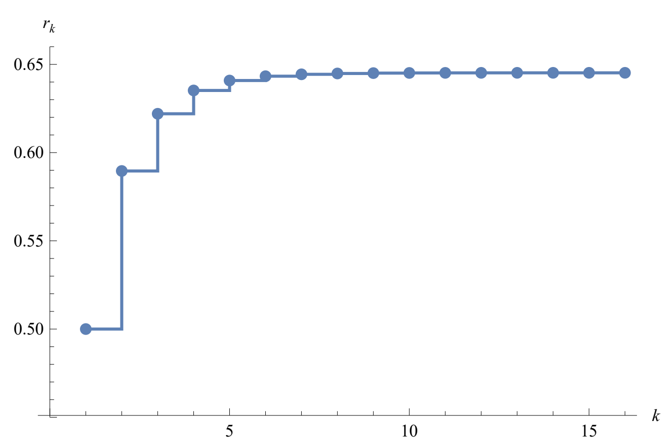

Figure 1.

Iteration 2:

$$r_{k+1} = r_k + \frac{1}{3}(2 - \psi(r_k)) = r_k + \frac{1}{3}\left(1 - \sum_{n=1}^{\infty} r_k^{n(n+1)/2}\right), \quad k = 1, 2, 3, \ldots \tag{6}$$

$$r_1 = \frac{1}{2}, \quad r_k \longrightarrow q = 0.64522 \ldots \tag{7}$$

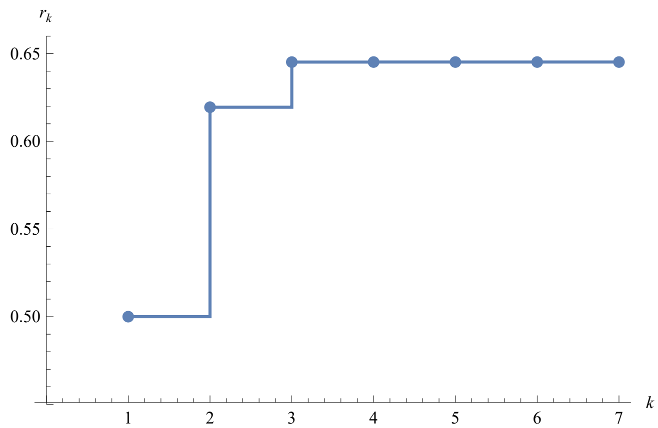

Figure 2.

III. Integral

If $q = 0.64522 \ldots$, then

$$\frac{\pi}{4} = \int_0^{\infty} \frac{1}{(1 + x^2)(1 + q^2 x^2)(1 + q^4 x^2)(1 + q^6 x^2) \ldots} \, dx \tag{8}$$

References

A. S. Ramanujan: Notebooks (2 Volumes) , Tata Institute of Fundamental Research, Bombay, 1957.

B. S. Ramanujan: The Lost notebook and other unpublished papers, Narosa, New Delhi , 1988.