On the Normalizer of a Proper Subgroup of a Group of Prime Power Order

Henry Wong

Abstract

In this paper we lay out the proof of this result in group theory using only elementary facts in group theory.

Definition Let G be a group, p be a prime number, and $Z(G)$ be the center of G.

Theorem 1 If $|G| = p^n$ and $H \neq G$ is a subgroup of G, then G has a subgroup of order $p|H|$ that contains H.
Proof.
Use induction on n. Suppose that the result is correct for $n-1$. Let G be a group of order p^n and $H \neq G$ be a subgroup of G. By Lagrange's theorem, $|Z(G)| = p^k$ for some integer $0 \leqq k \leqq n$. Since $Z(G) \neq (e)$, p divides $|Z(G)|$ and so $Z(G)$ has an element a of order p. Let N be the subgroup of G generated by a. Then N is of order p. Since $a \in Z(G)$, N must be normal in G. Moreover, $|N \cap H|$ divides $|N|$. So $|N \cap H|$ divides p. Thus $|N \cap H| = 1$ or p. Suppose $|N \cap H| = 1$. Then

$$|NH| = \frac{|N||H|}{|N \cap H|} = p|H|.$$

So NH is a subgroup of G of order $p|H|$ that contains H. Now suppose $|N \cap H| = p$. Since $N \cap H \subset N$ and $|N \cap H| = |N|$, it follows that $N \cap H = N$ and hence $N \subset H$. Since $H \neq G$, there is an $x \in G$, $x \notin H$. Clearly $xN \in G/N$. Suppose $xN \in H/N$. Then $xN = hN$ for some $h \in H$. Since $x \in xN$ and $xN \subset hN$, so $x \in hN$. Hence $x = hn$ for some $n \in N$. Since $h \in H$ and $N \subset H$, so $x \in H$, a contradiction. To conclude $xN \notin H/N$ and thus $H/N \neq G/N$. Since G/N is a group of order p^{n-1} and $H/N \neq G/N$ is a subgroup of G/N, by the induction hypothesis, G/N has a subgroup $\bar{P}$ of order $p|H/N| = |H|$ that contains H/N. Let $P = \{x \in G \mid xN \in \bar{P}\}$. Thus P is a subgroup of G and $\bar{P} \cong P/N$. As the result of

$$|H| = |\bar{P}| = \frac{|P|}{|N|} = \frac{|P|}{p},$$

so $|P| = p|H|$. Let $h \in H$. Then $hN \in H/N$. Moreover, $H/N \subset \bar{P}$. To conclude $hN \in \bar{P}$ and thus $h \in P$. As the result, $H \subset P$.

Theorem 2 Any subgroup of order p^{n-1} in a group G of order p^n is normal in G.

Theorem 3 If $|G| = p^n$ and $H \neq G$ is a subgroup of G, then there exists an $x \in G$, $x \notin H$ such that $x^{-1}Hx = H$.
Proof.
By Theorem 1, G has a subgroup K of order $p|H|$ that contains H. By Lagrange's theorem, $|H| = p^i$ for some integer $0 \leqq i \leqq n-1$. So $|K| = p^{i+1}$ and hence H is normal in K by Theorem 2. Since $|K - H| = |K| - |H| = p^{i+1} - p^i > 0$, there is an $x \in K$, $x \notin H$. Since H is normal in K,

$$x^{-1}Hx = x^{-1}H(x^{-1})^{-1} = H.$$

Finally $x \in K \subset G$ as required.

References
[1] I. N. Herstein, *Abstract Algebra*, Macmillan Publishing Company, New York, 1990.
[2] I. M. Isaacs, *Algebra: A Graduate Course*, American Mathematical Society, Providence, 1994.
[3] J. J. Rotman, *An Introduction to the Theory of Groups*, Springer-Verlag, New York, 1995.