Braid logic for mass condensation

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Abstract

In quantum logic, the emergence of spacetime and related symmetries goes hand in hand with the emergence of the real and complex numbers themselves. In this paper, we show how finite fields are surprisingly sufficient for most physical questions, once we throw away classical geometrical models in favour of categorical axioms. In particular, generalised Pauli matrix algebras are closely related to braid and ribbon diagrams, and holographic information for mass localisation gains its intuition from algebras for anyon condensation. We discuss definitions of homology and cohomology associated to braids, recalling the twistor construction of massive solutions in H^2.

Contents

1 Introduction

Historically, quantum field theory involved a great deal of real or complex analysis. But the quantities of physical interest are often captured by universality, and the abstract axioms themselves. Braided categories and other higher dimensional structures are an algebraic foundation for condensed matter physics, quantum computation and quantum gravity. In motivic quantum gravity, we turn the old story around, aiming to derive even complex geometry itself from a fundamental set of axioms for quantum logic.

Classical logic is governed by the tensor category of sets, which is a topos [1]. In quantum mechanics, the cardinality of a finite set is replaced with the dimension of a Hilbert space [2]. We interpret dimension categorically, insisting that the correct category of Hilbert spaces is infinite dimensional, although symmetric monoidal categories [3][4] suffice for most purposes. Considering gravity however, a non trivial braiding is required for chiral particle states in the Standard Model, breaking time reversal symmetry, and the question then is: how is rest mass localised by quantum information in the vacuum? We employ the inverse Higgs see-saw [5][6], which pairs a cosmological IR neutrino scale with a UV scale using a principle of quantum inertia [7][8], and identify right handed neutrinos with CMB photons [9][10][11][12].

From this perspective, the holographic principle is about boundary states for topological systems, as in condensed matter physics. Electric and magnetic charges can form dyon states [13], extending Levin-Wen type models [14][15] using geometric duality in the categorical axioms. Since our Standard Model charges consist of anyons, we consider the categorical structure of topological phases for anyon condensation [16][17][18][19].

The main difficulty faced in applications of fusion categories is the recovery of concrete physical data beyond that provided by structural parameters. In quantum gravity, this problem is exacerbated to the point that we question even the use of the real number system as a basis for geometry. We imagine generating classical geometry itself from quantised spaces, rather than quantising a classical theory [2]. In this paper we focus on the discrete information content of matter-spacetime, which attaches numbers directly to categorical geometry.

Initially, operators for spacetime are closely related to finite fields $\mathbb{F}_q$ for $q = p^r$ a prime power, through Schwinger's theory of mutually unbiased bases [20][21][22] for quantum measurement. In particular, multiplication for the finite field $\mathbb{F}_9$ is given by a set of three unitary matrices, as follows. Given the Pauli matrices

$$I = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J = i\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad K = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1}$$

their three basis sets of normed eigenvectors are

$$F_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad R_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \quad 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2}$$

and we have $R_2^8 = 1$ for the cyclic group generator. Taking instead $R = e^{-\pi i/4} R_2$, we have $R^4 = 1$ as a representation for $\mathbb{F}_5$. For ω the primitive cubed root of unity $\exp(2\pi i/3)$, the qutrit analog (for $\mathbb{F}_{13}$) uses the four bases

$$F_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \overline{\omega} \\ 1 & \overline{\omega} & \omega \end{pmatrix}, \quad R_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & \omega & 1 \\ 1 & 1 & \omega \\ \omega & 1 & 1 \end{pmatrix}, \quad R_3^{-1} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & \overline{\omega} \\ \overline{\omega} & 1 & 1 \\ 1 & \overline{\omega} & 1 \end{pmatrix} \tag{3}$$

and 1, where now $R_3^{12} = 1$. Taking $e^{\pi i/6} R_3$, we can work in $\mathbb{F}_4$. Thus the matrices R_2 and R_3 carry the redundant phases of $\pi/4$ and $\pi/6$, the basic arithmetic phases for the modular group [5]. A tensor product in six dimensions introduces the neutrino phase $\pi/12$ (with $R_2 \otimes R_3$ now associated to $\mathbb{F}_{25}$). The matrix

$$I + J + K = i \begin{pmatrix} 1 & \sqrt{2}e^{\pi i/4} \\ \sqrt{2}e^{-\pi i/4} & -1 \end{pmatrix}, \tag{4}$$

up to sign variations, is often useful.

We work over integral rings. The 3-vector columns of both the Fourier transform F_3 and 1 form the hexacode H_6 [23] over $\mathbb{F}_4 = \{0, 1, \omega, \overline{\omega}\}$, which is important in the construction of the 24 dimensional Leech lattice [24]. The 24 bit Golay code may be defined in terms of H_6 using vectors of the form $x_1 + \omega x_2 + \overline{\omega} x_3$ for each x_i a binary 6-vector. Here are the ingredients of the classification of finite simple groups [24]. Instead of taking a commutative space over a finite field, the mutually unbiased operators have in some sense quantised the field. When the field is of type $\mathbb{F}_{q^2}$, like $\mathbb{F}_9$, there is a Frobenius automorphism $x \mapsto x^q$ of order 2, generalising complex conjugation. A norm of the form $x\overline{x}$ in an algebra with conjugation exists for the integers in any quadratic number field [25].

The next section explains the connection between algebraic units and braid and ribbon diagrams, and introduces quandles, which are a natural route to the cohomology defined in section 3. Sections 4 and 5 look at the deeper categorical structure behind rest mass using the concepts of anyon condensation and quantum inertia, and section 6 then summarises this information in the Standard Model.

2 Algebraic structure of spacetime

The Pauli matrices of (1) satisfy $I^2 = J^2 = K^2 = -1$. As is well known, a Minkowski space vector (t, x, y, z) is represented by a complex quaternion of the form

$$q = t + xiI + yiJ + ziK, \tag{5}$$

where $i^2 = -1$ comes from another copy of $\mathbb{C}$. In fact, there will be two related copies of the quaternions in the braids that we use.

In particle physics we replace the coordinates of (5) with a spinor pair in $\mathbb{C}^4$, up to scalings giving the projective twistor space $\mathbb{CP}^3$. The spinor pair naturally identifies the helicities of massless states, and solutions to the Dirac equation are cohomology. For us, everything is motivated by categorical axioms, and cohomological structures are more fundamental than spacetime itself, which we build point by point starting from integral multiples of I, J and K.

A representation of the braid group [26] B_3 on three strands is given [27] by

$$\sigma_1 = \frac{1}{\sqrt{2}}(1 + I), \quad \sigma_2 = \frac{1}{\sqrt{2}}(1 + J), \tag{6}$$

so that $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$. The unit K then appears in the relation

$$\frac{1}{\sqrt{2}}(1 + K) = \sigma_1\sigma_2\sigma_1^{-1}, \tag{7}$$

where σ_1^{-1} uses a minus sign. Here group conjugation $\sigma_1\sigma_2\sigma_1^{-1}$ is a quandle (or rack) [28] product (see figure 3) on a distinct set of diagrams with strings labeled by braid elements, but in our copy of B_3 it denotes a generator which crosses the first and third strands. Quandle conjugation for any group is written

$$A \circ B \equiv ABA^{-1}, \tag{8}$$

but more general quandle products label arc segments at a crossing in a knot diagram. Note that $A \circ A = A$ defines idempotents.

Since a crossing uses up to three separate arcs, the general product is of the form $A \circ B = C$, where C is the under arc coming out of the crossing when lines are directed. In particular, a trefoil knot with three arcs A, B and C is given by the union of the quandle rules

$$A \circ B = C, \quad B \circ C = A, \quad C \circ A = B. \tag{9}$$

Once again, these rules are clearly represented by the Pauli matrices, and the conjugation quandle and trefoil quandles are related by a kind of triality $(A, B, C) \mapsto (B, C, A)$ on the product. A map from the standard $-\sigma_1^{-1}$ to the trefoil is shown in figure 1. Observe that $I = \sigma_1^2$ and $J = \sigma_2^2$ are full twists in B_3. Thus Pauli matrices give either quaternion braids or trefoil knots. Since $\sigma_i^8 = 1$, the braid group is truncated and $\sigma_i^4 = \sigma_i^{-4}$ restricts the number of twists in any local region to 0, ± 1. Similarly, the complex i may be used to represent a B_2 generator [29]

$$\tau_1 = \frac{1}{\sqrt{2}}(1 + i), \tag{10}$$

where complex conjugation takes particles to antiparticles in the ideals underlying Lorentz transformations for the Standard Model [30][31]. Alternatively, i itself gives τ_1, so that a full ribbon twist diagram in B_2 is the charge -1, and there are no double full twists. The Tutte graph [32] for a trefoil knot is a triangle, with one node for each crossing, giving us an I, J, K triangle.

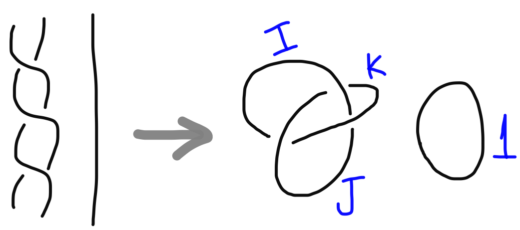

Figure 1: traced braid for the trefoil

A ribbon diagram belongs to a ribbon category [33][34][35], which for us will be a braided monoidal category with twists and duals on objects, along with fusion. The primary example is the Fibonacci anyon [36][37][38], which is universal [39] for quantum computation. Its 2×2 B_3 representation, related to the quaternion representation by a rotation, fills $SU(2)$ using the golden ratio $\phi = (1+\sqrt{5})/2$. For $A = \sqrt{\phi}^{-1}$, the matrix of fusion coefficients is

$$\mathbf{F} \equiv \begin{pmatrix} F_{11} & F_{1\tau} \\ F_{\tau 1} & F_{\tau\tau} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} A & 1 \\ 1 & A \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \tag{11}$$

where 1 and τ are the objects in the category [37][38]. The anyon spin of τ is $4\pi/5$ and 1 is the vacuum. The fusion rules, including $\tau \bullet \tau = 1 + \tau$, define trivalent vertices for diagrams. The B_3 generators are defined by

$$\sigma_1 = \begin{pmatrix} e^{6\pi i/5} & 0 \\ 0 & e^{3\pi i/5} \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} e^{-3\pi i/5} & 0 \\ 0 & 1 \end{pmatrix} \mathbf{F} \begin{pmatrix} e^{-3\pi i/5} & 0 \\ 0 & 1 \end{pmatrix}. \tag{12}$$

In applications to the electroweak interaction, a quantum trefoil carries an $SL_q(2)$ representation [40] for $j = 3/2$. It's four representations are labeled by $(\pm 3/2, 3/2)$ and $(\pm 3/2, -1/2)$ where the first parameter is the knot writhe and the second measures the projection to two dimensions. The sum of labels takes values in $\{0,1,2,3\}$, which are the anyon electric charges for ribbon leptons and quarks in the $\mathbb{C} \otimes \mathbb{O}$ picture [6][30][31]. Below we introduce the three dimensional parity cubes whose grading agrees with these charges. For us, the cubes are more fundamental than $SL_q(2)$. A quantum plane relation $y \otimes x - qx \otimes y$ arises [34] for finite dimensional vector spaces over any field, with q in the field. For example, in an $SL_q(2)$ matrix, let $q^{-1} = \phi$ and $ab = \phi ba$. Then $bc = cb$ and $da - ad = bc$.

Let $\rho = \sqrt{\phi + 2}$ be the diagonal of the golden rectangle. The integers $\mathbb{Z}$ give coordinates for a cubic integral lattice in 4 dimensions, which define a dense subset of $\mathbb{R}^2$ using the symplectic map [41]. This is the $\mathbb{Z}$ span of the four vectors

$$\frac{1}{2}(\phi - 1, \rho), \;\; \frac{1}{2}(-\phi, (\phi - 1)\rho), \;\; \frac{1}{2}(-\phi, (1-\phi)\rho), \;\; \frac{1}{2}(\phi - 1, -\rho). \tag{13}$$

The phases defining these vectors are $\pm 2\pi/5$ and $\pm 4\pi/5$. It follows that $\mathbb{Z}^8/2$ also fills the plane under the real integral maps

$$(a,b,c,d) \mapsto a + b\phi + c\rho + d\phi\rho. \tag{14}$$

A copy of $\mathbb{R}^6 \simeq \mathbb{C}^3$ as a discrete space over the ring $\mathbb{Z}[\rho]$ thus uses vectors in $\mathbb{Z}^{24}$, which is enough to define the cover of the Lorentz group $SL(2,\mathbb{C})$. For the twistor $\mathbb{C}^4$ we need $\mathbb{Z}^{32}$, which we will mention below.

The Minkowski metric of (5) is given by the quaternion norm $q\overline{q}$. Such a product of conjugates is the norm for any integral ring in a quadratic field. The ring $\mathbb{Z}[\phi]$ contains elements of negative norm, such as the norm -1 numbers ϕ^3 and ϕ, which is why ϕ is important in distinguishing space and time coordinates.

The quaternions also define idempotents of the form

$$P_I \equiv \frac{1}{\sqrt{2}}(1 - iI), \quad P_J \equiv \frac{1}{\sqrt{2}}(1 - iJ), \quad P_K \equiv \frac{1}{\sqrt{2}}(1 - iK). \tag{15}$$

Observe that $P_j\overline{P}_j = 0$ defines a null vector. This is interpreted like the statement that the intersection of a Boolean subset and its complement is the empty set, just as $PP = P$ says that the intersection of a set with itself is the same set. Similarly, $(P_j + \overline{P}_j)/\sqrt{2} = 1$ means that the union gives the full set 1. Idempotency applies to any object in a Heyting algebra, generalising open sets for a topological space to a topos [1]. Here we quantise the cardinality of a finite set, turning it into the dimension of an operator space. Then the set of all subsets of an n point set defines a cube in dimension n. For example, the set $\{I, J\}$ defines a square with vertices $1, I, J, IJ$.

The pattern continues in higher dimensions. The octonion units [42][43][44]

$$1, I, J, K, IL, JL, KL, L \tag{16}$$

define a representation of braids in B_7, as in (6), and a subset cube in dimension 3 based on I, J and KL. The 7 dimensional cube of 128 formal subsets for all 7 units is associated with magnetic information for a 128-spinor, studied in the higher dimensional algebras of exceptional periodicity [45][46][47]. On another cube, described below, ideals for $\mathbb{C} \otimes \mathbb{O}$ [30][31] define the $SU(3)$ color group, along with the $U(1)$ for electric charge.

There are various different cubes that appear here. A subset 3-cube with basis I, J and K is *not* a quaternion basis, because in $\mathbb{O}$ the quaternions lie on a line in the Fano plane. This cube is obtained from the octonion unit cube by the maps $K \mapsto IJ$ and $L \mapsto IJK$.

The vertices of a subset cube also carry spinor labels, as a string of n $\pm$ signs in dimension n. For example, the 32 dimensional integers required for a twistor in $\mathbb{C}^4$ are included in a 3×3 nonassociative integral matrix algebra [46][48] of shape

$$\begin{pmatrix} - & 32 & 2^{15} \\ & - & 2^{15} \\ & & - \end{pmatrix} \sim \begin{pmatrix} - & \begin{pmatrix} 8 & 8 \\ 8 & 8 \end{pmatrix} & \begin{pmatrix} 2^{11} & 2^{11} \\ 2^{11} & 2^{11} \end{pmatrix} \\ & - & \begin{pmatrix} 2^{11} & 2^{11} \\ 2^{11} & 2^{11} \end{pmatrix} \\ & & - \end{pmatrix}, \tag{17}$$

including a spinor cube of size 2^{16} [45][46][49]. Here we may use 2×2 matrices over $\mathbb{C}$ for $\mathbb{C}^4$, along with the cubes of the Golay code, which are also 24 dimensional over $\mathbb{F}_2$.

3 Sheaf, knot and cubic cohomology

Quantum gravity is motivic because its algebra comes from universal cohomology, which we aim to build with canonical geometric axioms, using the philosophy of higher dimensional topos theory [2]. As in twistor physics, we understand rest mass using sheaf cohomology and homology groups, particularly H_1, H_2 and H_3.

Now as a young woman, my favourite textbook was Bott and Tu's [50] introduction to algebraic topology. Early on, it launches into a discussion of the differential form functor for a manifold. Then it moves onto sheaves and cohomology. In the computation of the cohomology, the intersection of sets may be represented by a geometric point, and the edge between two points is a set that belongs to both intersections. It is the computational geometry that interests us, rather than manifolds.

Two sets I and J define the Mayer-Vietoris square

$$I \cap J \to I \coprod J \to I \cup J \tag{18}$$

in the lattice of all subsets, through the disjoint union of I and J. A contravariant functor C of forms reverses inclusion to restriction maps, such as $C(I \cap J) \to C(J)$. For sets I, J and S, there are two 3 dimensional cubes: one with inclusion edges and unions, and the other with restriction edges and intersections. However, we include only one map from the empty set into each of I, J and S, giving a basis for the cube, and these three edges are neither inclusions nor restrictions. Without the empty set, the seven objects resemble a Fano plane basis for $\mathbb{O}$ [6][49].

The 196560 vectors of the Leech lattice [24] come from three copies of $196560/3 = (2^{16} - 16)$ on a 16 dimensional spinor cube, with the 16 basis points removed. There are $\binom{16}{2} = 120$ 2-forms $e_i \wedge e_j$ associated to points of type IJ on the cube.

Although triangular simplices abound in algebraic topology, category theorists often prefer to use cubes for the natural higher dimensional compositions. The 2-cube in (18) appears in the Seifert van Kampen theorem [51] for the covariant fundamental groupoid $\pi_1(I \cup J, X)$. Here X is any subset of $I \cap J$, and the theorem states that the square is a pushout. A similar pushout holds for the second groupoid $\pi_2(I \cup J, X)$, where X is any subset of $I \cup J$. Groupoids are natural to a category theorist, because a group is merely a groupoid with one object. Given an aspherical space, meaning trivial higher homotopy groups, the Hopf formula [52] for the second cohomology H_2 of $\pi_1(X)$ (in terms of a presentation for π_1) comes from this two dimensional Seifert van Kampen theorem [51]. That is, for any exact sequence

$$1 \to R \to F \to G \to 1 \tag{19}$$

of groups, with F free, there is an exact sequence

$$0 \to H_2(G) \to \frac{R}{[F,R]} \to H_1(F) \to H_1(G) \to 0. \tag{20}$$

Bearing this in mind, we step away from classical spaces and consider cubes as basic categorical gadgets. A *cubical set* is a **Set** valued functor from the collection $\{C_n\}_{n \geq 0}$ of all n-cubes, with nice edge and face maps. This functor can then *define* a strict n-category [53] as a cubical set with composition. Our cubes in dimension n are generically associated to quantum spaces in dimension n, so that I and J label basis directions for a qubit Hilbert space. Rest mass, of course, is about three dimensional spaces.

In the quantum mechanical Khovanov homology [54][55], a 3-cube is now naturally associated to the trefoil knot. A smoothing of the three crossings is denoted by a sign triplet, like $++-$, which selects one of two smoothings for each of the three crossings IJK, JKI and KIJ. Mapping the $\{I, J, S\}$ cube to the $\{IJK, JKI, KIJ\}$ cube, we select a set of brackets for a Jacobi type identity, noting that a planar projection of a subdivided 3-cube gives the algebraic magic star [45][46], a basis for enveloping Lie algebras and nonassociative generalisations. Here a triality action on 3×3 matrix elements IJ, JK and IK in the exceptional Jordan algebra $J_3(\mathbb{O})$ [44] is associated to a basic trivalent vertex in the Jordan pair representation of an associator tree [2].

The four grades of the Khovanov complex on the 3-cube correspond to the four quantum group representations [40], because anyon charge gives the signed vertices. The smoothing choices for a link crossing [32] also correspond to either the deletion or contraction operation in a Tutte graph for the link diagram. The trefoil Tutte graph is the triangle, dual to a basic trivalent vertex. Tutte recursion computes the Jones invariant of the link.

Quandle homology [28] (or cohomology) is defined using a chain complex C_* for C_n the set of n-tuples $(A_1, A_2, \cdots, A_n)$ of elements A_i in the quandle. The operator $\partial_n : C_n \to C_{n-1}$ is defined by $\partial_n \equiv 0$ when $n \leq 1$ and for $n \geq 2$,

$$\partial_n(A_1, \cdots, A_n) = \sum_{i=2}^{n} (-1)^i ((A_1, \cdots, A_{i-1}, A_{i+1}, \cdots, A_n) \tag{21}$$

$$-(A_1 \circ A_i, A_2 \circ A_i, \cdots, A_{i-1} \circ A_i, A_{i+1}, \cdots, A_n)).$$

To add an abelian group G of coefficients, work with $C_* \otimes G$ and $\partial \otimes 1$.

For example, assume that $I \circ J = -J \circ I$ in an extended trefoil quandle. Over $\mathbb{Z}$, we have three H_2 class representatives of the form $\partial(I, J, K) = (I, K) - (I, J) - (J, I)$. This cuts the S_3 permutations of (I, J, K) down using the chains $2((K, I) + (I, K))$, $2((I, J) + (J, I))$ and $2((J, K) + (K, J))$. In other words, homology breaks symmetry. For a general quandle, a crossing diagram at (I, J) is weighted by a 2-cocycle [56].

As is well known, knot operators appear in hyperbolic volumes for state sum models [57] and fusion category approaches to loop quantum gravity. Khovanov homology appears in supersymmetric gauge theories, where knots occur as brane intersections [58][59], and in that setting we also see an emphasis on condensates [58][60]. We emphasise here that the motivic approach is top down, rather than bottom up, and starts with knotty axioms rather than ill defined continua.

4 The localisation of rest mass

The creation of rest mass hinges on the inverse neutrino see-saw [5], which pairs the neutrino CMB cutoff in the IR with the familiar UV Planck scale. This is a cosmological principle of quantum inertia [7][8], which replaces the cold dark matter paradigm with quantum causality, implementing Mach's principle in the physical scales of quantum field theory rather than in a ficticious classical universe. Particle states in the Standard Model are given by ribbon diagrams, where four basic braids in B_3 account for left and right handed neutrinos.

We interpret the existence of mirror braids as dyonic information. The ribbon categories that we need for dyons will be discussed in section 5. First we discuss the general categorical structures needed for mass generation, and a more complete description of the Standard Model.

Operads, like the n-cube operad or the associahedra, are precisely multicategories with one object [53]. That is, the trees of the associahedra operad do not carry labels on vertices. Figure 2 shows a generic multicategory arrow, which is a map $(a_2, \cdots, a_n) \to a_1$. The idea is familiar in particle physics, where analytic data is attached to components of a tree, and operads encode renormalisation to all loop orders.

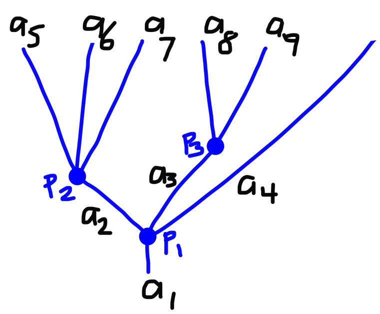

Figure 2: A multicategory arrow

Any (strict) monoidal category $\mathcal{V}$ defines an underlying multicategory, where a tensor product in $\mathcal{V}(A_2 \otimes A_3 \otimes \cdots \otimes A_n, A_1)$ gives an arrow $(A_1, \cdots, A_n; A_1)$. Similarly, if $\mathcal{V}$ has finite coproducts [53], the multicategory arrow comes from $\mathcal{V}(A_2, A_1) \times \cdots \times \mathcal{V}(A_n, A_1)$. Given a set of possible bracketings on the tensor product in a weak category, we may still define a multicategory arrow. Thus we can look at Hopf algebras or modules in the context of multicategories.

Given a cubical set, there is an endomorphism multicategory with an arrow $(a_2, \cdots, a_n; a_1)$ given by the function $C_2 \times C_3 \times \cdots \times C_n \to C_1$ on sets. Here the objects are labeled by $n \in \mathbb{N}$, and there is an operad for any fixed value of n. At $n = 1$, trees resemble the usual associahedra trees, but we imagine that the particle propagators are potentially carrying along cubes in higher dimensions. And of course these cubes carry the fermionic data that we require to build the QFT.

Electroweak doublets and mixing phenomenology have long suggested a quark lepton complementarity. The Khovanov complex for the 3-cube collapses the up quark colors into a singlet, and similarly for down quarks, so that C_0 and C_3 for the leptons must correspond to C_1 and C_2. That is, the complement of (e^-, ν) should be the proton and neutron. What is the neutrino analog of QGP gluons? In a condensate approach [61], there are two Chern-Simons theories: one for quarks and one for neutrino gravity. On the charge cubes, neutrinos specify the vacuum, and we expect instanton exchange. Conveniently, the instanton number is given [58] by the first argument n for a torus knot $T(n,m)$, where the trefoil is a $T(3,2)$ knot. Each torus knot of type $T(n,2)$ defines a Tutte polygon graph, from which the Jones invariant may be computed.

Fermion mass is traditionally obtained [62][63] from a twistor $H_2(\mathbb{T} \times \mathbb{T})$ pairing of two H_1 solutions to the massless Dirac equation [64], where coefficients lie in a helicity twisted sheaf of holomorphic functions, and $\mathbb{T}$ is the positive cone in $\mathbb{CP}^3$. On each copy of $\mathbb{T}$ the solution is massless, which for us is directly analogous to the masslessness of neutrinos when only one helicity is localised. A simple massive solution of helicity type $(+1,-1)$ (called type $(-4,0)$ in [62]) exists for spin 2.

In section 6 we write down the B_3 braid states for Standard Model fermions, where the underlying neutrino braids come in both left and right handed varieties, so that mass arises as a pairing involving non local states [5]. Quandle homology justifies the study of 3×3 or 6×6 mass operators indexed by I, J and K. A triality scheme for mass fits naturally into 3×3 matrices, starting with the exceptional Jordan algebra $J_3(\mathbb{O})$ and its three off diagonal copies of $\mathbb{O}$ [5]. The required higher dimensional algebras [46][49] replace the 8 dimensional spinors with higher dimensional cubes, as noted above, and weaker forms of triality come into play.

The non local neutrino is associated to the background thermal state, explaining the 2010 discovery [9][10] of the exact correspondence between a 0.00117 eV neutrino mass and the present day temperature of the CMB. Then the inverse see-saw rule $m_H = \sqrt{m_\nu m_P}$ derives the Higgs scale from the fundamental neutrino and Planck scales, and this coupling of two horizons is justified by quantum inertia [7][8].

The Fourier supersymmetry [11] between neutrinos and photons suggests that we look at categorical condensation in topological systems, since this is already modeled by braided fusion categories. Holography demands anyon statistics. Our picture for anyon condensation starts with the remarkable paper by Davydov and Booker [65], which shows that Fibonacci ribbon categories are *completely anisotropic*, roughly meaning that only the vacuum survives in a Fibonacci condensate. Then we look at more general categorical anyon models [16][17][19].

We are interested in a theory for two phases separated by a common boundary, which puts an unconfined phase in the bulk inside a mixed phase boundary for the global initial phase [18][19]. In this setting, the Fourier transform F_3 of (3) can appear as an S matrix for the $\mathbf{su(3)_1}$ anyon unconfined phase in the $\mathbf{su(2)_4}$ WZW theory [18], which has a deformation parameter a sixth root of unity. The initial objects $0,1,2,3,4$ for $\mathbf{su(2)_4}$ include a fusion rule $4 \circ 4 = 0$ which defines a boson object 4, which equals the vacuum after condensation. The basis of F_3 for the unconfined phase is given by this vacuum object along with two objects 2_+ and 2_- from the splitting of 2, while the objects 1 and 3 get confined.

Section 6 introduces the Koide mass operator as the F_3 Fourier transform of a diagonal. This F_3 acts on a copy of $\mathbf{su(3)}$ in the magic star [46], permuting three copies of $J_3(\mathbb{O})$. But first, let us consider the fundamental dyon categories, particularly that of the quantum double for the cyclic group C_3, which combines with our F_3 supersymmetry to implement electric magnetic duality [12].

5 Dyons and quantum double categories

Quandle objects A_1 and A_2 can represent the braiding of flux lines (Dirac strings) in the Aharanov Bohm effect [18], as indicated in figure 3. We think of our electric ribbon charges concretely as the motion of the charges around flux lines, so that categorical axioms literally represent the physics. Our cosmological dyons are associated to a mirror pair of states at an abstract topological surface [13], and we represent them categorically starting with the Hopf symmetry breaking mechanism of [16].

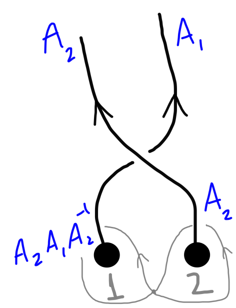

Figure 3: Dirac strings crossing for two test charges

Let $\mathbb{F}$ be a field. For a finite group G, usually non abelian, its quantum double Hopf algebra is $\text{Fun}(G) \otimes \mathbb{F}G$ with product

$$(f_1 \otimes g_1)(f_2 \otimes g_2)(x) = f_1(x) f_2(g_1 x g_1^{-1}) \otimes g_1 g_2. \tag{22}$$

Let f_g denote the function that equals 1 only on g and zero elsewhere. The coproduct is

$$\Delta(f_g \otimes h) = \sum_{xy=g} (f_x \otimes h) \otimes (f_y \otimes h). \tag{23}$$

A dyonic representation $R_{\Phi,e}$ carries a magnetic flux quantum number Φ and an electric quantum number e, where the electric symmetry is given by the elements $1 \otimes g$ and the magnetic symmetry by $f \otimes 1$, in the quantum double [66].

The desired electric magnetic duality becomes manifest when we make the function algebra $\text{Fun}(G)$ look like the group algebra $\mathbb{F}G$. Consider the simple case of the cyclic group C_3 inside S_3, given by the permutation matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \omega = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \overline{\omega} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{24}$$

Table 1: Standard Model electric braid states

ν_L	e_L^-	$\overline{u}_L(1)$	$\overline{u}_L(2)$	$\overline{u}_L(3)$	$d_L(1)$	$d_L(2)$	$d_L(3)$
$\sigma_1\sigma_2^{-1}$	$---$	$0--$	$-0-$	$--0$	-00	$0-0$	$00-$
$\overline{\nu}_R$	e_R^+	$u_R(1)$	$u_R(2)$	$u_R(3)$	$\overline{d}_R(1)$	$\overline{d}_R(2)$	$\overline{d}_R(3)$
$\sigma_2\sigma_1^{-1}$	$+++$	$0++$	$+0+$	$++0$	$+00$	$0+0$	$00+$
$\overline{\nu}_L$	e_L^+	$u_L(1)$	$u_L(2)$	$u_L(3)$	$\overline{d}_L(1)$	$\overline{d}_L(2)$	$\overline{d}_L(3)$
$\sigma_1^{-1}\sigma_2$	$+++$	$0++$	$+0+$	$++0$	$+00$	$0+0$	$00+$
ν_R	e_R^-	$\overline{u}_R(1)$	$\overline{u}_R(2)$	$\overline{u}_R(3)$	$d_R(1)$	$d_R(2)$	$d_R(3)$
$\sigma_2^{-1}\sigma_1$	$---$	$0--$	$-0-$	$--0$	-00	$0-0$	$00-$

The 3×3 diagonal matrices with entries (f_1, f_2, f_3) act as functions on C_3, and the Fourier transform F_3 of these diagonals gives a circulant element of the group algebra [12].

The modular group $SL_2(\mathbb{C})$ of electric magnetic duality is represented by diagrams in a modular ribbon category, and there are connections to mirror symmetry [67].

6 The Standard Model

The trefoil of figure 1 might correspond to four spacetime dimensions, but quantum gravity starts with a $(3,3)$ three time theory, using a mirror pair of knots. The composition of a left handed braid with its right handed mirror does *not* result in the trivial braid, while swapping all crossings on the right handed diagram does give the antiparticle braid.

Table 1 lists the B_3 braids for the neutrino, along with anyon ribbon charges for the three strands of the diagram [68]. Right handed singlets in the three time picture are also B_3 diagrams. Massless neutrinos have a fixed helicity, but both states are possible when neutrinos gain mass. The transformation from trefoil arcs to positively charged particle diagrams is given by an octonion cube map [29][30][31]. Starting with an I, J, K subset cube, in terms of a new set of units e_i for $i \in 1, \cdots, 7$, the new cube is given by the eight vertices indexed by I, J, K in products of the variables

$$\alpha_I = \frac{1}{2}(-e_5 + ie_4), \quad \alpha_J = \frac{1}{2}(-e_3 + ie_1), \quad \alpha_K = \frac{1}{2}(-e_6 + ie_2), \tag{25}$$

where the product $-i\alpha_I\alpha_J\alpha_K$ equals the idempotent $(1+ie_7)/2$. The conjugates under $i \mapsto -i$ give the negative (antiparticle) cube.

The non local braids for charged leptons and quarks, with opposite charges for a given neutrino diagram, are *not* included in Table I. Braid composition of a particle and antiparticle annihilates to a neutral photon identity diagram. In the quaternion representation for neutrino braids, $\sigma_1\sigma_2^{-1} = (1 + I - J - K)/2$, and $\pm K$ gives helicity while the sign on $I - J$ is the mirror. These particle braids have writhe zero, making them a basis for planar invariants.

Assuming that each local particle state defines a mass triplet, the double set of neutrino helicities in Table I allows for two distinct triplets of mass states. We assign the $+\pi/12$ phase to the correct helicity neutrinos and the $-\pi/12$ phase to the wrong helicity ones. Using oscillation data for the normal hierarchy, both mass triplets sum to a scale of 0.06 eV.

The 3×3 Fourier transform of the diagonal triplet $(\sqrt{m_1}, \sqrt{m_2}, \sqrt{m_3})$ of square root charged lepton masses is defined by the Koide matrix [69][70][71]

$$\sqrt{M} = \frac{\sqrt{\mu}}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & \delta & \overline{\delta} \\ \overline{\delta} & \sqrt{2} & \delta \\ \delta & \overline{\delta} & \sqrt{2} \end{pmatrix} \tag{26}$$

for a dimensionful scale μ close to the dynamical quark mass, and a complex phase δ. One is able to select the neutrino phases $\delta + \pi/12$ and $\delta - \pi/12$ relative to the charged lepton δ, which is very close to $2/9$. The neutrino parameters are fixed by two constraints. The $\sqrt{2}$ corresponds to Koide's original relation, and follows from a number of simple geometric arguments. A second constraint, first considered in [72], uses the cubic determinant of $\sqrt{M}$,

$$\sqrt{m_1 m_2 m_3} = \lambda(\sqrt{m_1} + \sqrt{m_2} - \sqrt{m_3})^3 \tag{27}$$

for a constant $\lambda \simeq 1/27$ at $2/9 - \pi/12$. Using the magic stars [46] for $\mathbb{R}$ and $\mathbb{O}$, the star tips define a ratio of $2/9$ and the centre point defines a ratio of $1/12$.

Let us consider also an alternative origin for the $2/9$. Noting that ϕ and ρ both define rectangles in our integral lattices, let

$$\delta = \frac{\pi}{4} - \tan^{-1}(\rho^{-1}) = 0.2222, \tag{28}$$

where the modular phases $\pi/4$ and $\pi/6$ define the two dimensional tribimaximal approximation [73] to the PMNS neutrino mixing matrix. Observe [6] that the angles

$$\frac{\pi}{4} - \tan^{-1}(\phi^{-1}) = 13.28^\circ, \quad \frac{\pi}{6} - \tan^{-1}(\rho^{-1}) = \delta - \frac{\pi}{12} = 2.3^\circ \tag{29}$$

approximate two of the three CKM Euler angles. The electron rest mass then corresponds to an eigenvalue phase of $11\pi/12 - \tan^{-1}(\rho^{-1})$, and the tangent rule gives

$$\tan^{-1}(\rho) + \tan^{-1}(\rho^3) = \frac{4\pi}{5}, \tag{30}$$
$$\tan^{-1}(\phi) + \tan^{-1}(\phi^3) = \frac{3\pi}{4} = -i.$$

The τ spin of $4\pi/5$ is special under Jordan triality because $4\pi/5 = (2/3)(-4\pi/5)$.

The new neutrino phase $\delta - \pi/12$ defines the present day CMB temperature [9][10] and a non local 1.3 eV sterile neutrino [74]. There are no 3D local particle states beyond those listed. A candidate for the third PMNS mixing parameter is the triality action angle $4/27 = (2/3)(2/9)$ [12].

Under the inverse Higgs see-saw [5], the neutrino mass and Planck scale fix the parameters of the Standard Model. The braid picture also fits well with skyrmion models for proton phenomenology.

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