On the Emergence of Spacetime Dimensions from the Kolmogorov Entropy

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Abstract

This short pedagogical report is based on a couple of premises. First, it was recently shown that the long run of non-equilibrium Renormalization Group flows is prone to end up on strange attractors. As a result, multifractals are likely to provide the proper framework for the characterization of effective field theories. Secondly, it is known that multifractal analysis uses the Kolmogorov entropy (K-entropy) to quantify the degree of disorder in chaotic systems and turbulent flows. Building on these premises, the report details the remarkable connection between K-entropy, multifractal sets and spacetime dimensions. It also supports the proposal that near and above the Fermi scale, spacetime is defined by continuous and arbitrarily small deviations from four-dimensions.

Key words: Kolmogorov entropy, multifractals, minimal fractal manifold, effective field theory.

We have conjectured in [1, 2] that the flow from the ultraviolet (UV) to the infrared (IR) sector of any multidimensional nonlinear field theory approaches chaotic dynamics in a universal way. This result stems from several independent routes to aperiodic behavior and implies that the IR attractor of effective field theories is likely to replicate the properties of *strange attractors* (SA) and *multifractal sets*. In particular, the chaotic behavior of the Renormalization Group flow near or above the Fermi scale suggests that phenomena on or above this scale mimic the dynamics on the SA [3-5, 9]. A counterintuitive property of these attractors is that adjacent trajectories separate exponentially fast, yet they remain forever trapped to a bounded region of phase-space. This is to say that, while microscopic SA dynamics is locally unstable, it stays stable at the global attractor scale. Elaborating further, it can be shown that the global stability of SA trajectories provides the foundation for classical statistical mechanics, which is based on invariant measures and the ergodic hypothesis [10].

Let a generic UV to IR trajectory be described by the n- dimensional phase-space flow $x(\tau)$. Here, τ denotes the evolution parameter ("time") corresponding to the Renormalization Group scale μ

$$\tau = \log(\mu/\mu_0) \tag{1}$$

The random behavior of the flow near the strange attractor can be characterized by dividing the phase-space into n- dimensional hypercubes of side r, which are sampled at discrete time intervals $\Delta\tau$. The generalized *K-entropy* of order $q \neq 1$ is given by the equation [6]

$$K_q(X) = -\lim_{r\to 0}\lim_{\Delta\tau\to 0}\lim_{N\to\infty}\frac{1}{N\Delta\tau}\frac{1}{q-1}\ln\sum_{i_1,i_2,\ldots i_N}^{M(r)} p_{i_1,i_2,\ldots i_N}^{q} \tag{2}$$

where $X = x_i$ is the discrete random variable, that is, $x_i = x(\tau = i\Delta\tau)$, and $p_{i_1,i_2,\ldots,i_M}$ stands for the joint probability that the trajectory $x(\tau = \Delta\tau)$ is in i_1, $x(\tau = 2\Delta\tau)$ is in i_2 and $x(\tau = M\Delta\tau)$ is in i_M. The K-entropy defines the asymptotic scenario where $r \to 0$ and the phase-space is sampled with an infinite number of steps ($N \to \infty$) at vanishing time intervals $(\Delta\tau \to 0)$. In the special case $N\Delta\tau = 1$ and when the joint probability is constant across all hypercubes ($M(r) = const.$, $p_{i_1,i_2,\ldots,i_M} = p_i$), (2) turns into the *Rényi entropy* in the logarithm base b, which assumes the form

$$S_q(X) = \frac{1}{1-q}\log_b(\sum_{i=1}^{M} p_i^q) \tag{3}$$

Furthermore, (3) reduces to the familiar *thermodynamic entropy* when $q \to 1$ and Boltzmann constant is set to $k_B = 1$ [7]

$$S(X) = -\sum_{i=1}^{M} p_i \ln p_i \tag{4}$$

Finally, the concept of *generalized dimension* of order q is introduced in conjunction with (3) according to

$$D_q = \lim_{r \to 0} \frac{1}{1-q} \frac{\log_b(\sum_{i=1}^{M} p_i^q)}{\log r} \tag{5}$$

A particularly straightforward expression of (3) is obtained for the null order $q = 0$ and the natural logarithm base. It is referred to as *topological entropy* and is given by

$$S_0(r) = \ln \sum_{i=1}^{M} p_i^0 = \ln M \tag{6}$$

It is known that the *box-counting dimension* of a fractal object of normalized size r is defined as

$$D_0 \approx \frac{\ln M}{\ln r} \Rightarrow M \approx r^{D_0} = \varepsilon^{-D_0} \tag{7}$$

in which M denotes the number of covering boxes and $\varepsilon = r^{-1}$ is the normalized size of the box. The dimension of ordinary Euclidean space corresponds to integer and positive-definite values of the box-counting dimension, $D_0 = k$, $k = 0,1,2\ldots$.

Comparing (6) to (7) leads to the connection between the box-counting dimension and topological entropy via

$$\boxed{\varepsilon^{-D_0} = \exp[S_0(r)]} \qquad (8)$$

Two straightforward conclusions may be drawn from (8):

- Maximal topological entropy ($S_0(r) \to \infty$) matches the limit $\varepsilon \to 0$ and corresponds to the four-dimensional continuum of both General Relativity and Quantum Field Theory.
- The steady increase of topological entropy along the flow implies that, near or above the Fermi scale, spacetime is endowed with a *continuous* spectrum of dimensions ($\varepsilon = 4 - D_0 << 1$), asymptotically reaching $D_0 = 4$ as $\varepsilon \to 0$ [8, 4].

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