# FREE QUANTUM GROUPS AND RELATED TOPICS

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Abstract. The unitary group $U_N$ has a free analogue $U_N^+$, and the closed subgroups $G\subset U_N^+$ can be thought of as being the "compact quantum Lie groups". We review here the general theory of such quantum groups. We discuss as well a number of more advanced topics, selected for their beauty, and potential importance.

## Contents



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## Introduction

One important discovery, going back to the beginning of the 20th century, is that at the subatomic level the "coordinates" of the various moving objects do not necessarily commute. In fact, at this level, our ambient space $\mathbb{R}^3$ gets replaced with something not commutative, and infinite dimensional - typically a space of infinite matrices.

Understanding why is it so, and working out all the details, remains an open problem, belonging of course to physics. However, mathematically speaking, the problem makes sense as well. To be more precise, the challenge is that of developing a theory of "non-commutative geometry", as nice and beautiful as the classical geometry. With a bit of luck, such a theory could be exactly what the physicists are looking for.

The quantum groups belong to this circle of ideas. They are meant to play the role of "symmetry groups" in this hypothetical noncommutative geometry theory.

There is no simple way of introducing the quantum groups. Indeed, these objects are of "quantum" nature, in the sense that, as for the elementary particles, their coordinates do not necessarily commute. This is not much of an issue in the long run, after getting used to the "think quantum" philosophy, but in order to get started, some sort of algebraic geometry formalism is definitely needed. We will use here the operator algebra one:

(1) A $C^*$-algebra is a complex algebra $A$, given with an involution $a\to a^*$, and with a Banach space norm $||.||$, related by the formula $||aa^*||=||a||^2$.

(2) Given a compact space $X$, the algebra $C(X)$ of continuous functions $f:X\to\mathbb{C}$ is such an algebra, with involution $f^*(x)=\overline{f(x)}$, and norm $||f||=\sup_{x\in X}|f(x)|$.

(3) This latter algebra is commutative, $fg=gf$, and one can prove, using complex analysis, that any commutative $C^*$-algebra is of this form, $A=C(X)$.

(4) In view of this, we agree to write any $C^*$-algebra $A$, not necessarily commutative, as $A=C(X)$, with $X$ being a "compact quantum space".

This was for the basic theory, that we will use all the time. Further results include the basic fact that the algebra $B(H)$ of bounded linear operators $T:H\to H$ on a Hilbert space $H$ is a $C^*$-algebra, with its usual norm and involution, and the more advanced fact that any $C^*$-algebra can be realized as a subalgebra $A\subset B(H)$. In the case of the commutative algebras, these embeddings appear as $C(X)\subset B(L^2(X))$.

Summarizing, we know what a compact quantum space is. All that is left now is to understand when such spaces have a group-theoretical structure.

In order to deal with this latter question, let us look first at the classical case. It is well-known, and non-trivial, that the compact Lie groups appear as closed subgroups of the unitary groups, $G\subset U_N$. Thus, our first objective will be that of understanding the commutative $C^*$-algebras of type $C(G)$, with $G\subset U_N$ being a closed subgroup:

(1) Given such a closed subgroup $G\subset U_N$, the first observation is that the corresponding algebra $C(G)$ is generated by the coordinate functions $u_{ij}(g)=g_{ij}$:

$$C(G)=< u_{ij} >$$

Indeed, these coordinate functions separate the points of $G$, so by the Stone-Weierstrasss theorem, they generate the whole algebra of continuous functions $f:G\to\mathbb C$.

(2) Regarding now the group structure on $G\subset U_N$, this comes from the usual group operations on the unitary matrices, namely:

$$(UV)_{ij}=\sum_k U_{ik}V_{kj}$$

$$(1_N)_{ij}=\delta_{ij}$$

$$(U^{-1})_{ij}=U_{ji}^*$$

(3) Thus, at the dual level, the group structure comes from maps as follows:

$$\Delta(u_{ij})=\sum_k u_{ik}\otimes u_{kj}$$

$$\varepsilon(u_{ij})=\delta_{ij}$$

$$S(u_{ij})=u_{ji}^*$$

(4) With a bit more work, one can show that the algebras of type $C(G)$, with $G\subset U_N$ being a closed subgroup, are exactly the commutative $C^*$-algebras $A=< u_{ij} >$ generated by the entries of a unitary matrix, having maps $\Delta,\varepsilon,S$ as above.

Summarizing, we have a nice description of the algebras of type $C(G)$, with $G$ being a compact Lie group. Getting back now to our quantum space philosophy, we can say that we have a nice description of the "compact quantum Lie groups" which are classical.

In order to define now the compact quantum Lie groups, in general, all that is left is to take the above, and remove the assumption that $A=C(G)$ is commutative. We are led in this way into the notion of Woronowicz algebra. Such an algebra is a $C^*$-algebra $A=< u_{ij} >$ generated by the entries of a unitary matrix $u=(u_{ij})$, having maps $\Delta,\varepsilon,S$ given by the same formulae as those for the compact Lie groups, (3) above.

These maps, called comultiplication, counit and antipode, automatically satisfy the following conditions, called coassociativity, counitality and coinversality:

$$(\Delta\otimes id)\Delta=(id\otimes\Delta)\Delta$$

$$(id\otimes\varepsilon)\Delta=(\varepsilon\otimes id)\Delta=id$$

$$m(id\otimes S)\Delta=m(S\otimes id)\Delta=\varepsilon(.)1$$

We recognize here the usual group theory axioms, written in dual form. In addition, we have $S^2 = id$, corresponding to the group theory formula $(g^{-1})^{-1} = g$.

Given such an algebra, we write $A = C(G)$, and call $G$ a compact quantum group. This compact quantum group is by definition of "matrix" or "Lie" type. Inspired by Pontrjagin duality, we can write as well $A = C^*(\Gamma)$, and call $\Gamma$ a discrete quantum group. This discrete quantum group is by definition "finitely generated".

As a conclusion, we have axiomatized both the compact and discrete quantum groups, under the mild assumption that we are in the Lie/finitely generated case.

As a basic, central example of a compact quantum group, we have the free analogue $U_N^+$ of the unitary group $U_N$. This quantum group appears as follows:

$$C(U_N^+) = C^* \left( (u_{ij})_{i,j=1,\ldots,N} \Big| u^* = u^{-1}, u^t = \bar{u}^{-1} \right)$$

To be more precise, on the right we have a certain universal algebra, constructed with generators and relations. Our claim is that if we call this algebra $C(U_N^+)$, then $U_N^+$ is a compact quantum group, which can be thought of as being a "free analogue" of $U_N$.

Our first task is that of explaining the construction of the universal algebra on the right, which definitely needs some discussion. The details here are as follows:

(1) Consider a square matrix $u = (u_{ij})$. Assuming that the entries $u_{ij}$ live in some complex algebra having an involution $*$, we can form the adjoint matrix, $u^* = (u_{ji}^*)$. With this convention, $u^* = u^{-1}$ is a shorthand for the condition $uu^* = u^*u = 1$.

(2) For the usual matrices $U \in M_N(\mathbb{C})$ the transpose of a unitary matrix is unitary too. However, this implication fails for the abstract matrices $u = (u_{ij})$ that we are interested in, and this is why we have to impose the condition $u^t = \bar{u}^{-1}$ as well.

(3) We can consider the universal complex $*$-algebra generated by $N^2$ abstract variables $(u_{ij})_{i,j=1,\ldots,N}$, subject to the $4N^2$ relations coming from the equalities $uu^* = u^*u = u^t\bar{u} = \bar{u}u^t = 1$, making our unitarity conditions $u^* = u^{-1}, u^t = \bar{u}^{-1}$ hold.

(4) Finally, in order to have a $C^*$-algebra, we can consider the abstract biggest $C^*$-norm on our $*$-algebra, and complete with respect to this norm. We obtain in this way the universal $C^*$-algebra that we are interested in.

Now observe that, by universality of the algebra that we constructed, we have morphisms $\Delta, \varepsilon, S$ as above. Thus $C(U_N^+)$ is a Woronowicz algebra, and the underlying compact quantum space $U_N^+$ is a compact quantum group, called "free unitary group".

All this might seem a bit mysterious, but will be explained in great detail, in this book. We will first review the operator algebra theory, then the Woronowicz algebra formalism, and then we will talk about $U_N^+$, and other compact quantum groups.

The compact quantum Lie groups, as defined before, appear exactly as the closed quantum subgroups $G \subset U_N^+$. The main examples are as follows:

(1) The compact Lie groups, $G \subset U_N$.

(2) The duals $G = \widehat{\Gamma}$ of the finitely generated groups $\Gamma = < g_1, \ldots, g_N >$.

(3) Deformations of the compact Lie groups, with parameter $q = -1$.

(4) Liberations, half-liberations, quantum permutation groups, and more.

We will present here the main tools for dealing with such quantum groups, and we will discuss as well a number of more advanced topics. The general idea will be that that such quantum groups do not have a Lie algebra, or much differential geometric structure, but one can study them via representation theory, with a mix of algebraic geometry and probability techniques. Also, we will mostly focus on the examples, with the idea in mind that, as in the case of the finite groups, or discrete groups, or compact Lie groups, there is a hierarchy between our objects, with some being more important that some other.

This point of view is particularly needed in connection with physics and applications. There might be many quantum groups, and other mathematical objects, but whether we want it or not, it is not up to us to decide what is useful, and what is not.

There are about 20 compact quantum groups which are of particular importance, at least at the starting level. Among them, we have 8 quantum groups which are really central, in connection with everything, forming a nice cubic diagram, as follows:

$$\begin{tikzcd}
 & K_N^+ \arrow[rr] & & U_N^+ \\
H_N^+ \arrow[rr] \arrow[ur] & & O_N^+ \arrow[ur] & \\
 & K_N \arrow[rr] \arrow[uu] & & U_N \arrow[uu] \\
H_N \arrow[rr] \arrow[ur] \arrow[uu] & & O_N \arrow[ur] \arrow[uu] &
\end{tikzcd}$$

Here on the right we have $O_N, U_N$ and their free analogues $O_N^+, U_N^+$, with $O_N^+ \subset U_N^+$ being constructed by imposing the relations $u_{ij} = u_{ij}^*$ to the standard coordinates.

On the bottom left we have the "discrete versions" of $O_N, U_N$, namely the hyperoctahedral group $H_N = \mathbb{Z}_2 \wr S_N$, and the full complex reflection group $K_N = \mathbb{T} \wr S_N$.

Finally, on top left we have the quantum groups $H_N^+ = \mathbb{Z}_2 \wr_* S_N^+$ and $K_N^+ = \mathbb{T} \wr_* S_N^+$, constructed by using the quantum permutation group $S_N^+$, which is something quite tricky, and the free analogue $\wr_*$ of the wreath product operation $\wr$.

We will discuss in detail the construction and main properties of these quantum groups, and of some other quantum groups of same type, and of potential importance as well.

Regarding the possible applications of this, the problem is open. The closed subgroups $G\subset U_N^+$ are potentially related to many things, and can normally be of help in connection with a number of questions in quantum physics. This remains to be confirmed.

This book is organized in four parts, as follows:

(1) Sections 1-4 are an introduction to the closed subgroups $G\subset U_N^+$, with the main examples $(O_N,O_N^*,O_N^+,U_N,U_N^*,U_N^+)$ explained in detail.

(2) Sections 5-8 contain basic theory, with the main examples, their bistochastic versions $(B_N,B_N^+,C_N,C_N^+)$ and their twists $(\bar{O}_N,\bar{O}_N^*,\bar{U}_N,\bar{U}_N^*)$ worked out.

(3) Sections 9-12 are concerned with quantum permutations $(S_N,S_N^+)$, quantum reflections $(H_N,H_N^*,H_N^+,K_N,K_N^*,K_N^+)$, and other related quantum groups.

(4) Sections 13-16 deal with more specialized topics, namely toral subgroups, amenability and growth, homogeneous spaces and modelling questions.

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## 1. **Quantum spaces**

In order to introduce the quantum groups, we will use the space/algebra correspondence coming from operator algebra theory. Here by "operator" we mean bounded linear operator $T : H \to H$ on a Hilbert space, and as a starting point, we have:

**Definition 1.1.** *A Hilbert space is a complex vector space $H$ given with a scalar product $< x, y >$, satisfying the following conditions:*

(1) *$< x, y >$ is linear in $x$, and antilinear in $y$.*
(2) *$\overline{< x, y >} =< y, x >$, for any $x, y$.*
(3) *$< x, x >> 0$, for any $x \neq 0$.*
(4) *$H$ is complete with respect to the norm $||x|| = \sqrt{< x, x >}$.*

Here the fact that $||.||$ is indeed a norm comes from the Cauchy-Schwarz inequality, which states that if (1,2,3) above are satisfied, then we have:

$$| < x, y > | \leq ||x|| \cdot ||y||$$

Indeed, this inequality comes from the fact that the following degree 2 polynomial, with $t \in \mathbb{R}$ and $w \in \mathbb{T}$, being positive, its discriminant must be negative:

$$f(t) = ||x + twy||^2$$

In finite dimensions, any algebraic basis $\{f_1, \ldots, f_N\}$ can be turned into an orthonormal basis $\{e_1, \ldots, e_N\}$, by using the Gram-Schmidt procedure. Thus, we have $H \simeq \mathbb{C}^N$, with this latter space being endowed with its usual scalar product:

$$< x, y >= \sum_i x_i \bar{y}_i$$

The same happens in infinite dimensions, once again by Gram-Schmidt, coupled if needed with the Zorn lemma, in case our space is really very big. In other words, any Hilbert space has an orthonormal basis $\{e_i\}_{i \in I}$, and we have $H \simeq l^2(I)$.

Of particular interest is the "separable" case, where $I$ is countable. According to the above, there is up to isomorphism only one Hilbert space here, namely $H = l^2(\mathbb{N})$.

All this is, however, quite tricky, and can be a bit misleading. Consider for instance the space $H = L^2[0, 1]$ of square-summable functions $f : [0, 1] \to \mathbb{C}$, with:

$$< f, g >= \int_0^1 f(x)\overline{g(x)}dx$$

This space is of course separable, because we can use the basis $f_n = x^n$ with $n \in \mathbb{N}$, orthogonalized by Gram-Schmidt. However, the orthogonalization procedure is something non-trivial, and so the isomorphism $H \simeq l^2(\mathbb{N})$ that we obtain is something non-trivial as well. Doing some computations here is actually a very good exercise.

Let us get now into the study of operators. We first have:

**Proposition 1.2.** *Let $H$ be a Hilbert space, with orthonormal basis $\{e_i\}_{i\in I}$. The algebra $\mathcal{L}(H)$ of linear operators $T : H \to H$ embeds then into the matrix algebra $M_I(\mathbb{C})$, with $T$ corresponding to the matrix $M_{ij} =< Te_j, e_i >$. In particular:*

1. *In the finite dimensional case, where $\dim(H) = N < \infty$, we obtain in this way a usual matrix algebra, $\mathcal{L}(H) \simeq M_N(\mathbb{C})$.*
2. *In the separable infinite dimensional case, where $I \simeq \mathbb{N}$, we obtain in this way a subalgebra of the infinite matrices, $\mathcal{L}(H) \subset M_\infty(\mathbb{C})$.*

*Proof.* The correspondence $T \to M$ in the statement is indeed linear, and its kernel is $\{0\}$. As for the last two assertions, these are clear as well. □

The above result is something quite theoretical, because for basic spaces like $L^2[0, 1]$, which do not have a simple orthonormal basis, the embedding $\mathcal{L}(H) \subset M_\infty(\mathbb{C})$ that we obtain is not very useful. Thus, while the operators $T : H \to H$ are basically some infinite matrices, it is better to think of these operators as being objects on their own.

In what follows we will be interested in the operators $T : H \to H$ which are bounded. Regarding such operators, we have the following result:

**Theorem 1.3.** *Given a Hilbert space $H$, the linear operators $T : H \to H$ which are bounded, in the sense that $||T|| = \sup_{||x||\leq 1} ||Tx||$ is finite, form a complex algebra with unit, denoted $B(H)$. This algebra has the following properties:*

1. *$B(H)$ is complete with respect to $||.||$, and so we have a Banach algebra.*
2. *$B(H)$ has an involution $T \to T^*$, given by $< Tx, y >=< x, T^*y >$.*

*In addition, the norm and the involution are related by the formula $||TT^*|| = ||T||^2$.*

*Proof.* The fact that we have indeed an algebra follows from:

$$||S + T|| \leq ||S|| + ||T||$$
$$||\lambda T|| = |\lambda| \cdot ||T||$$
$$||ST|| \leq ||S|| \cdot ||T||$$

(1) Assuming that $\{T_n\} \subset B(H)$ is Cauchy, the sequence $\{T_n x\}$ is Cauchy for any $x \in H$, so we can define the limit $T = \lim_{n\to\infty} T_n$ by setting:

$$Tx = \lim_{n\to\infty} T_n x$$

It is routine to check that this formula defines indeed a bounded operator $T \in B(H)$, and that we have $T_n \to T$ in norm, and this gives the result.

(2) Here the existence of $T^*$ comes from the fact that $\varphi(x) =< Tx, y >$ being a linear map $H \to \mathbb{C}$, we must have a formula as follows, for a certain vector $T^*y \in H$:

$$\varphi(x) =< x, T^*y >$$

Moreover, since this vector is unique, $T^*$ is unique too, and we have as well:

$$(S+T)^* = S^* + T^*$$
$$(\lambda T)^* = \bar{\lambda} T^*$$
$$(ST)^* = T^* S^*$$
$$(T^*)^* = T$$

Observe also that we have indeed $T^* \in B(H)$, because:

$$\begin{aligned} ||T|| &= \sup_{||x||=1} \sup_{||y||=1} < Tx, y > \\ &= \sup_{||y||=1} \sup_{||x||=1} < x, T^* y > \\ &= ||T^*|| \end{aligned}$$

Regarding now the last assertion, we have:

$$||TT^*|| \leq ||T|| \cdot ||T^*|| = ||T||^2$$

Also, we have the following estimate:

$$\begin{aligned} ||T||^2 &= \sup_{||x||=1} | < Tx, Tx > | \\ &= \sup_{||x||=1} | < x, T^* T x > | \\ &\leq ||T^* T|| \end{aligned}$$

By replacing in this formula $T \to T^*$ we obtain $||T||^2 \leq ||TT^*||$. Thus, we have proved both the needed inequalities, and we are done. □

Observe that, in view of Proposition 1.2, we embeddings of $*$-algebras, as follows:

$$B(H) \subset \mathcal{L}(H) \subset M_I(\mathbb{C})$$

In this picture the adjoint operation $T \to T^*$ constructed above takes a very simple form, namely $(M^*)_{ij} = \overline{M}_{ji}$ at the level of the associated matrices.

We will be interested here in the algebras of operators, rather than in the operators themselves. The axioms here, coming from Theorem 1.3, are as follows:

**Definition 1.4.** *A unital $C^*$-algebra is a complex algebra with unit $A$, having:*

1. *A norm $a \to ||a||$, making it a Banach algebra (the Cauchy sequences converge).*
2. *An involution $a \to a^*$, which satisfies $||aa^*|| = ||a||^2$, for any $a \in A$.*

We know from Theorem 1.3 that the full operator algebra $B(H)$ is a $C^*$-algebra, for any Hilbert space $H$. More generally, any closed $*$-subalgebra $A \subset B(H)$ is a $C^*$-algebra. The celebrated Gelfand-Naimark-Segal (GNS) theorem states that any $C^*$-algebra appears in fact in this way. This is something non-trivial, and we will be back to it later on.

For the moment, we will be interested in developing the theory of $C^*$-algebras, without reference to operators, or Hilbert spaces. Our first task will be that of understanding the structure of the commutative $C^*$-algebras. As a first observation, we have:

**Proposition 1.5.** *If $X$ is an abstract compact space, the algebra $C(X)$ of continuous functions $f : X \to \mathbb{C}$ is a $C^*$-algebra, with structure as follows:*

(1) *The norm is the usual sup norm, $||f|| = \sup_{x\in X} |f(x)|$.*
(2) *The involution is the usual involution, $f^*(x) = \overline{f(x)}$.*

*This algebra is commutative, in the sense that $fg = gf$, for any $f, g \in C(X)$.*

*Proof.* Almost everything here is trivial. Observe also that we have indeed:

$$\begin{aligned} ||ff^*|| &= \sup_{x\in X} |f(x)\overline{f(x)}| \\ &= \sup_{x\in X} |f(x)|^2 \\ &= ||f||^2 \end{aligned}$$

Finally, we have $fg = gf$, since $f(x)g(x) = g(x)f(x)$ for any $x \in X$. □

Our claim now is that any commutative $C^*$-algebra appears in this way. This is a non-trivial result, which requires a number of preliminaries. Let us begin with:

**Definition 1.6.** *The spectrum of an element $a \in A$ is the set*

$$\sigma(a) = \left\{\lambda \in \mathbb{C}\Big|a - \lambda \notin A^{-1}\right\}$$

*where $A^{-1} \subset A$ is the set of invertible elements.*

As a basic example, the spectrum of a usual matrix $M \in M_N(\mathbb{C})$ is the collection of its eigenvalues. Also, the spectrum of a continuous function $f \in C(X)$ is its image. In the case of the trivial algebra $A = \mathbb{C}$, the spectrum of an element is the element itself.

As a first, basic result regarding spectra, we have:

**Proposition 1.7.** *We have the following formula, valid for any $a, b \in A$:*

$$\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}$$

*Moreover, there are examples where $\sigma(ab) \neq \sigma(ba)$.*

*Proof.* We first prove that we have the following implication:

$$1 \notin \sigma(ab) \implies 1 \notin \sigma(ba)$$

Assume indeed that $1 - ab$ is invertible, with inverse $c = (1 - ab)^{-1}$. We have then $abc = cab = c - 1$, and by using these identities, we obtain:

$$\begin{aligned}(1 + bca)(1 - ba) &= 1 + bca - ba - bcaba\\ &= 1 + bca - ba - bca + ba\\ &= 1\end{aligned}$$

A similar computation shows that we have as well $(1 - ba)(1 + bca) = 1$. We conclude that $1 - ba$ is invertible, with inverse $1 + bca$, which proves our claim. By multiplying by scalars, we deduce from this that we have, for any $\lambda \in \mathbb{C} - \{0\}$, as desired:

$$\lambda \notin \sigma(ab) \implies \lambda \notin \sigma(ba)$$

Regarding now the last claim, let us first recall that for usual matrices $a, b \in M_N(\mathbb{C})$ we have $0 \in \sigma(ab) \iff 0 \in \sigma(ba)$, because $ab$ is invertible if any only if $ba$ is.

However, this latter fact fails for general operators on Hilbert spaces. As a basic example, we can take $a, b$ to be the shift $S(e_i) = e_{i+1}$ on the space $l^2(\mathbb{N})$, and its adjoint. Indeed, we have $S^*S = 1$, and $SS^*$ being the projection onto $e_0^\perp$, it is not invertible. □

Given an element $a \in A$, and a rational function $f = P/Q$ having poles outside $\sigma(a)$, we can construct the element $f(a) = P(a)Q(a)^{-1}$. For simplicity, we write:

$$f(a) = \frac{P(a)}{Q(a)}$$

With this convention, we have the following result:

**Theorem 1.8.** *We have the "rational functional calculus" formula*

$$\sigma(f(a)) = f(\sigma(a))$$

*valid for any rational function $f \in \mathbb{C}(X)$ having poles outside $\sigma(a)$.*

*Proof.* In order to prove this result, we can proceed in two steps, as follows:

(1) Assume first that we are in the polynomial case, $f \in \mathbb{C}[X]$. We pick $\lambda \in \mathbb{C}$, and we write $f(X) - \lambda = c(X - r_1)\ldots(X - r_n)$. We have then, as desired:

$$\begin{aligned}\lambda \notin \sigma(f(a)) &\iff f(a) - \lambda \in A^{-1}\\ &\iff c(a - r_1)\ldots(a - r_n) \in A^{-1}\\ &\iff a - r_1, \ldots, a - r_n \in A^{-1}\\ &\iff r_1, \ldots, r_n \notin \sigma(a)\\ &\iff \lambda \notin f(\sigma(a))\end{aligned}$$

(2) Assume now that we are in the general case, $f \in \mathbb{C}(X)$. We pick $\lambda \in \mathbb{C}$, we write $f = P/Q$, and we set $F = P - \lambda Q$. By using (1), we obtain:

$$\begin{aligned}
\lambda \in \sigma(f(a)) &\iff F(a) \notin A^{-1}\\
&\iff 0 \in \sigma(F(a))\\
&\iff 0 \in F(\sigma(a))\\
&\iff \exists \mu \in \sigma(a), F(\mu) = 0\\
&\iff \lambda \in f(\sigma(a))
\end{aligned}$$

Thus, we have obtained the formula in the statement. □

Given an element $a \in A$, its spectral radius $\rho(a)$ is the radius of the smallest disk centered at 0 containing $\sigma(a)$. We have the following key result:

**Theorem 1.9.** *Let $A$ be a $C^*$-algebra.*

1. *The spectrum of a norm one element is in the unit disk.*
2. *The spectrum of a unitary element ($a^* = a^{-1}$) is on the unit circle.*
3. *The spectrum of a self-adjoint element ($a = a^*$) consists of real numbers.*
4. *The spectral radius of a normal element ($aa^* = a^*a$) is equal to its norm.*

*Proof.* We use the various results established above.

(1) This comes from the following formula, valid when $||a|| < 1$:

$$\frac{1}{1-a} = 1 + a + a^2 + \ldots$$

(2) Assuming $a^* = a^{-1}$, we have the following norm computations:

$$||a|| = \sqrt{||aa^*||} = \sqrt{1} = 1$$

$$||a^{-1}|| = ||a^*|| = ||a|| = 1$$

If we denote by $D$ the unit disk, we obtain from this, by using (1):

$$||a|| = 1 \implies \sigma(a) \subset D$$

$$||a^{-1}|| = 1 \implies \sigma(a^{-1}) \subset D$$

On the other hand, by using the rational function $f(z) = z^{-1}$, we have:

$$\sigma(a^{-1}) \subset D \implies \sigma(a) \subset D^{-1}$$

Now by putting everything together we obtain, as desired:

$$\sigma(a) \subset D \cap D^{-1} = \mathbb{T}$$

(3) This follows by using (2), and the rational function $f(z) = (z + it)/(z - it)$, with $t \in \mathbb{R}$. Indeed, for $t >> 0$ the element $f(a)$ is well-defined, and we have:

$$\left(\frac{a+it}{a-it}\right)^* = \frac{a-it}{a+it} = \left(\frac{a+it}{a-it}\right)^{-1}$$

Thus $f(a)$ is a unitary, and by (2) its spectrum is contained in $\mathbb{T}$. We conclude that we have $f(\sigma(a)) = \sigma(f(a)) \subset \mathbb{T}$, and so $\sigma(a) \subset f^{-1}(\mathbb{T}) = \mathbb{R}$, as desired.

(4) We have $\rho(a) \leq ||a||$ from (1). Conversely, given $\rho > \rho(a)$, we have:

$$\int_{|z|=\rho} \frac{z^n}{z-a}\, dz = \sum_{k=0}^{\infty} \left( \int_{|z|=\rho} z^{n-k-1} dz \right) a^k = a^{n-1}$$

By applying the norm and taking $n$-th roots we obtain:

$$\rho \geq \lim_{n\to\infty} ||a^n||^{1/n}$$

In the case $a = a^*$ we have $||a^n|| = ||a||^n$ for any exponent of the form $n = 2^k$, and by taking $n$-th roots we get $\rho \geq ||a||$. This gives the missing inequality, namely:

$$\rho(a) \geq ||a||$$

In the general case, $aa^* = a^*a$, we have $a^n(a^n)^* = (aa^*)^n$. We obtain from this $\rho(a)^2 = \rho(aa^*)$, and since $aa^*$ is self-adjoint, we get $\rho(aa^*) = ||a||^2$, and we are done. □

Summarizing, we have so far a collection of useful results regarding the spectra of the elements in $C^*$-algebras, which are quite similar to the results regarding the eigenvalues of the usual matrices. We will heavily use these results, in what follows.

We are now in position of proving a key result, from [84], namely:

**Theorem 1.10** (Gelfand)**.** *Any commutative $C^*$-algebra is the form $C(X)$, with its "spectrum" $X = Spec(A)$ appearing as the space of characters $\chi : A \to \mathbb{C}$.*

*Proof.* Given a commutative $C^*$-algebra $A$, we can define indeed $X$ to be the set of characters $\chi : A \to \mathbb{C}$, with the topology making continuous all the evaluation maps $ev_a : \chi \to \chi(a)$. Then $X$ is a compact space, and $a \to ev_a$ is a morphism of algebras:

$$ev : A \to C(X)$$

(1) We first prove that $ev$ is involutive. We use the following formula, which is similar to the $z = Re(z) + iIm(z)$ formula for the usual complex numbers:

$$a = \frac{a+a^*}{2} - i \cdot \frac{i(a-a^*)}{2}$$

Thus it is enough to prove the equality $ev_{a^*} = ev_a^*$ for self-adjoint elements $a$. But this is the same as proving that $a = a^*$ implies that $ev_a$ is a real function, which is in turn true, because $ev_a(\chi) = \chi(a)$ is an element of $\sigma(a)$, contained in $\mathbb{R}$.

(2) Since $A$ is commutative, each element is normal, so $ev$ is isometric:

$$||ev_a|| = \rho(a) = ||a||$$

(3) It remains to prove that $ev$ is surjective. But this follows from the Stone-Weierstrass theorem, because $ev(A)$ is a closed subalgebra of $C(X)$, which separates the points. □

As a first consequence of the Gelfand theorem, we can extend Theorem 1.8 above to the case of the normal elements ($aa^* = a^*a$), in the following way:

**Theorem 1.11.** *Assume that $a \in A$ is normal, and let $f \in C(\sigma(a))$.*

1. *We can define $f(a) \in A$, with $f \to f(a)$ being a morphism of $C^*$-algebras.*
2. *We have the "continuous functional calculus" formula $\sigma(f(a)) = f(\sigma(a))$.*

*Proof.* Since $a$ is normal, the $C^*$-algebra $< a >$ that is generates is commutative, so if we denote by $X$ the space formed by the characters $\chi :< a >\to \mathbb{C}$, we have:

$$< a >= C(X)$$

Now since the map $X \to \sigma(a)$ given by evaluation at $a$ is bijective, we obtain:

$$< a >= C(\sigma(a))$$

Thus, we are dealing with usual functions, and this gives all the assertions. □

We can develop as well the theory of positive elements, as follows:

**Theorem 1.12.** *For a normal element $a \in A$, the following are equivalent:*

1. *$a$ is positive, in the sense that $\sigma(a) \subset [0, \infty)$.*
2. *$a = b^2$, for some $b \in A$ satisfying $b = b^*$.*
3. *$a = cc^*$, for some $c \in A$.*

*Proof.* This is something very standard, as follows:

(1) $\implies$ (2) This follows from Theorem 1.11, because we can use the function $f(z) = \sqrt{z}$, which is well-defined on $\sigma(a) \subset [0, \infty)$, and so set $b = \sqrt{a}$.

(2) $\implies$ (3) This is trivial, because we can set $c = b$.

(2) $\implies$ (1) Observe that this is clear too, because we have:

$$\begin{aligned}
\sigma(a) &= \sigma(b^2) \\
&= \sigma(b)^2 \\
&\subset [0, \infty)
\end{aligned}$$

(3) $\implies$ (1) We proceed by contradiction. By multiplying $c$ by a suitable element of $< cc^* >$, we are led to the existence of an element $d \neq 0$ satisfying:

$$-dd^* \geq 0$$

By writing now $d = x + iy$ with $x = x^*, y = y^*$ we have:

$$dd^* + d^*d = 2(x^2 + y^2) \geq 0$$

Thus $d^*d \geq 0$. But this contradicts the elementary fact that $\sigma(dd^*), \sigma(d^*d)$ must coincide outside $\{0\}$, coming from Proposition 1.7 above. □

The Gelfand theorem has as well some important philosophical consequences. Indeed, in view of this theorem, we can formulate the following definition:

**Definition 1.13.** *Given an arbitrary $C^*$-algebra $A$, we write $A = C(X)$, and call $X$ a compact quantum space. Equivalently, the category of the compact quantum spaces is the category of the $C^*$-algebras, with the arrows reversed.*

When $A$ is commutative, the space $X$ considered above exists indeed, as a Gelfand spectrum, $X = Spec(A)$. In general, $X$ is something rather abstract, and our philosophy here will be that of studying of course $A$, but formulating our results in terms of $X$. For instance whenever we have a morphism $\Phi : A \to B$, we will write $A = C(X), B = C(Y)$, and rather speak of the corresponding morphism $\phi : Y \to X$. And so on.

We will see later on, after developing some more theory, that this formalism has its limitations, and needs a fix. For the moment, however, let us explore the possibilities that it opens up. Inspired by the Connes philosophy [59], we have the following definition, which is something quite recent, coming from the work in [4], [32], [86]:

**Definition 1.14.** *We have compact quantum spaces, constructed as follows,*

$$C(S^{N-1}_{\mathbb{R},+}) = C^*\left(x_1, \ldots, x_N \Big| x_i = x_i^*, \sum_i x_i^2 = 1\right)$$

$$C(S^{N-1}_{\mathbb{C},+}) = C^*\left(x_1, \ldots, x_N \Big| \sum_i x_i x_i^* = \sum_i x_i^* x_i = 1\right)$$

*called respectively the free real sphere, and the free complex sphere.*

Here the $C^*$ symbols on the right stand for "universal $C^*$-algebra generated by". The fact that such universal $C^*$-algebras exist indeed follows by considering the corresponding universal $*$-algebras, and then completing with respect to the biggest $C^*$-norm. Observe that this biggest $C^*$-norm exists indeed, because the quadratic conditions give:

$$\begin{aligned} ||x_i||^2 &= ||x_i x_i^*|| \\ &\leq ||\sum_i x_i x_i^*|| \\ &= 1 \end{aligned}$$

Given a compact quantum space $X$, its classical version is the compact space $X_{class}$ obtained by dividing $C(X)$ by its commutator ideal, and using the Gelfand theorem:

$$C(X_{class}) = C(X)/I \quad , \quad I = < [a, b] >$$

Observe that we have an embedding of compact quantum spaces $X_{class} \subset X$. In this situation, we also say that $X$ appears as a "liberation" of $X$.

As a first result regarding the above free spheres, we have:

**Proposition 1.15.** *We have embeddings of compact quantum spaces, as follows,*

$$\begin{array}{ccc}S^{N-1}_{\mathbb{C}}&\rightarrow&S^{N-1}_{\mathbb{C},+}\\ \uparrow&&\uparrow\\ S^{N-1}_{\mathbb{R}}&\rightarrow&S^{N-1}_{\mathbb{R},+}\end{array}$$

*and the spaces on the right appear as liberations of the spaces of the left.*

*Proof.* The first assertion is clear. For the second one, we must establish the following isomorphisms, where $C^*_{comm}$ stands for "universal commutative $C^*$-algebra":

$$C(S^{N-1}_{\mathbb{R}})=C^*_{comm}\left(x_1,\ldots,x_N\Big|x_i=x_i^*,\sum_ix_i^2=1\right)$$

$$C(S^{N-1}_{\mathbb{C}})=C^*_{comm}\left(x_1,\ldots,x_N\Big|\sum_ix_ix_i^*=\sum_ix_i^*x_i=1\right)$$

But these isomorphisms are both clear, by using the Gelfand theorem. □

We can enlarge our class of basic manifolds by introducing tori, as follows:

**Definition 1.16.** *Given a closed subspace $S\subset S^{N-1}_{\mathbb{C},+}$, the subspace $T\subset S$ given by*

$$C(T)=C(S)\Big/\left\langle x_ix_i^*=x_i^*x_i=\frac{1}{N}\right\rangle$$

*is called associated torus. In the real case, $S\subset S^{N-1}_{\mathbb{R},+}$, we also call $T$ cube.*

As a basic example here, for $S=S^{N-1}_{\mathbb{C}}$ the corresponding submanifold $T\subset S$ appears by imposing the relations $|x_i|=\frac{1}{\sqrt{N}}$ to the coordinates, so we obtain a torus:

$$S=S^{N-1}_{\mathbb{C}}\implies T=\left\{x\in\mathbb{C}^N\Big||x_i|=\frac{1}{\sqrt{N}}\right\}$$

As for the case of the real sphere, $S=S^{N-1}_{\mathbb{R}}$, here the submanifold $T\subset S$ appears by imposing the relations $x_i=\pm\frac{1}{\sqrt{N}}$ to the coordinates, so we obtain a cube:

$$S=S^{N-1}_{\mathbb{R}}\implies T=\left\{x\in\mathbb{R}^N\Big|x_i=\pm\frac{1}{\sqrt{N}}\right\}$$

Observe that we have a relation here with group theory, because the complex torus computed above is the group $\mathbb{T}^N$, and the cube is the finite group $\mathbb{Z}_2^N$.

In general now, in order to compute $T$, we can use the following simple fact:

**Proposition 1.17.** *When $S\subset S^{N-1}_{\mathbb C,+}$ is an algebraic manifold, in the sense that*

$$C(S)=C(S^{N-1}_{\mathbb C,+})\Big/\Big\langle f_i(x_1,\ldots,x_N)=0\Big\rangle$$

*for certain noncommutative polynomials $f_i\in\mathbb C<x_1,\ldots,x_N>$, we have*

$$C(T)=C^*\left(u_1,\ldots,u_N\Big|u_i^*=u_i^{-1},g_i(u_1,\ldots,u_N)=0\right)$$

*with the poynomials $g_i$ being given by $g_i(u_1,\ldots,u_N)=f_i(\sqrt{N}u_1,\ldots,\sqrt{N}u_N)$.*

*Proof.* According to our definition of the torus $T\subset S$, the following variables must be unitaries, in the quotient algebra $C(S)\to C(T)$:

$$u_i=\frac{x_i}{\sqrt{N}}$$

Now if we assume that these elements are unitaries, the quadratic conditions $\sum_i x_ix_i^*=\sum_i x_i^*x_i=1$ are automatic. Thus, we obtain the space in the statement. □

Summarizing, we are led to the question of computing certain algebras generated by unitaries. In order to deal with this latter problem, let us start with:

**Proposition 1.18.** *Let $\Gamma$ be a discrete group, and consider the complex group algebra $\mathbb C[\Gamma]$, with involution given by the fact that all group elements are unitaries:*

$$g^*=g^{-1}\quad,\quad\forall g\in\Gamma$$

*The maximal $C^*$-seminorm on $\mathbb C[\Gamma]$ is then a $C^*$-norm, and the closure of $\mathbb C[\Gamma]$ with respect to this norm is a $C^*$-algebra, denoted $C^*(\Gamma)$.*

*Proof.* In order to prove the result, we must find a $*$-algebra embedding $\mathbb C[\Gamma]\subset B(H)$, with $H$ being a Hilbert space. For this purpose, consider the space $H=l^2(\Gamma)$, having $\{h\}_{h\in\Gamma}$ as orthonormal basis. Our claim is that we have an embedding, as follows:

$$\pi:\mathbb C[\Gamma]\subset B(H)\quad,\quad\pi(g)(h)=gh$$

Indeed, since $\pi(g)$ maps the basis $\{h\}_{h\in\Gamma}$ into itself, this operator is well-defined, bounded, and is an isometry. It is also clear from the formula $\pi(g)(h)=gh$ that $g\to\pi(g)$ is a morphism of algebras, and since this morphism maps the unitaries $g\in\Gamma$ into isometries, this is a morphism of $*$-algebras. Finally, the faithfulness of $\pi$ is clear. □

In the abelian group case, we have the following result:

**Theorem 1.19.** *Given an abelian discrete group $\Gamma$, we have an isomorphism*

$$C^*(\Gamma)\simeq C(G)$$

*where $G=\widehat{\Gamma}$ is its Pontrjagin dual, formed by the characters $\chi:\Gamma\to\mathbb T$.*

*Proof.* Since $\Gamma$ is abelian, the corresponding group algebra $A = C^*(\Gamma)$ is commutative. Thus, we can apply the Gelfand theorem, and we obtain $A = C(X)$, with $X = Spec(A)$. But the spectrum $X = Spec(A)$, consisting of the characters $\chi : C^*(\Gamma) \to \mathbb{C}$, can be identified with the Pontrjagin dual $G = \widehat{\Gamma}$, and this gives the result. $\square$

The above result suggests the following definition:

**Definition 1.20.** *Given a discrete group $\Gamma$, the compact quantum space $G$ given by*

$$C(G) = C^*(\Gamma)$$

*is called abstract dual of $\Gamma$, and is denoted $G = \widehat{\Gamma}$.*

This notion should be taken in the general sense of Definition 1.13. The same warning as there applies, because there is a functoriality problem here, which needs a fix. To be more precise, in the context of Proposition 1.18, we can see that the closure $C^*_{red}(\Gamma)$ of the group algebra $\mathbb{C}[\Gamma]$ in the regular representation is a $C^*$-algebra as well. We have a quotient map $C^*(\Gamma) \to C^*_{red}(\Gamma)$, and if this map is not an isomorphism, which is something that can happen, we are in trouble. We will be back to this, later on.

By getting back now to the spheres, we have the following result:

**Theorem 1.21.** *The tori of the basic spheres are all group duals, as follows,*

$$\begin{tikzcd}
\mathbb{T}^N \arrow[r] & \widehat{F_N} \\
\mathbb{Z}_2^N \arrow[u] \arrow[r] & \widehat{\mathbb{Z}_2^{*N}} \arrow[u]
\end{tikzcd}$$

*where $F_N$ is the free group on $N$ generators, and $*$ is a group-theoretical free product.*

*Proof.* By using the presentation result in Proposition 1.17 above, we obtain that the diagram formed by the algebras $C(T)$ is as follows:

$$\begin{tikzcd}
C^*(\mathbb{Z}^N) \arrow[d] & C^*(\mathbb{Z}^{*N}) \arrow[l] \arrow[d] \\
C^*(\mathbb{Z}_2^N) & C^*(\mathbb{Z}_2^{*N}) \arrow[l]
\end{tikzcd}$$

According to Definition 1.20, and together with the Fourier transform identifications from Theorem 1.19, and with our convention $F_N = \mathbb{Z}^{*N}$, this gives the result. $\square$

As a conclusion to these considerations, the Gelfand theorem alone produces out of nothing, or at least out of some basic common sense, some potentially interesting mathematics. We will be back later on to all this, on several occasions.

Let us review now the other fundamental result regarding the $C^*$-algebras, namely the representation theorem of Gelfand, Naimark and Segal. We first have:

**Proposition 1.22.** *Let $A$ be a commutative $C^*$-algebra, write $A = C(X)$, with $X$ being a compact space, and let $\mu$ be a positive measure on $X$. We have then an embedding*

$$A \subset B(H)$$

*where $H = L^2(X)$, with $f \in A$ corresponding to the operator $g \to fg$.*

*Proof.* Given $f \in C(X)$, consider the following operator, on the space $H = L^2(X)$:

$$T_f(g) = fg$$

Observe that $T_f$ is indeed well-defined, and bounded as well, because:

$$\begin{aligned} ||fg||_2 &= \sqrt{\int_X |f(x)|^2 |g(x)|^2 d\mu(x)} \\ &\leq ||f||_\infty ||g||_2 \end{aligned}$$

The application $f \to T_f$ being linear, involutive, continuous, and injective as well, we obtain in this way a $C^*$-algebra embedding $A \subset B(H)$, as claimed. □

In general, the idea will be that of extending this construction. We will need:

**Definition 1.23.** *Consider a linear map $\varphi : A \to \mathbb{C}$.*

(1) *$\varphi$ is called positive when $a \geq 0 \implies \varphi(a) \geq 0$.*
(2) *$\varphi$ is called faithful and positive when $a > 0 \implies \varphi(a) > 0$.*

In the commutative case, $A = C(X)$, the positive linear forms appear as follows, with $\mu$ being positive, and strictly positive if we want $\varphi$ to be faithful and positive:

$$\varphi(f) = \int_X f(x) d\mu(x)$$

In general, the positive linear forms can be thought of as being integration functionals with respect to some underlying "positive measures". We can use them as follows:

**Proposition 1.24.** *Let $\varphi : A \to \mathbb{C}$ be a positive linear form.*

(1) *$< a, b >= \varphi(ab^*)$ defines a generalized scalar product on $A$.*
(2) *By separating and completing we obtain a Hilbert space $H$.*
(3) *$\pi(a) : b \to ab$ defines a representation $\pi : A \to B(H)$.*
(4) *If $\varphi$ is faithful in the above sense, then $\pi$ is faithful.*

*Proof.* Almost everything here is straightforward, as follows:

(1) This is clear from definitions, and from Theorem 1.12.

(2) This is a standard procedure, which works for any scalar product.

(3) All the verifications here are standard algebraic computations.

(4) This follows indeed from $a \neq 0 \implies \pi(aa^*) \neq 0 \implies \pi(a) \neq 0$. □

In order to establish the GNS theorem, it remains to prove that any $C^*$-algebra has a faithful and positive linear form $\varphi : A \to \mathbb{C}$. This is something more technical:

**Proposition 1.25.** *Let $A$ be a $C^*$-algebra.*

1. *Any positive linear form $\varphi : A \to \mathbb{C}$ is continuous.*
2. *A linear form $\varphi$ is positive iff there is a norm one $h \in A_+$ such that $||\varphi|| = \varphi(h)$.*
3. *For any $a \in A$ there exists a positive norm one form $\varphi$ such that $\varphi(aa^*) = ||a||^2$.*
4. *If $A$ is separable there is a faithful positive form $\varphi : A \to \mathbb{C}$.*

*Proof.* The proof here, which is quite technical, inspired from the existence proof of the probability measures on abstract compact spaces, goes as follows:

(1) This follows from Proposition 1.24, via the following inequality:

$$\begin{aligned}|\varphi(a)| &\leq ||\pi(a)||\varphi(1)\\ &\leq ||a||\varphi(1)\end{aligned}$$

(2) In one sense we can take $h = 1$. Conversely, let $a \in A_+$, $||a|| \leq 1$. We have:

$$\begin{aligned}|\varphi(h) - \varphi(a)| &\leq ||\varphi|| \cdot ||h - a||\\ &\leq \varphi(h)1\\ &= \varphi(h)\end{aligned}$$

Thus we have $Re(\varphi(a)) \geq 0$, and it remains to prove that the following holds:

$$a = a^* \implies \varphi(a) \in \mathbb{R}$$

By using $1 - h \geq 0$ we can apply the above to $a = 1 - h$ and we obtain:

$$Re(\varphi(1 - h)) \geq 0$$

We conclude that $Re(\varphi(1)) \geq Re(\varphi(h)) = ||\varphi||$, and so $\varphi(1) = ||\varphi||$.

Summing up, we can assume $h = 1$. Now observe that for any self-adjoint element $a$, and any $t \in \mathbb{R}$ we have the following inequality:

$$\begin{aligned}|\varphi(1 + ita)|^2 &\leq ||\varphi||^2 \cdot ||1 + ita||^2\\ &= \varphi(1)^2||1 + t^2a^2||\\ &\leq \varphi(1)^2(1 + t^2||a||^2)\end{aligned}$$

On the other hand with $\varphi(a) = x + iy$ we have:

$$\begin{aligned}|\varphi(1+ita)| &= |\varphi(1) - ty + itx| \\ &\geq (\varphi(1) - ty)^2\end{aligned}$$

We therefore obtain that for any $t \in \mathbb{R}$ we have:

$$\varphi(1)^2(1 + t^2||a||^2) \geq (\varphi(1) - ty)^2$$

Thus we have $y = 0$, and this finishes the proof of our remaining claim.

(3) Consider the linear subspace of $A$ spanned by the element $aa^*$. We can define here a linear form by the following formula:

$$\varphi(\lambda aa^*) = \lambda||a||^2$$

This linear form has norm one, and by Hahn-Banach we get a norm one extension to the whole $A$. The positivity of $\varphi$ follows from (2).

(4) Let $(a_n)$ be a dense sequence inside $A$. For any $n$ we can construct as in (3) a positive form satisfying $\varphi_n(a_n a_n^*) = ||a_n||^2$, and then define $\varphi$ in the following way:

$$\varphi = \sum_{n=1}^{\infty} \frac{\varphi_n}{2^n}$$

Let $a \in A$ be a nonzero element. Pick $a_n$ close to $a$ and consider the pair $(H, \pi)$ associated to the pair $(A, \varphi_n)$, as in Proposition 1.24. We have then:

$$\begin{aligned}\varphi_n(aa^*) &= ||\pi(a)1|| \\ &\geq ||\pi(a_n)1|| - ||a - a_n|| \\ &= ||a_n|| - ||a - a_n|| \\ &> 0\end{aligned}$$

Thus $\varphi_n(aa^*) > 0$. It follows that we have $\varphi(aa^*) > 0$, and we are done. □

With these ingredients in hand, we can now state and prove:

**Theorem 1.26** (GNS theorem)**.** *Let $A$ be a $C^*$-algebra.*

(1) *$A$ appears as a closed $*$-subalgebra $A \subset B(H)$, for some Hilbert space $H$.*
(2) *When $A$ is separable (usually the case), $H$ can be chosen to be separable.*
(3) *When $A$ is finite dimensional, $H$ can be chosen to be finite dimensional.*

*Proof.* This result, from [85], follows indeed by combining the construction from Proposition 1.24 above with the existence result from Proposition 1.25. □

Generally speaking, the GNS theorem is something very powerful and concrete, which perfectly complements the Gelfand theorem, and the resulting compact quantum space formalism. We can go back to good old Hilbert spaces, whenever we get lost.

As a first application, let us get back to the bad functoriality properties of the Gelfand correspondence. We can fix these issues by using the GNS theorem, as follows:

**Definition 1.27.** *The category of compact quantum measured spaces $(X,\mu)$ is the category of the $C^*$-algebras with faithful traces $(A,tr)$, with the arrows reversed. In the case where we have a $C^*$-algebra $A$ with a non-faithful trace $tr$, we can still talk about the corresponding space $(X,\mu)$, by performing the GNS construction.*

Observe that this definition fixes the functoriality problem with Gelfand duality, at least for the group algebras. Indeed, in the context of the comments following Definition 1.20, consider an arbitrary intermediate $C^*$-algebra, as follows:

$$C^*(\Gamma)\to A\to C^*_{red}(\Gamma)$$

If we perform the GNS construction with respect to the canonical trace, we obtain the reduced algebra $C^*_{red}(\Gamma)$. Thus, all these algebras $A$ correspond to a unique compact quantum measured space in the above sense, which is the abstract group dual $\widehat{\Gamma}$. Let us record a statement about this finding, as follows:

**Proposition 1.28.** *The category of group duals $\widehat{\Gamma}$ is a well-defined subcategory of the category of compact quantum measured spaces, with each $\widehat{\Gamma}$ corresponding to the full group algebra $C^*(\Gamma)$, or the reduced group algebra $C^*_{red}(\Gamma)$, or any algebra in between.*

*Proof.* This is indeed more of an empty statement, coming from the above discussion. □

With this in hand, it is tempting to go even further, namely forgetting about the $C^*$-algebras, and trying to axiomatize instead the operator algebras of type $L^\infty(X)$. Such an axiomatization is possible, and the resulting class of operator algebras consists of a certain special type of $C^*$-algebras, called "finite von Neumann algebras". However, and here comes our point, doing so would be bad, and would lead to a weak theory, because many spaces such as the compact groups, or the compact homogeneous spaces, do not come with a measure by definition, but rather by theorem.

In short, our "fix" is not a very good fix, and if we want a really strong theory, we must invent something else. In order to do so, our idea will be that of restricting the attention to certain special classes of quantum algebraic manifolds, as follows:

**Definition 1.29.** *A real algebraic submanifold $X\subset S^{N-1}_{\mathbb{C},+}$ is a closed quantum subspace defined, at the level of the corresponding $C^*$-algebra, by a formula of type*

$$C(X)=C(S^{N-1}_{\mathbb{C},+})\Big/\Big\langle f_i(x_1,\ldots,x_N)=0\Big\rangle$$

*for certain noncommutative polynomials $f_i\in\mathbb{C}<x_1,\ldots,x_N>$. We denote by $\mathcal{C}(X)$ the $*$-subalgebra of $C(X)$ generated by the coordinate functions $x_1,\ldots,x_N$.*

Observe that any family of noncommutative polynomials $f_i \in \mathbb{C} < x_1, \ldots, x_N >$ produces such a manifold $X$, simply by defining an algebra $C(X)$ as above. Observe also that the use of the free complex sphere is essential in all this, because the quadratic condition $\sum_i x_i x_i^* = \sum_i x_i^* x_i = 1$ guarantees the fact that the universal $C^*$-norm is bounded.

We have already met such manifolds, in the context of the free spheres, free tori, and more generally in Proposition 1.17 above. Here is a list of examples:

**Proposition 1.30.** *The following are algebraic submanifolds $X \subset S^{N-1}_{\mathbb{C},+}$:*

1. *The spheres $S^{N-1}_{\mathbb{R}} \subset S^{N-1}_{\mathbb{C}}, S^{N-1}_{\mathbb{R},+} \subset S^{N-1}_{\mathbb{C},+}$.*
2. *Any compact Lie group, $G \subset U_n$, when $N = n^2$.*
3. *The duals $\widehat{\Gamma}$ of finitely generated groups, $\Gamma =< g_1, \ldots, g_N >$.*

*Proof.* These facts are all well-known, the proof being as follows:

(1) This is true by definition of our various spheres.

(2) Given a closed subgroup $G \subset U_n$, we have indeed an embedding $G \subset S^{N-1}_{\mathbb{C}}$, with $N = n^2$, given in double indices by $x_{ij} = \frac{u_{ij}}{\sqrt{n}}$, that we can further compose with the standard embedding $S^{N-1}_{\mathbb{C}} \subset S^{N-1}_{\mathbb{C},+}$. As for the fact that we obtain indeed a real algebraic manifold, this is well-known, coming either from Lie theory or from Tannakian duality. We will be back to this fact later on, in a more general context.

(3) This follows from the fact that the variables $x_i = \frac{g_i}{\sqrt{N}}$ satisfy the quadratic relations $\sum_i x_i x_i^* = \sum_i x_i^* x_i = 1$, with the algebricity claim of the manifold being clear. $\square$

At the level of the general theory, we have the following version of the Gelfand theorem, which is something very useful, and that we will use many times in what follows:

**Theorem 1.31.** *When $X \subset S^{N-1}_{\mathbb{C},+}$ is an algebraic manifold, given by*

$$C(X) = C(S^{N-1}_{\mathbb{C},+}) \Big/ \Big\langle f_i(x_1, \ldots, x_N) = 0 \Big\rangle$$

*for certain noncommutative polynomials $f_i \in \mathbb{C} < x_1, \ldots, x_N >$, we have*

$$X_{class} = \left\{ x \in S^{N-1}_{\mathbb{C}} \Big| f_i(x_1, \ldots, x_N) = 0 \right\}$$

*and $X$ appears as a liberation of $X_{class}$.*

*Proof.* This is something that already met, in the context of the free spheres. In general, the proof is similar, by using the Gelfand theorem. Indeed, if we denote by $X'_{class}$ the manifold constructed in the statement, then we have a quotient map of $C^*$-algebras as follows, mapping standard coordinates to standard coordinates:

$$C(X_{class}) \to C(X'_{class})$$

Conversely now, from $X\subset S^{N-1}_{\mathbb{C},+}$ we obtain $X_{class}\subset S^{N-1}_{\mathbb{C}}$, and since the relations defining $X'_{class}$ are satisfied by $X_{class}$, we obtain an inclusion of subspaces $X_{class}\subset X'_{class}$. Thus, at the level of algebras of continuous functions, we have a quotient map of $C^*$-algebras as follows, mapping standard coordinates to standard coordinates:

$$C(X'_{class})\to C(X_{class})$$

Thus, we have constructed a pair of inverse morphisms, and we are done. □

With these results in hand, we are now ready for formulating our second "fix" for the functoriality issues of the Gelfand correspondence, as follows:

**Definition 1.32.** *The category of the real algebraic submanifolds $X\subset S^{N-1}_{\mathbb{C},+}$ is the category of the universal $C^*$-algebras of type*

$$C(X)=C(S^{N-1}_{\mathbb{C},+})\Big/\Big\langle f_i(x_1,\ldots,x_N)=0\Big\rangle$$

*with $f_i\in\mathbb{C}<x_1,\ldots,x_N>$ being noncommutative polynomials, with the arrows $X\to Y$ being the $*$-algebra morphisms between $*$-algebras of coordinates*

$$\mathcal{C}(Y)\to\mathcal{C}(X)$$

*mapping standard coordinates to standard coordinates.*

In other words, what we are doing here is that of proposing a definition for the morphisms between the compact quantum spaces, in the particular case where these compact quantum spaces are algebraic submanifolds of the free complex sphere $S^{N-1}_{\mathbb{C},+}$. The point is that this "fix" perfectly works for the group duals, as follows:

**Theorem 1.33.** *The category of finitely generated groups $\Gamma=<g_1,\ldots,g_N>$, with the morphisms being the group morphisms mapping generators to generators, embeds contravariantly via $\Gamma\to\widehat{\Gamma}$ into the category of real algebraic submanifolds $X\subset S^{N-1}_{\mathbb{C},+}$.*

*Proof.* We know from Proposition 1.30 that, given a group $\Gamma=<g_1,\ldots,g_N>$, we have an embedding $\widehat{\Gamma}\subset S^{N-1}_{\mathbb{C},+}$ given by $x_i=\frac{g_i}{\sqrt{N}}$. Now since a morphism $C[\Gamma]\to C[\Lambda]$ mapping coordinates to coordinates means a morphism of groups $\Gamma\to\Lambda$ mapping generators to generators, our notion of isomorphism is indeed the correct one, as claimed. □

We will see later on that Theorem 1.33 has various extensions to the quantum groups and quantum homogeneous spaces that we will be interested in, which are all algebraic submanifolds $X\subset S^{N-1}_{\mathbb{C},+}$. We will also see that all these manifolds have Haar integration functionals, which are traces, and so that for these manifolds, our functoriality fix from Definition 1.32 coincides with the "von Neumann" fix from Definition 1.27.

So, this will be our formalism, and operator algebra knowledge required. We should mention that our approach heavily relies on Woronowicz's philosophy in [148]. Also, part of the above has been folklore for a long time, with the details worked out in [16].

## 2. Quantum groups

We have seen so far that the Gelfand philosophy, based on the operator algebra formalism, allows the construction of a number of interesting compact quantum spaces, such as the free versions $S^{N-1}_{\mathbb{R},+}$ and $S^{N-1}_{\mathbb{C},+}$ of the real and complex spheres. We have as well the duals $\widehat{\Gamma}$ of the discrete groups $\Gamma$, which can be thought of as being "quantum tori".

In this section we keep building on this, by introducing the compact quantum groups. The idea is very simple, coming from the usual formulae for unitary matrices:

$$(UV)_{ij}=\sum_k U_{ik}V_{kj}$$
$$(1_N)_{ij}=\delta_{ij}$$
$$(U^{-1})_{ij}=U_{ji}^*$$

A bit of Gelfand duality thinking, to be explained in the proof of Proposition 2.2 below, leads from this to the following definition, basically due to Woronowicz [148]:

**Definition 2.1.** *A Woronowicz algebra is a $C^*$-algebra $A$, given with a unitary matrix $u\in M_N(A)$ whose coefficients generate $A$, such that:*

1. *$\Delta(u_{ij})=\sum_k u_{ik}\otimes u_{kj}$ defines a morphism of $C^*$-algebras $A\to A\otimes A$.*
2. *$\varepsilon(u_{ij})=\delta_{ij}$ defines a morphism of $C^*$-algebras $A\to\mathbb{C}$.*
3. *$S(u_{ij})=u_{ji}^*$ defines a morphism of $C^*$-algebras $A\to A^{opp}$.*

*In this case, we write $A=C(G)$, and call $G$ a compact matrix quantum group.*

In this definition $A\otimes A$ is the universal $C^*$-algebraic completion of the usual algebraic tensor product of $A$ with itself, and $A^{opp}$ is the opposite $C^*$-algebra, with multiplication $a\cdot b=ba$. The above morphisms $\Delta,\varepsilon,S$ are called comultiplication, counit and antipode. Observe that if these morphisms exist, they are unique. This is analogous to the fact that a closed set of unitary matrices $G\subset U_N$ is either a compact group, or not.

The motivating examples are as follows:

**Proposition 2.2.** *Given a closed subgroup $G\subset U_N$, the algebra $A=C(G)$, with the matrix formed by the standard coordinates $u_{ij}(g)=g_{ij}$, is a Woronowicz algebra, and:*

1. *For this algebra, the morphisms $\Delta,\varepsilon,S$ appear as functional analytic transposes of the multiplication, unit and inverse maps $m,u,i$ of the group $G$.*
2. *This Woronowicz algebra is commutative, and conversely, any Woronowicz algebra which is commutative appears in this way.*

*Proof.* Since we have $G\subset U_N$, the matrix $u=(u_{ij})$ is unitary. Also, since the coordinate functions $u_{ij}$ separate the points of $G$, by the Stone-Weierstrass theorem we obtain that the $*$-subalgebra $\mathcal{A}\subset C(G)$ generated by them is dense. Finally, the fact that we have morphisms $\Delta,\varepsilon,S$ as in Definition 2.1 follows from the proof of (1) below.

(1) We use the previous formulae for unitary matrices. The fact that $m^t$ satisfies the condition in Definition 2.1 (1) follows from the following computation, with $U,V\in G$:

$$\begin{aligned}m^t(u_{ij})(U\otimes V)&=(UV)_{ij}\\&=\sum_kU_{ik}V_{kj}\\&=\sum_k(u_{ik}\otimes u_{kj})(U\otimes V)\end{aligned}$$

Regarding now the morphism $u^t$, the verification of the condition in Definition 2.1 (2) is trivial, coming from the following equalities:

$$u^t(u_{ij})=1_{ij}=\delta_{ij}$$

Finally, the morphism $i^t$ verifies the condition in Definition 2.1 (3) as well, because we have the following computation, valid for any $U\in G$:

$$i^t(u_{ij})(U)=(U^{-1})_{ij}=\bar{U}_{ji}=u_{ji}^*(U)$$

(2) By using the Gelfand theorem, we can write $A=C(G)$, with $G$ being a certain compact space. By using now the coordinates $u_{ij}$, we obtain an embedding as follows:

$$G\subset U_N$$

Finally, by using the maps $\Delta,\varepsilon,S$, it follows that the subspace $G\subset U_N$ that we have obtained is in fact a closed subgroup, and we are done. □

Let us go back now to the general setting of Definition 2.1. According to Proposition 2.2, and to the general $C^*$-algebra philosophy, the morphisms $\Delta,\varepsilon,S$ can be thought of as coming from a multiplication, unit map and inverse map, as follows:

$$m:G\times G\to G$$

$$u:\{.\}\to G$$

$$i:G\to G$$

Here is a first result of this type, expressing in terms of $\Delta,\varepsilon,S$ the fact that the underlying maps $m,u,i$ should satisfy the usual group theory axioms:

**Proposition 2.3.** *The comultiplication, counit and antipode have the following properties, on the dense $*$-subalgebra $\mathcal{A}\subset A$ generated by the variables $u_{ij}$:*

1. *Coassociativity: $(\Delta\otimes id)\Delta=(id\otimes\Delta)\Delta$.*
2. *Counitality: $(id\otimes\varepsilon)\Delta=(\varepsilon\otimes id)\Delta=id$.*
3. *Coinversality: $m(id\otimes S)\Delta=m(S\otimes id)\Delta=\varepsilon(.)1$.*

*In addition, the square of the antipode is the identity, $S^2=id$.*

*Proof.* Observe first that the result holds in the case where $A$ is commutative. Indeed, by using Proposition 2.2 we can write:

$$\Delta = m^t \quad , \quad \varepsilon = u^t \quad , \quad S = i^T$$

The above 3 conditions come then by transposition from the basic 3 group theory conditions satisfied by $m, u, i$, which are as follows, with $\delta(g) = (g, g)$:

$$m(m \times id) = m(id \times m)$$

$$m(id \times u) = m(u \times id) = id$$

$$m(id \times i)\delta = m(i \times id)\delta = 1$$

Observe that $S^2 = id$ is satisfied as well, coming from $i^2 = id$, which is a consequence of the group axioms. In general now, the proof goes as follows:

(1) We have indeed the following computation:

$$\begin{aligned}(\Delta \otimes id)\Delta(u_{ij}) &= \sum_l \Delta(u_{il}) \otimes u_{lj} \\ &= \sum_{kl} u_{ik} \otimes u_{kl} \otimes u_{lj}\end{aligned}$$

We have as well the following computation:

$$\begin{aligned}(id \otimes \Delta)\Delta(u_{ij}) &= \sum_k u_{ik} \otimes \Delta(u_{kj}) \\ &= \sum_{kl} u_{ik} \otimes u_{kl} \otimes u_{lj}\end{aligned}$$

(2) The proof here is quite similar. We first have:

$$(id \otimes \varepsilon)\Delta(u_{ij}) = \sum_k u_{ik} \otimes \varepsilon(u_{kj}) = u_{ij}$$

On the other hand, we have as well the following computation:

$$(\varepsilon \otimes id)\Delta(u_{ij}) = \sum_k \varepsilon(u_{ik}) \otimes u_{kj} = u_{ij}$$

(3) By using the fact that the matrix $u = (u_{ij})$ is unitary, we obtain:

$$\begin{aligned}m(id \otimes S)\Delta(u_{ij}) &= \sum_k u_{ik} S(u_{kj}) \\ &= \sum_k u_{ik} u_{jk}^* \\ &= (uu^*)_{ij} \\ &= \delta_{ij}\end{aligned}$$

Similarly, we have the following computation:

$$\begin{aligned} m(S\otimes id)\Delta(u_{ij}) &= \sum_k S(u_{ik})u_{kj}\\ &= \sum_k u_{ki}^*u_{kj}\\ &= (u^*u)_{ij}\\ &= \delta_{ij} \end{aligned}$$

Finally, the formula $S^2=id$ holds as well on the generators, and we are done. □

Let us discuss now another class of basic examples, namely the group duals:

**Proposition 2.4.** *Given a finitely generated discrete group $\Gamma=<g_1,\ldots,g_N>$, the group algebra $A=C^*(\Gamma)$, together with the diagonal matrix formed by the standard generators, $u=diag(g_1,\ldots,g_N)$, is a Woronowicz algebra, with $\Delta,\varepsilon,S$ given by:*

$$\Delta(g)=g\otimes g$$

$$\varepsilon(g)=1$$

$$S(g)=g^{-1}$$

*This Woronowicz algebra is cocommutative, in the sense that $\Sigma\Delta=\Delta$.*

*Proof.* Since the involution on $C^*(\Gamma)$ is given by $g^*=g^{-1}$ for any group element $g\in\Gamma$, all these group elements are unitaries. In particular the standard generators $g_1,\ldots,g_N$ are unitaries, and so must be the diagonal matrix formed by them:

$$u=\begin{pmatrix}g_1&&\\&\ddots&\\&&g_N\end{pmatrix}$$

Also, since $g_1,\ldots,g_N$ generate $\Gamma$, these elements generate the group algebra $C^*(\Gamma)$ as well, in the algebraic sense. Let us verify now the axioms in Definition 2.1:

(1) Consider the following map, which is a unitary representation:

$$\Gamma\to C^*(\Gamma)\otimes C^*(\Gamma)$$

$$g\to g\otimes g$$

This representation extends, as desired, into a morphism of algebras, as follows:

$$\Delta:C^*(\Gamma)\to C^*(\Gamma)\otimes C^*(\Gamma)$$

$$\Delta(g)=g\otimes g$$

(2) The situation for $\varepsilon$ is similar, because this comes from the trivial representation:

$$\Gamma\to\{1\}$$

$$g\to 1$$

(3) Finally, the antipode $S$ comes from the following unitary representation:

$$\Gamma \to C^*(\Gamma)^{opp}$$

$$g \to g^{-1}$$

Summarizing, we have shown that we have a Woronowicz algebra, with $\Delta, \varepsilon, S$ being as in the statement. Regarding now the last assertion, observe that we have:

$$\begin{aligned}\Sigma\Delta(g) &= \Sigma(g \otimes g)\\ &= g \otimes g\\ &= \Delta(g)\end{aligned}$$

Thus $\Sigma\Delta = \Delta$ holds on the group elements $g \in \Gamma$, and by linearity and continuity, this formula must hold on the whole algebra $C^*(\Gamma)$, as desired. $\square$

We will see later on that any cocommutative Woronowicz algebra appears in fact as above, up to a standard equivalence relation for such algebras. In the abelian group case now, we have a more precise result, as follows:

**Proposition 2.5.** *Assume that $\Gamma$ as above is abelian, and let $G = \widehat{\Gamma}$ be its Pontrjagin dual, formed by the characters $\chi : \Gamma \to \mathbb{T}$. The canonical isomorphism*

$$C^*(\Gamma) \simeq C(G)$$

*transforms then the comultiplication, counit and antipode of $C^*(\Gamma)$, given by*

$$\Delta(g) = g \otimes g$$

$$\varepsilon(g) = 1$$

$$S(g) = g^{-1}$$

*into the comultiplication, counit and antipode of $C(G)$, given by:*

$$\Delta\varphi(g,h) = \varphi(gh)$$

$$\varepsilon(\varphi) = \varphi(1)$$

$$S\varphi(g) = \varphi(g^{-1})$$

*Thus, the identification $G = \widehat{\Gamma}$ is a compact quantum group isomorphism.*

*Proof.* Assume indeed that $\Gamma =< g_1, \ldots, g_N >$ is abelian. Our claim is that with $G = \widehat{\Gamma}$ we have a group embedding $G \subset U_N$, constructed as follows:

$$\chi \to \begin{pmatrix}\chi(g_1)&&\\&\ddots&\\&&\chi(g_N)\end{pmatrix}$$

Indeed, this formula defines a unitary group representation, whose kernel is $\{1\}$.

Summarizing, we have two Woronowicz algebras to be compared, namely $C(G)$, constructed as in Proposition 2.2, and $C^*(\Gamma)$, constructed as in Proposition 2.4.

We already know from section 1 above that the underlying $C^*$-algebras are isomorphic. Now since the morphisms $\Delta,\varepsilon,S$ agree on the standard generators $g_1,\ldots,g_N$, they agree everywhere, and we are led to the conclusions in the statement. □

As a conclusion to all this, we can supplement Definition 2.1 with:

**Definition 2.6.** *Given a Woronowicz algebra $A=C(G)$, we write as well*

$$A=C^*(\Gamma)$$

*and call $\Gamma=\widehat{G}$ a finitely generated discrete quantum group.*

Let us develop now some further general theory. We first have:

**Proposition 2.7.** *Given a Woronowicz algebra $(A,u)$, we have*

$$u^t=\bar{u}^{-1}$$

*and so $u=(u_{ij})$ is a biunitary, meaning unitary, with unitary transpose.*

*Proof.* The idea is that $u^t=\bar{u}^{-1}$ comes from $u^*=u^{-1}$, by applying the antipode. Indeed, by denoting $(a,b)\to a\cdot b$ the multiplication of $A^{opp}$, we have:

$$\begin{aligned}
(uu^*)_{ij}=\delta_{ij}\quad&\implies\quad\sum_k u_{ik}u_{jk}^*=\delta_{ij}\\
&\implies\quad\sum_k S(u_{ik})\cdot S(u_{jk}^*)=\delta_{ij}\\
&\implies\quad\sum_k u_{ki}^*\cdot u_{kj}=\delta_{ij}\\
&\implies\quad\sum_k u_{kj}u_{ki}^*=\delta_{ij}\\
&\implies\quad(u^t\bar{u})_{ji}=\delta_{ij}
\end{aligned}$$

Similarly, we have the following computation:

$$\begin{aligned}
(u^*u)_{ij}=\delta_{ij}\quad&\implies\quad\sum_k u_{ki}^*u_{kj}=\delta_{ij}\\
&\implies\quad\sum_k S(u_{ki}^*)\cdot S(u_{kj})=\delta_{ij}\\
&\implies\quad\sum_k u_{ik}\cdot u_{jk}^*=\delta_{ij}\\
&\implies\quad\sum_k u_{jk}^*u_{ik}=\delta_{ij}\\
&\implies\quad(\bar{u}u^t)_{ji}=\delta_{ij}
\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

We have now the following theoretical result:

**Proposition 2.8.** *Given a Woronowicz algebra $A = C(G)$, we have an embedding*

$$G \subset S^{N^2-1}_{\mathbb{C},+}$$

*given in double indices by $x_{ij} = \frac{u_{ij}}{\sqrt{N}}$, where $u_{ij}$ are the standard coordinates of $G$.*

*Proof.* This is something that we already know for the classical groups, and for the group duals as well, from section 1 above. In general, the proof is similar, coming from the fact that the matrices $u, \bar{u}$ are both unitaries, that we know from Proposition 2.7. $\square$

In view of the above result, we can take some inspiration from the Gelfand correspondence "fix" presented in section 1, and formulate:

**Definition 2.9.** *Given two Woronowicz algebras $(A, u)$ and $(B, v)$, we write*

$$A \simeq B$$

*and indentify as well the corresponding compact and discrete quantum groups, when we have an isomorphism of $*$-algebras*

$$\mathcal{A} \simeq \mathcal{B}$$

*mapping standard coordinates to standard coordinates.*

In view of the various results and comments from section 1, the functoriality problem for the compact and discrete quantum groups is therefore fixed. Let us get now into a more exciting question, namely the construction of examples. We first have:

**Proposition 2.10.** *Given two compact quantum groups $G, H$, so is their product $G \times H$, constructed according to the following formula:*

$$C(G \times H) = C(G) \otimes C(H)$$

*Equivalently, at the level of the associated discrete duals $\Gamma, \Lambda$, we can set*

$$C^*(\Gamma \times \Lambda) = C^*(\Gamma) \otimes C^*(\Lambda)$$

*and we obtain the same equality of Woronowicz algebras as above.*

*Proof.* Assume indeed that we have two Woronowicz algebras, $(A, u)$ and $(B, v)$. Our claim is that the following construction produces a Woronowicz algebra:

$$C = A \otimes B \quad , \quad w = diag(u, v)$$

Indeed, the matrix $w$ is unitary, and its coefficients generate $C$. As for the existence of the maps $\Delta, \varepsilon, S$, this follows from the functoriality properties of $\otimes$, which is here, as usual, the universal $C^*$-algebraic completion of the algebraic tensor product.

With this claim in hand, the first assertion is clear. As for the second assertion, let us recall that when $G, H$ are classical and abelian, we have the following formula:

$$\widehat{G \times H} = \widehat{G} \times \widehat{H}$$

Thus, our second assertion is simply a reformulation of the first assertion, with the $\times$ symbol used there being justified by this well-known group theory formula. □

Here is now a more subtle construction, due to Wang [139]:

**Proposition 2.11.** *Given two compact quantum groups $G,H$, so is their dual free product $G\,\hat{*}\,H$, constructed according to the following formula:*

$$C(G\,\hat{*}\,H)=C(G)*C(H)$$

*Equivalently, at the level of the associated discrete duals $\Gamma,\Lambda$, we can set*

$$C^*(\Gamma*\Lambda)=C^*(\Gamma)*C^*(\Lambda)$$

*and we obtain the same equality of Woronowicz algebras as above.*

*Proof.* The proof here is identical with the proof of Proposition 2.10, by replacing everywhere the tensor product $\otimes$ with the free product $*$, with this latter product being by definition the universal $C^*$-algebraic completion of the algebraic free product. □

Here is another construction, which once again, has no classical counterpart:

**Proposition 2.12.** *Given a compact quantum group $G$, so is its free complexification $\widetilde{G}$, constructed according to the following formula, where $z=id\in C(\mathbb{T})$:*

$$C(\widetilde{G})\subset C(\mathbb{T})*C(G)\quad,\quad\tilde{u}=zu$$

*Equivalently, at the level of the associated discrete dual $\Gamma$, we can set*

$$C^*(\widetilde{\Gamma})\subset C^*(\mathbb{Z})*C^*(\Gamma)\quad,\quad\tilde{u}=zu$$

*where $z=1\in\mathbb{Z}$, and we obtain the same Woronowicz algebra as above.*

*Proof.* This follows from Proposition 2.11. Indeed, we know from there that $C(\mathbb{T})*C(G)$ is a Woronowicz algebra, with matrix of coordinates $w=diag(z,u)$. Now, let us try to replace this matrix with the matrix $\tilde{u}=zu$. This matrix is unitary, and we have:

$$\Delta(\tilde{u}_{ij})=(z\otimes z)\sum_k u_{ik}\otimes u_{kj}=\sum_k\tilde{u}_{ik}\otimes\tilde{u}_{kj}$$

Similarly, in what regards the counit, we have the following formula:

$$\varepsilon(\tilde{u}_{ij})=1\cdot\delta_{ij}=\delta_{ij}$$

Finally, recalling that $S$ takes values in the opposite algebra, we have as well:

$$S(\tilde{u}_{ij})=u_{ji}^*\cdot\bar{z}=\tilde{u}_{ji}^*$$

Summarizing, the conditions in Definition 2.1 are satisfied, except for the fact that the entries of $\tilde{u}=zu$ do not generate the whole algebra $C(\mathbb{T})*C(G)$. We conclude that if we let $C(\widetilde{G})\subset C(\mathbb{T})*C(G)$ be the subalgebra generated by the entries of $\tilde{u}=zu$, as in the statement, then the conditions in Definition 2.1 are satisfied, as desired. □

Another standard operation is that of taking subgroups:

**Proposition 2.13.** *Let $G$ be compact quantum group, and let $I \subset C(G)$ be a closed $*$-ideal satisfying the following condition:*

$$\Delta(I) \subset C(G) \otimes I + I \otimes C(G)$$

*We have then a closed quantum subgroup $H \subset G$, constructed as follows:*

$$C(H) = C(G)/I$$

*At the dual level we obtain a quotient of discrete quantum groups, $\widehat{\Gamma} \to \widehat{\Lambda}$.*

*Proof.* This follows indeed from the above conditions on $I$, which are designed precisely as for $\Delta, \varepsilon, S$ to factorize through the quotient. As for the last assertion, this is just a reformulation, coming from the functoriality properties of the Pontrjagin duality. □

In order to discuss now the quotient operation, let us agree to call "corepresentation" of a Woronowicz algebra $A$ any unitary matrix $v \in M_n(\mathcal{A})$ satisfying:

$$\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj} \quad , \quad \varepsilon(v_{ij}) = \delta_{ij} \quad , \quad S(v_{ij}) = v_{ji}^*$$

We will study in detail such corepresentations in section 3 below. For the moment, we just need their definition, in order to formulate the following result:

**Proposition 2.14.** *Let $G$ be a compact quantum group, and $v = (v_{ij})$ be a corepresentation of $C(G)$. We have then a quotient quantum group $G \to H$, given by:*

$$C(H) = < v_{ij} >$$

*At the dual level we obtain a discrete quantum subgroup, $\widehat{\Lambda} \subset \widehat{\Gamma}$.*

*Proof.* Here the first assertion follows from definitions, and the second assertion is just a reformulation, coming from the functoriality properties of the Pontrjagin duality. □

Finally, here is one more construction, which will be of importance in what follows:

**Theorem 2.15.** *Given a compact quantum group $G$, with fundamental corepresentation denoted $u = (u_{ij})$, the $N^2 \times N^2$ matrix given in double index notation by*

$$v_{ia,jb} = u_{ij} u_{ab}^*$$

*is a corepresentation in the above sense, and we have the following results:*

1. *The corresponding quotient $G \to PG$ is a compact quantum group.*
2. *Via the standard embedding $G \subset S^{N^2-1}_{\mathbb{C},+}$, this is the projective version.*
3. *In the classical group case, $G \subset U_N$, we have $PG = G/(G \cap \mathbb{T}^N)$.*
4. *In the group dual case, with $\Gamma = < g_i >$, we have $\widehat{P\Gamma} = < g_i g_j^{-1} >$.*

*Proof.* The fact that $v$ is indeed a corepresentation is routine, and follows as well from the general properties of such corepresentations, to be discussed in section 3 below.

(1) This follows from Proposition 2.14 above.

(2) Observe first that, since the matrix $v = (v_{ia,jb})$ is biunitary, we have indeed an embedding $G \subset S^{N^2-1}_{\mathbb{C},+}$ as in the statement, given in double index notation by $x_{ia,jb} = \frac{v_{ia,jb}}{N}$. Now with this formula in hand, the assertion is clear from definitions.

(3) This follows from the elementary fact that, via Gelfand duality, $w$ is the matrix of coefficients of the adjoint representation of $G$, whose kernel is the subgroup $G \cap \mathbb{T}^N$, where $\mathbb{T}^N \subset U_N$ denotes the subgroup formed by the diagonal matrices.

(4) This is something trivial, which follows from definitions. □

At the level of the really "new" examples now, we have basic liberation constructions, going back to the pioneering work of Wang [139], [140], and to the subsequent papers [1], [2] as well as several more recent constructions. We first have, following Wang [139]:

**Theorem 2.16.** *The following universal algebras are Woronowicz algebras,*

$$\begin{aligned} C(O_N^+) &= C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\bar{u}, u^t=u^{-1}\right) \\ C(U_N^+) &= C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big|u^*=u^{-1}, u^t=\bar{u}^{-1}\right) \end{aligned}$$

*so the underlying compact quantum spaces $O_N^+, U_N^+$ are compact quantum groups.*

*Proof.* This follows from the elementary fact that if a matrix $u = (u_{ij})$ is orthogonal or biunitary, then so must be the following matrices:

$$u_{ij}^\Delta = \sum_k u_{ik} \otimes u_{kj} \quad , \quad u_{ij}^\varepsilon = \delta_{ij} \quad , \quad u_{ij}^S = u_{ji}^*$$

Consider indeed the matrix $U = u^\Delta$. We have then:

$$\begin{aligned} (UU^*)_{ij} &= \sum_{klm} u_{il}u_{jm}^* \otimes u_{lk}u_{mk}^* \\ &= \sum_{lm} u_{il}u_{jm}^* \otimes \delta_{lm} \\ &= \delta_{ij} \end{aligned}$$

In the other sense the computation is similar, as follows:

$$\begin{aligned} (U^*U)_{ij} &= \sum_{klm} u_{kl}^*u_{km} \otimes u_{li}^*u_{mj} \\ &= \sum_{lm} \delta_{lm} \otimes u_{li}^*u_{mj} \\ &= \delta_{ij} \end{aligned}$$

The verification of the unitarity of $\bar{U}$ is similar. We first have:

$$\begin{aligned}(\bar{U}U^t)_{ij} &= \sum_{klm} u_{il}^* u_{jm} \otimes u_{lk}^* u_{mk}\\ &= \sum_{lm} u_{il}^* u_{jm} \otimes \delta_{lm}\\ &= \delta_{ij}\end{aligned}$$

In the other sense the computation is similar, as follows:

$$\begin{aligned}(U^t\bar{U})_{ij} &= \sum_{klm} u_{kl} u_{km}^* \otimes u_{li} u_{mj}^*\\ &= \sum_{lm} \delta_{lm} \otimes u_{li} u_{mj}^*\\ &= \delta_{ij}\end{aligned}$$

Regarding now the matrix $u^\varepsilon = 1_N$, this is clearly biunitary. Regarding the matrix $u^S$, there is nothing to prove here either, because its unitarity its clear too.

Thus, we can indeed define morphisms $\Delta, \varepsilon, S$ as in Definition 2.1, by using the universal properties of $C(O_N^+)$, $C(U_N^+)$, and this gives the result. □

Let us study now the above quantum groups, with the techniques that we have. As a first observation, we have embeddings of compact quantum groups, as follows:

$$\begin{tikzcd} U_N \arrow[r] & U_N^+ \\ O_N \arrow[u] \arrow[r] & O_N^+ \arrow[u] \end{tikzcd}$$

The basic properties of $O_N^+, U_N^+$ can be summarized as follows:

**Theorem 2.17.** *The quantum groups $O_N^+, U_N^+$ have the following properties:*

1. *The closed subgroups $G \subset U_N^+$ are exactly the $N \times N$ compact quantum groups. As for the closed subgroups $G \subset O_N^+$, these are those satisfying $u = \bar{u}$.*
2. *We have liberation embeddings $O_N \subset O_N^+$ and $U_N \subset U_N^+$, obtained by dividing the algebras $C(O_N^+), C(U_N^+)$ by their respective commutator ideals.*
3. *We have as well embeddings $\widehat{L}_N \subset O_N^+$ and $\widehat{F}_N \subset U_N^+$, where $L_N$ is the free product of $N$ copies of $\mathbb{Z}_2$, and where $F_N$ is the free group on $N$ generators.*

*Proof.* All these assertions are elementary, as follows:

(1) This is clear from definitions, and from Proposition 2.7.

(2) This follows from the Gelfand theorem, which shows that we have presentation results for $C(O_N),C(U_N)$ as follows, similar to those in Theorem 2.16:

$$\begin{aligned} C(O_N) &= C^*_{comm}\left((u_{ij})_{i,j=1,\ldots,N}\Big\|u=\bar{u},u^t=u^{-1}\right)\\ C(U_N) &= C^*_{comm}\left((u_{ij})_{i,j=1,\ldots,N}\Big\|u^*=u^{-1},u^t=\bar{u}^{-1}\right)\end{aligned}$$

(3) This follows from (1) and from Proposition 2.4 above, with the remark that with $u=diag(g_1,\ldots,g_N)$, the condition $u=\bar{u}$ is equivalent to $g_i^2=1$, for any $i$. □

The last assertion in Theorem 2.17 suggests the following construction:

**Proposition 2.18.** *Given a closed subgroup $G\subset U_N^+$, consider its "diagonal torus", which is the closed subgroup $T\subset G$ constructed as follows:*

$$C(T)=C(G)\Big/\left\langle u_{ij}=0\Big\|\forall i\neq j\right\rangle$$

*This torus is then a group dual, $T=\widehat{\Lambda}$, where $\Lambda=<g_1,\ldots,g_N>$ is the discrete group generated by the elements $g_i=u_{ii}$, which are unitaries inside $C(T)$.*

*Proof.* Since $u$ is unitary, its diagonal entries $g_i=u_{ii}$ are unitaries inside $C(T)$. Moreover, from $\Delta(u_{ij})=\sum_k u_{ik}\otimes u_{kj}$ we obtain, when passing inside the quotient:

$$\Delta(g_i)=g_i\otimes g_i$$

It follows that we have $C(T)=C^*(\Lambda)$, modulo identifying as usual the $C^*$-completions of the various group algebras, and so that we have $T=\widehat{\Lambda}$, as claimed. □

With this notion in hand, Theorem 2.17 (3) tells us that the diagonal tori of $O_N^+,U_N^+$ are the group duals $\widehat{L}_N,\widehat{F}_N$. There is an obvious relation here with the noncommutative geometry considerations from section 1 above, that we will analyse later on. Here is now a more subtle result on $O_N^+,U_N^+$, having no classical counterpart:

**Proposition 2.19.** *Consider the quantum groups $O_N^+,U_N^+$, with the corresponding fundamental corepresentations denoted $v,u$, and let $z=id\in C(\mathbb{T})$.*

1. *We have a morphism $C(U_N^+)\to C(\mathbb{T})*C(O_N^+)$, given by $u=zv$.*
2. *In other words, we have a quantum group embedding $\widetilde{O_N^+}\subset U_N^+$.*
3. *This embedding is an isomorphism at the level of the diagonal tori.*

*Proof.* The first two assertions follow from Proposition 2.12 above, or simply from the fact that $u=zv$ is biunitary. As for the third assertion, the idea here is that we have a similar model for the free group $F_N$, which is well-known to be faithful, $F_N\subset\mathbb{Z}*L_N$. □

We will be back to the above morphism later on, with a proof of its faithfulness, after performing a suitable GNS construction, with respect to the Haar functionals.

Let us construct now some more examples of compact quantum groups. As a basic construction here, coming however from the work in [27], [38], [41], [49], which is quite advanced, we can introduce some intermediate liberations, as follows:

**Proposition 2.20.** *We have intermediate quantum groups as follows,*

$$\begin{tikzcd} U_N \arrow[r] & U_N^* \arrow[r] & U_N^+ \\ O_N \arrow[r] \arrow[u] & O_N^* \arrow[r] \arrow[u] & O_N^+ \arrow[u] \end{tikzcd}$$

*with $*$ standing for the fact that $u_{ij}, u_{ij}^*$ must satisfy the relations $abc = cba$.*

*Proof.* This is elementary, by using the fact that if the entries of $u = (u_{ij})$ half-commute, then so do the entries of the following matrices:

$$u_{ij}^\Delta = \sum_k u_{ik} \otimes u_{kj} \quad , \quad u_{ij}^\varepsilon = \delta_{ij} \quad , \quad u_{ij}^S = u_{ji}^*$$

Thus, we have indeed morphisms $\Delta, \varepsilon, S$, as in Definition 2.1. See [38], [41]. ☐

In the same spirit, we have as well intermediate spheres as follows, with the symbol $*$ standing for the fact that $x_i, x_i^*$ must satisfy the relations $abc = cba$:

$$\begin{tikzcd} S^{N-1}_{\mathbb{C}} \arrow[r] & S^{N-1}_{\mathbb{C},*} \arrow[r] & S^{N-1}_{\mathbb{C},+} \\ S^{N-1}_{\mathbb{R}} \arrow[r] \arrow[u] & S^{N-1}_{\mathbb{R},*} \arrow[r] \arrow[u] & S^{N-1}_{\mathbb{R},+} \arrow[u] \end{tikzcd}$$

At the level of the diagonal tori, we have the following result:

**Theorem 2.21.** *The tori of the basic spheres and quantum groups are as follows,*

$$\begin{tikzcd} \widehat{\mathbb{Z}^N} \arrow[r] & \widehat{\mathbb{Z}^{\circ N}} \arrow[r] & \widehat{\mathbb{Z}^{*N}} \\ \widehat{\mathbb{Z}_2^N} \arrow[r] \arrow[u] & \widehat{\mathbb{Z}_2^{\circ N}} \arrow[r] \arrow[u] & \widehat{\mathbb{Z}_2^{*N}} \arrow[u] \end{tikzcd}$$

*with $\circ$ standing for the half-classical product operation for groups.*

*Proof.* The idea here is as follows:

(1) The result on the left is well-known.

(2) The result on the right follows from Theorem 2.17 (3).

(3) The middle result follows as well, by imposing the relations $abc = cba$. $\square$

Let us discuss now the relation with the noncommutative spheres. Having the things started here is a bit tricky, and as a main source of inspiration, we have:

**Proposition 2.22.** *Given an algebraic manifold $X \subset S^{N-1}_{\mathbb{C}}$, the formula*

$$G(X) = \left\{U \in U_N \Big| U(X) = X\right\}$$

*defines a compact group of unitary matrices (a.k.a. isometries), called affine isometry group of $X$. For the spheres $S^{N-1}_{\mathbb{R}}, S^{N-1}_{\mathbb{C}}$ we obtain in this way the groups $O_N, U_N$.*

*Proof.* The fact that $G(X)$ as defined above is indeed a group is clear, its compactness is clear as well, and finally the last assertion is clear as well. In fact, all this works for any closed subset $X \subset \mathbb{C}^N$, but we are not interested here in such general spaces. $\square$

We have the following quantum analogue of the above construction:

**Proposition 2.23.** *Given an algebraic manifold $X \subset S^{N-1}_{\mathbb{C},+}$, the category of the closed subgroups $G \subset U_N^+$ acting affinely on $X$, in the sense that the formula*

$$\Phi(x_i) = \sum_j x_j \otimes u_{ji}$$

*defines a morphism of $C^*$-algebras as follows,*

$$\Phi : C(X) \to C(X) \otimes C(G)$$

*has a universal object, denoted $G^+(X)$, and called affine quantum isometry group of $X$.*

*Proof.* Observe first that in the case where the above morphism $\Phi$ exists, this morphism is automatically a coaction, in the sense that it satisfies the following conditions:

$$(\Phi \otimes id)\Phi = (id \otimes \Delta)\Phi$$

$$(id \otimes \varepsilon)\Phi = id$$

In order to prove now the result, assume that $X \subset S^{N-1}_{\mathbb{C},+}$ comes as follows:

$$C(X) = C(S^{N-1}_{\mathbb{C},+}) \Big/ \Big\langle f_\alpha(x_1, \ldots, x_N) = 0 \Big\rangle$$

Consider now the following variables:

$$X_i = \sum_j x_j \otimes u_{ji} \in C(X) \otimes C(U_N^+)$$

Our claim is that $G = G^+(X)$ in the statement appears as follows:

$$C(G) = C(U_N^+) \Big/ \Big\langle f_\alpha(X_1, \ldots, X_N) = 0 \Big\rangle$$

In order to prove this claim, we have to clarify how the relations $f_\alpha(X_1,\ldots,X_N)=0$ are interpreted inside $C(U_N^+)$, and then show that $G$ is indeed a quantum group.

So, pick one of the defining polynomials, $f=f_\alpha$, and write it as follows:

$$f(x_1,\ldots,x_N)=\sum_r\sum_{i_1^r\ldots i_{s_r}^r}\lambda_r\cdot x_{i_1^r}\ldots x_{i_{s_r}^r}$$

With $X_i=\sum_j x_j\otimes u_{ji}$ as above, we have the following formula:

$$f(X_1,\ldots,X_N)=\sum_r\sum_{i_1^r\ldots i_{s_r}^r}\lambda_r\sum_{j_1^r\ldots j_{s_r}^r}x_{j_1^r}\ldots x_{j_{s_r}^r}\otimes u_{j_1^r i_1^r}\ldots u_{j_{s_r}^r i_{s_r}^r}$$

Since the variables on the right span a certain finite dimensional space, the relations $f(X_1,\ldots,X_N)=0$ correspond to certain relations between the variables $u_{ij}$.

Thus, we have indeed a closed subspace $G\subset U_N^+$, coming with a universal map:

$$\Phi:C(X)\to C(X)\otimes C(G)$$

In order to show now that $G$ is a quantum group, consider the following elements:

$$u_{ij}^\Delta=\sum_k u_{ik}\otimes u_{kj}\quad,\quad u_{ij}^\varepsilon=\delta_{ij}\quad,\quad u_{ij}^S=u_{ji}^*$$

Consider the following associated elements, with $\gamma\in\{\Delta,\varepsilon,S\}$:

$$X_i^\gamma=\sum_j x_j\otimes u_{ji}^\gamma$$

From the relations $f(X_1,\ldots,X_N)=0$ we deduce that we have:

$$f(X_1^\gamma,\ldots,X_N^\gamma)=(id\otimes\gamma)f(X_1,\ldots,X_N)=0$$

Thus, for any $\gamma\in\{\Delta,\varepsilon,S\}$, we can map $u_{ij}\to u_{ij}^\gamma$. It follows that $G$ is indeed a quantum group, and we are done. $\square$

We can formulate a quantum isometry group result, from [4], as follows:

**Theorem 2.24.** *The quantum isometry groups of the basic spheres are*

$$\begin{array}{ccccc}U_N&\longrightarrow&U_N^*&\longrightarrow&U_N^+\\ \uparrow&&\uparrow&&\uparrow\\ O_N&\longrightarrow&O_N^*&\longrightarrow&O_N^+\end{array}$$

*modulo identifying, as usual, the various $C^*$-algebraic completions.*

*Proof.* Let us first construct an action $U_N^+ \curvearrowright S^{N-1}_{\mathbb{C},+}$. We must prove here that the variables $X_i=\sum_j x_j\otimes u_{ji}$ satisfy the defining relations for $S^{N-1}_{\mathbb{C},+}$, namely:

$$\sum_i x_ix_i^*=\sum_i x_i^*x_i=1$$

But this follows from the biunitarity of $u$. We have indeed:

$$\begin{aligned}\sum_i X_iX_i^* &= \sum_{ijk}x_jx_k^*\otimes u_{ji}u_{ki}^*\\ &= \sum_j x_jx_j^*\otimes 1\\ &= 1\otimes 1\end{aligned}$$

In the other sense the computation is similar, as follows:

$$\begin{aligned}\sum_i X_i^*X_i &= \sum_{ijk}x_j^*x_k\otimes u_{ji}^*u_{ki}\\ &= \sum_j x_j^*x_j\otimes 1\\ &= 1\otimes 1\end{aligned}$$

Regarding now $O_N^+ \curvearrowright S^{N-1}_{\mathbb{R},+}$, here we must check the extra relations $X_i=X_i^*$, and these are clear from $u_{ia}=u_{ia}^*$. Finally, regarding the remaining actions, the verifications are clear as well, because if the coordinates $u_{ia}$ and $x_a$ are subject to commutation relations of type $ab=ba$, or of type $abc=cba$, then so are the variables $X_i=\sum_j x_j\otimes u_{ji}$.

We must prove now that all these actions are universal:

$\underline{S^{N-1}_{\mathbb{R},+},S^{N-1}_{\mathbb{C},+}}$. The universality of $U_N^+ \curvearrowright S^{N-1}_{\mathbb{C},+}$ is trivial by definition. As for the universality of $O_N^+ \curvearrowright S^{N-1}_{\mathbb{R},+}$, this comes from the fact that $X_i=X_i^*$, with $X_i=\sum_j x_j\otimes u_{ji}$ as above, gives $u_{ia}=u_{ia}^*$. Thus $G \curvearrowright S^{N-1}_{\mathbb{R},+}$ implies $G\subset O_N^+$, as desired.

$\underline{S^{N-1}_{\mathbb{R}},S^{N-1}_{\mathbb{C}}}$. We use here a trick from [43]. Assuming first that we have an action $G \curvearrowright S^{N-1}_{\mathbb{R}}$, consider the following variables:

$$w_{kl,ij}=u_{ki}u_{lj}$$

$$p_{ij}=x_ix_j$$

In terms of these variables, which can be thought of as being projective coordinates, the corresponding projective coaction map is given by:

$$\Phi(p_{ij})=\sum_{kl}p_{kl}\otimes w_{kl,ij}$$

We have the following formulae:

$$\begin{aligned}\Phi(p_{ij}) &= \sum_{k<l} p_{kl}\otimes(w_{kl,ij}+w_{lk,ij})+\sum_k p_{kk}\otimes w_{kk,ij}\\ \Phi(p_{ji}) &= \sum_{k<l} p_{kl}\otimes(w_{kl,ji}+w_{lk,ji})+\sum_k p_{kk}\otimes w_{kk,ji}\end{aligned}$$

By comparing these two formulae, and then by using the linear independence of the variables $p_{kl}=x_kx_l$ with $k\leq l$, we conclude that we must have:

$$w_{kl,ij}+w_{lk,ij}=w_{kl,ji}+w_{lk,ji}$$

Following now a well-known trick from [43], let us apply the antipode to this formula. For this purpose, observe first that we have:

$$S(w_{kl,ij})=S(u_{ki}u_{lj})=S(u_{lj})S(u_{ki})=u_{jl}u_{ik}=w_{ji,lk}$$

Thus by applying the antipode we obtain:

$$w_{ji,lk}+w_{ji,kl}=w_{ij,lk}+w_{ij,kl}$$

By relabelling the indices, we obtain from this:

$$w_{kl,ij}+w_{kl,ji}=w_{lk,ij}+w_{lk,ji}$$

Now by comparing with the original relation, we obtain:

$$w_{lk,ij}=w_{kl,ji}$$

But, recalling that we have $w_{kl,ij}=u_{ki}u_{lj}$, this formula reads:

$$u_{li}u_{kj}=u_{kj}u_{li}$$

We therefore conclude we have $G\subset O_N$, as claimed. The proof of the universality of the action $U_N\curvearrowright S^{N-1}_{\mathbb{C}}$ is similar.

<u>$S^{N-1}_{\mathbb{R},*},S^{N-1}_{\mathbb{C},*}$</u>. Assume that we have an action $G\curvearrowright S^{N-1}_{\mathbb{C},*}$. From $\Phi(x_a)=\sum_i x_i\otimes u_{ia}$ we obtain then that, with $p_{ab}=z_a\bar{z}_b$, we have:

$$\Phi(p_{ab})=\sum_{ij} p_{ij}\otimes u_{ia}u_{jb}^*$$

By multiplying these two formulae, we obtain:

$$\begin{aligned}\Phi(p_{ab}p_{cd}) &= \sum_{ijkl} p_{ij}p_{kl}\otimes u_{ia}u_{jb}^*u_{kc}u_{ld}^*\\ \Phi(p_{ad}p_{cb}) &= \sum_{ijkl} p_{il}p_{kj}\otimes u_{ia}u_{ld}^*u_{kc}u_{jb}^*\end{aligned}$$

The left terms being equal, and the first terms on the right being equal too, we deduce that, with $[a,b,c]=abc-cba$, we must have the following equality:

$$\sum_{ijkl}p_{ij}p_{kl}\otimes u_{ia}[u_{jb}^*,u_{kc},u_{ld}^*]=0$$

Since the variables $p_{ij}p_{kl}=z_i\bar{z}_jz_k\bar{z}_l$ depend only on $|\{i,k\}|,|\{j,l\}|\in\{1,2\}$, and this dependence produces the only relations between them, we are led to 4 equations:

(1) $u_{ia}[u_{jb}^*,u_{ka},u_{lb}^*]=0$, $\forall a,b$.

(2) $u_{ia}[u_{jb}^*,u_{ka},u_{ld}^*]+u_{ia}[u_{jd}^*,u_{ka},u_{lb}^*]=0$, $\forall a$, $\forall b\neq d$.

(3) $u_{ia}[u_{jb}^*,u_{kc},u_{lb}^*]+u_{ic}[u_{jb}^*,u_{ka},u_{lb}^*]=0$, $\forall a\neq c$, $\forall b$.

(4) $u_{ia}([u_{jb}^*,u_{kc},u_{ld}^*]+[u_{jd}^*,u_{kc},u_{lb}^*])+u_{ic}([u_{jb}^*,u_{ka},u_{ld}^*]+[u_{jd}^*,u_{ka},u_{lb}^*])=0,\forall a\neq c,\forall b\neq d$.

From (1,2) we conclude that (2) holds with no restriction on the indices. By multiplying now this formula to the left by $u_{ia}^*$, and then summing over $i$, we obtain:

$$[u_{jb}^*,u_{ka},u_{ld}^*]+[u_{jd}^*,u_{ka},u_{lb}^*]=0$$

By applying now the antipode, then the involution, and finally by suitably relabelling all the indices, we successively obtain from this formula:

$$\begin{aligned}
&[u_{dl},u_{ak}^*,u_{bj}]+[u_{bl},u_{ak}^*,u_{dj}]=0\\
\implies\quad&[u_{dl}^*,u_{ak},u_{bj}^*]+[u_{bl}^*,u_{ak},u_{dj}^*]=0\\
\implies\quad&[u_{ld}^*,u_{ka},u_{jb}^*]+[u_{jd}^*,u_{ka},u_{lb}^*]=0
\end{aligned}$$

Now by comparing with the original relation, above, we conclude that we have:

$$[u_{jb}^*,u_{ka},u_{ld}^*]=[u_{jd}^*,u_{ka},u_{lb}^*]=0$$

Thus we have reached to the formulae defining $U_N^*$, and we are done.

Finally, in what regards the universality of $O_N^*\curvearrowright S^{N-1}_{\mathbb{R},*}$, this follows from the universality of $U_N^*\curvearrowright S^{N-1}_{\mathbb{C},*}$ and of $O_N^+\curvearrowright S^{N-1}_{\mathbb{R},+}$, and from $U_N^*\cap O_N^+=O_N^*$. $\square$

As a conclusion to all this, we have now a simple and reliable definition for the compact quantum groups, in the Lie case, namely $G\subset U_N^+$, covering all the compact Lie groups, $G\subset U_N$, covering as well all the duals $\widehat{\Gamma}$ of the finitely generated groups, $F_N\to\Gamma$, and allowing the construction of several interesting examples, such as $O_N^+,U_N^+$.

In respect to the noncommutative geometry questions raised in section 1 above, we have some advances. In order to further advance, we would need representation theory results, in the spirit of [144], for our quantum isometry groups.

## 3. Representation theory

In order to reach to some more advanced insight into the structure of the compact quantum groups, we can use representation theory. We follow Woronowicz's paper [148], with a few simplifications coming from our $S^2 = id$ formalism. We first have:

**Definition 3.1.** *A corepresentation of a Woronowicz algebra $(A, u)$ is a unitary matrix $v \in M_n(\mathcal{A})$ over the dense $*$-algebra of smooth elements $\mathcal{A} =< u_{ij} >$, satisfying:*

$$\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj}$$

$$\varepsilon(v_{ij}) = \delta_{ij}$$

$$S(v_{ij}) = v_{ji}^*$$

*That is, $v$ must satisfy the same conditions as $u$.*

As basic examples here, we have the trivial corepresentation, having dimension 1, as well as the fundamental corepresentation, and its adjoint:

$$1 = (1) \quad , \quad u = (u_{ij}) \quad , \quad \bar{u} = (u_{ij}^*)$$

In the classical case, we recover in this way the usual representations of $G$:

**Proposition 3.2.** *Given a closed subgroup $G \subset U_N$, the corepresentations of the associated Woronowicz algebra $C(G)$ are in one-to-one correspondence, given by*

$$\pi(g) = \begin{pmatrix} v_{11}(g) & \ldots & v_{1n}(g) \\ \vdots & & \vdots \\ v_{n1}(g) & \ldots & v_{nn}(g) \end{pmatrix}$$

*with the finite dimensional unitary smooth representations of $G$.*

*Proof.* We recall from section 2 that any closed subgroup $G \subset U_N$ is a Lie group. Thus, the corepresentations that we are interested in are certain matrices $v \in M_n(C^\infty(G))$. With this observation in hand, the fact that we have a correspondence $v \leftrightarrow \pi$ as in the statement is clear, by using the computations from section 2, performed when proving that any closed subgroup $G \subset U_N$ is indeed a compact quantum group. □

In general now, we have the following operations on the corepresentations:

**Proposition 3.3.** *The corepresentations are subject to the following operations:*

1. *Making sums, $v + w = diag(v, w)$.*
2. *Making tensor products, $(v \otimes w)_{ia,jb} = v_{ij} w_{ab}$.*
3. *Taking conjugates, $(\bar{v})_{ij} = v_{ij}^*$.*

*Proof.* Observe that the result holds in the commutative case, where we obtain the usual operations on the representations of the corresponding group. In general now:

(1) Everything here is clear, as already mentioned in section 2 above, when using such corepresentations in order to construct quantum group quotients.

(2) First of all, the matrix $v\otimes w$ is unitary. Indeed, we have:

$$\begin{aligned}\sum_{jb}(v\otimes w)_{ia,jb}(v\otimes w)^*_{kc,jb} &= \sum_{jb}v_{ij}w_{ab}w^*_{cb}v^*_{kj}\\ &= \delta_{ac}\sum_j v_{ij}v^*_{kj}\\ &= \delta_{ik}\delta_{ac}\end{aligned}$$

In the other sense, the computation is similar, as follows:

$$\begin{aligned}\sum_{jb}(v\otimes w)^*_{jb,ia}(v\otimes w)_{jb,kc} &= \sum_{jb}w^*_{ba}v^*_{ji}v_{jk}w_{bc}\\ &= \delta_{ik}\sum_b w^*_{ba}w_{bc}\\ &= \delta_{ik}\delta_{ac}\end{aligned}$$

The comultiplicativity condition follows from the following computation:

$$\begin{aligned}\Delta((v\otimes w)_{ia,jb}) &= \sum_{kc}v_{ik}w_{ac}\otimes v_{kj}w_{cb}\\ &= \sum_{kc}(v\otimes w)_{ia,kc}\otimes(v\otimes w)_{kc,jb}\end{aligned}$$

The proof of the counitality condition is similar, as follows:

$$\varepsilon((v\otimes w)_{ia,jb})=\delta_{ij}\delta_{ab}=\delta_{ia,jb}$$

As for the condition involving the antipode, this can be checked as follows:

$$S((v\otimes w)_{ia,jb})=w^*_{ba}v^*_{ji}=(v\otimes w)^*_{jb,ia}$$

(3) In order to check that $\bar{v}$ is unitary, we can use the antipode, exactly as we did in section 2 above, for $\bar{u}$. As for the comultiplicativity axioms, these are all clear. $\square$

We have as well the following supplementary operation:

**Proposition 3.4.** *Given a corepresentation $v\in M_n(A)$, its spinned version*

$$w=UvU^*$$

*is a corepresentation as well, for any unitary matrix $U\in U_n$.*

*Proof.* The matrix $w$ is unitary, and its comultiplicativity properties can be checked by doing some computations. Here is however another proof of this fact, using a useful trick. In the context of Definition 3.1, if we write $v \in M_n(\mathbb{C}) \otimes A$, the axioms read:

$$(id \otimes \Delta)v = v_{12}v_{13} \quad , \quad (id \otimes \varepsilon)v = 1 \quad (id \otimes S)v = v^*$$

Here we use standard tensor calculus conventions. Now when spinning by a unitary the matrix that we obtain, with these conventions, is $w = U_1 v U_1^*$, and we have:

$$\begin{aligned}(id \otimes \Delta)w &= U_1 v_{12} v_{13} U_1^* \\ &= U_1 v_{12} U_1^* \cdot U_1 v_{13} U_1^* \\ &= w_{12} w_{13}\end{aligned}$$

The proof of the counitality condition is similar, as follows:

$$(id \otimes \varepsilon)w = U \cdot 1 \cdot U = 1$$

Finally, the last condition, involving the antipode, can be checked as follows:

$$(id \otimes S)w = U_1 v^* U_1^* = w^*$$

Thus, with usual notations, $w = UvU^*$ is a corepresentation, as claimed. □

As a philosophical comment, the above proof might suggest that the more abstract our notations and formalism, the easier our problems will become. This is wrong. Bases and indices are a blessing: they can be understood by undergraduate students, computers, fellow scientists, engineers, and of course also by yourself, when you're tired or so.

In addition, in the quantum group context, we will see later on, starting from section 4 below, that bases and indices can be turned into something very beautiful and powerful, allowing us to do some serious theory, well beyond the level of abstractions.

Back to work now, in the group dual case, we have the following result:

**Proposition 3.5.** *Assume $A = C^*(\Gamma)$, with $\Gamma =< g_1, \ldots, g_N >$ being a discrete group.*

1. *Any group element $h \in \Gamma$ is a 1-dimensional corepresentation of $A$, and the operations on corepresentations are the usual ones on group elements.*
2. *Any diagonal matrix of type $v = diag(h_1, \ldots, h_n)$, with $n \in \mathbb{N}$ arbitrary, and with $h_1, \ldots, h_n \in \Gamma$, is a corepresentation of $A$.*
3. *More generally, any matrix of type $w = Udiag(h_1, \ldots, h_n)U^*$ with $h_1, \ldots, h_n \in \Gamma$ and with $U \in U_n$, is a corepresentation of $A$.*

*Proof.* These assertions are all elementary, as follows:

(1) The first assertion is clear from definitions and from the comultiplication, counit and antipode formulae for the discrete group algebras, namely:

$$\Delta(h) = h \otimes h \quad , \quad \varepsilon(h) = 1 \quad , \quad S(h) = h^{-1}$$

The assertion on the operations is clear too, because we have:

$$(g)\otimes(h)=(gh)\quad,\quad\overline{(g)}=(g^{-1})$$

(2) This follows from (1) by performing sums, as in Proposition 3.3 above.

(3) This follows from (2) and from the fact that we can conjugate any corepresentation by a unitary matrix, as explained in Proposition 3.4 above. □

Observe that the class of corepresentations in (3) is stable under all the operations from Propositions 3.3 and 3.4. When $\Gamma$ is abelian we can apply Proposition 3.2 with $G=\widehat{\Gamma}$, and after performing a number of identifications, we conclude that these are all the corepresentations of $C^*(\Gamma)$. We will see later that this holds in fact for any $\Gamma$.

Let us go back now to the general case. Our next definition is:

**Definition 3.6.** *Given two corepresentations $v\in M_n(A),w\in M_m(A)$, we set*

$$Hom(v,w)=\left\{T\in M_{m\times n}(\mathbb{C})\Big|Tv=wT\right\}$$

*and we use the following conventions:*

1. *We use the notations $Fix(v)=Hom(1,v)$, and $End(v)=Hom(v,v)$.*
2. *We write $v\sim w$ when $Hom(v,w)$ contains an invertible element.*
3. *We say that $v$ is irreducible, and write $v\in Irr(G)$, when $End(v)=\mathbb{C}1$.*

In the classical case $A=C(G)$ we obtain the usual notions concerning the representations. Observe also that in the group dual case we have:

$$g\sim h\iff g=h$$

Finally, observe that $v\sim w$ means that $v,w$ are conjugated by an invertible matrix. Here are a few basic results, regarding the above Hom spaces:

**Proposition 3.7.** *We have the following results:*

1. *$T\in Hom(u,v),S\in Hom(v,w)\implies ST\in Hom(u,w)$.*
2. *$S\in Hom(p,q),T\in Hom(v,w)\implies S\otimes T\in Hom(p\otimes v,q\otimes w)$.*
3. *$T\in Hom(v,w)\implies T^*\in Hom(w,v)$.*

*In other words, the Hom spaces form a tensor $*$-category.*

*Proof.* The proofs are all elementary, as follows:

(1) By using our assumptions $Tu=vT$ and $Sv=Ws$ we obtain, as desired:

$$STu=SvT=wST$$

(2) Assume indeed that we have $Sp=qS$ and $Tv=wT$. With tensor product notations, as in the proof of Proposition 3.4 above, we have:

$$(S\otimes T)(p\otimes v)=S_1T_2p_{13}v_{23}=(Sp)_{13}(Tv)_{23}$$

We have as well the following computation:

$$(q \otimes w)(S \otimes T) = q_{13}w_{23}S_1T_2 = (qS)_{13}(wT)_{23}$$

The quantities on the right being equal, this gives the result.

(3) By conjugating, and then using the unitarity of $v, w$, we obtain, as desired:

$$\begin{aligned} Tv = wT &\implies v^*T^* = T^*w^* \\ &\implies vv^*T^*w = vT^*w^*w \\ &\implies T^*w = vT^* \end{aligned}$$

Finally, the last assertion follows from definitions, and from the obvious fact that, in addition to (1,2,3) above, the Hom spaces are linear spaces, and contain the units. In short, this is just a theoretical remark, that will be used only later on. $\square$

As a main consequence of the above result, the spaces $End(v) \subset M_n(\mathbb{C})$ are unital subalgebras stable under the involution $*$, and so are $C^*$-algebras. In order to exploit this fact, we will need a basic result, complementing the operator algebra theory presented in section 1 above, namely:

**Theorem 3.8.** *Let $B \subset M_n(\mathbb{C})$ be a $C^*$-algebra.*

1. *We can write $1 = p_1 + \ldots + p_k$, with $p_i \in B$ central minimal projections.*
2. *Each of the linear spaces $B_i = p_iBp_i$ is a non-unital $*$-subalgebra of $B$.*
3. *We have a non-unital $*$-algebra sum decomposition $B = B_1 \oplus \ldots \oplus B_k$.*
4. *We have unital $*$-algebra isomorphisms $B_i \simeq M_{r_i}(\mathbb{C})$, where $r_i = rank(p_i)$.*
5. *Thus, we have a $C^*$-algebra isomorphism $B \simeq M_{r_1}(\mathbb{C}) \oplus \ldots \oplus M_{r_k}(\mathbb{C})$.*

*In addition, the final conclusion holds for any finite dimensional $C^*$-algebra.*

*Proof.* This is something well-known, with the proof of the various assertions in the statement being something elementary, and routine:

(1) This is more of a definition.

(2) This is elementary, coming from $p_i^2 = p_i = p_i^*$.

(3) The verification of the direct sum conditions is indeed elementary.

(4) This follows from the fact that each $p_i$ was assumed to be central and minimal.

(5) This follows by putting everything together.

As for the last assertion, this follows from (5) by using the GNS representation theorem, which provides us with an embedding $B \subset M_n(\mathbb{C})$, for some $n \in \mathbb{N}$. $\square$

Following Woronowicz's paper [148], we can now formulate a first Peter-Weyl theorem, and to be more precise a first such theorem from a 4-series, as follows:

**Theorem 3.9** (PW1)**.** *Let $v \in M_n(A)$ be a corepresentation, consider the $C^*$-algebra $B = End(v)$, and write its unit as $1 = p_1 + \ldots + p_k$, as above. We have then*

$$v = v_1 + \ldots + v_k$$

*with each $v_i$ being an irreducible corepresentation, obtained by restricting $v$ to $Im(p_i)$.*

*Proof.* This can be deduced from Theorem 3.8 above, as follows:

(1) We first associate to our corepresentation $v \in M_n(A)$ the corresponding coaction map $\Phi : \mathbb{C}^n \to A \otimes \mathbb{C}^n$, given by $\Phi(e_i) = \sum_j v_{ij} \otimes e_j$. We say that a linear subspace $V \subset \mathbb{C}^n$ is invariant if $\Phi(V) \subset A \otimes V$. In this case, we can consider the restriction map $\Phi_{|V} : V \to A \otimes V$, which is a coaction too, coming from a subcorepresentation $w \subset v$.

(2) Consider now a projection $p \in End(v)$. From $pv = vp$ we obtain that the linear space $V = Im(p)$ is invariant under $v$, and so this space must come from a subcorepresentation $w \subset v$. It is routine to check that the operation $p \to w$ maps subprojections to subcorepresentations, and minimal projections to irreducible corepresentations.

(3) With these preliminaries in hand, let us decompose the algebra $End(v)$ as in Theorem 3.8 above. Consider now the vector spaces $V_i = Im(p_i)$. If we denote by $v_i \subset v$ the subcorepresentations coming from these vector spaces, then we obtain in this way a decomposition $v = v_1 + \ldots + v_k$, as in the statement. □

In order to formulate our second Peter-Weyl type theorem, we will need:

**Definition 3.10.** *We denote by $u^{\otimes k}$, with $k = \circ \bullet \bullet \circ \ldots$ being a colored integer, the various tensor products between $u, \bar{u}$, indexed according to the rules*

$$u^{\otimes \emptyset} = 1 \quad , \quad u^{\otimes \circ} = u \quad , \quad u^{\otimes \bullet} = \bar{u}$$

*and multiplicativity, $u^{\otimes kl} = u^{\otimes k} \otimes u^{\otimes l}$, and call them Peter-Weyl corepresentations.*

Here are a few examples of such corepresentations, namely those coming from the colored integers of length 2, to be often used in what follows:

$$u^{\otimes \circ\circ} = u \otimes u \quad , \quad u^{\otimes \circ\bullet} = u \otimes \bar{u}$$

$$u^{\otimes \bullet\circ} = \bar{u} \otimes u \quad , \quad u^{\otimes \bullet\bullet} = \bar{u} \otimes \bar{u}$$

There are some particular cases of interest, where simplifications appear:

**Proposition 3.11.** *The Peter-Weyl corepresentations $u^{\otimes k}$ are as follows:*

(1) *In the real case, $u = \bar{u}$, we can assume $k \in \mathbb{N}$.*
(2) *In the classical case, we can assume, up to equivalence, $k \in \mathbb{N} \times \mathbb{N}$.*

*Proof.* These assertions are both elementary, as follows:

(1) Here we have indeed $u^{\otimes k} = u^{\otimes |k|}$, where $|k| \in \mathbb{N}$ is the length. Thus the Peter-Weyl corepresentations are indexed by $\mathbb{N}$, as claimed.

(2) In the classical case, our claim is that we have equivalences $v \otimes w \sim w \otimes v$, implemented by the flip operator $\Sigma(a \otimes b) = b \otimes a$. Indeed, we have:

$$v \otimes w = v_{13}w_{23} = w_{23}v_{13} = \Sigma w_{13}v_{23}\Sigma = \Sigma(w \otimes v)\Sigma$$

In particular we have an equivalence $u \otimes \bar{u} \sim \bar{u} \otimes u$. We conclude that the Peter-Weyl corepresentations are the corepresentations of type $u^{\otimes k} \otimes \bar{u}^{\otimes l}$, with $k, l \in \mathbb{N}$. □

Here is the second Peter-Weyl theorem, also from [148], complementing Theorem 3.9:

**Theorem 3.12** (PW2)**.** *Each irreducible corepresentation of $A$ appears as:*

$$v \subset u^{\otimes k}$$

*That is, $v$ appears inside a certain Peter-Weyl corepresentation.*

*Proof.* Given an arbitrary corepresentation $v \in M_n(A)$, consider its space of coefficients, $C(v) = span(v_{ij})$. It is routine to check that the construction $v \to C(v)$ is functorial, in the sense that it maps subcorepresentations into subspaces.

By definition of the Peter-Weyl corepresentations, we have:

$$\mathcal{A} = \sum_{k \in \mathbb{N} * \mathbb{N}} C(u^{\otimes k})$$

Now given a corepresentation $v \in M_n(A)$, the corresponding coefficient space is a finite dimensional subspace $C(v) \subset \mathcal{A}$, and so we must have, for certain $k_1, \ldots, k_p$:

$$C(v) \subset C(u^{\otimes k_1} \oplus \ldots \oplus u^{\otimes k_p})$$

We deduce from this that we have an inclusion of corepresentations, as follows:

$$v \subset u^{\otimes k_1} \oplus \ldots \oplus u^{\otimes k_p}$$

Together with Theorem 3.9, this leads to the conclusion in the statement. □

In order to further advance, with some finer results, we need to integrate over $G$. In the classical case the existence of such an integration is well-known, as follows:

**Proposition 3.13.** *Any commutative Woronowicz algebra, $A = C(G)$ with $G \subset U_N$, has a unique faithful positive unital linear form $\int_G : A \to \mathbb{C}$ satisfying*

$$\int_G f(xy)dx = \int_G f(yx)dx = \int_G f(x)dx$$

*called Haar integration. This Haar integration functional can be constructed by starting with any faithful positive unital form $\varphi \in A^*$, and taking the Cesàro limit*

$$\int_G = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \varphi^{*k}$$

*where the convolution operation for linear forms is given by $\phi * \psi = (\phi \otimes \psi)\Delta$.*

*Proof.* This is the existence theorem for the Haar measure of $G$, in functional analytic formulation. Observe first that the invariance conditions in the statement read:

$$d(xy)=d(yx)=dx\quad,\quad\forall y\in G$$

Thus, we are looking indeed for the integration with respect to the Haar measure on $G$. Now recall that this Haar measure exists, is unique, and can be constructed by starting with any probability measure $\mu$, and performing the following Cesàro limit:

$$dx=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n d\mu^{*k}(x)$$

In functional analysis terms, this corresponds precisely to the second assertion. □

In general now, let us start with a definition, as follows:

**Definition 3.14.** *Given an arbitrary Woronowicz algebra $A=C(G)$, any positive unital tracial state $\int_G:A\to\mathbb C$ subject to the invariance conditions*

$$\left(\int_G\otimes id\right)\Delta=\left(id\otimes\int_G\right)\Delta=\int_G(.)1$$

*is called Haar integration over $G$.*

As a first observation, in the commutative case, this notion agrees with the one in Proposition 3.13. To be more precise, Proposition 3.13 tells us that any commutative Woronowicz algebra has a Haar integration in the above sense, which is unique, and which can be constructed by performing the Cesàro limiting procedure there. Let us discuss now the group dual case. We have here the following result:

**Proposition 3.15.** *Given a discrete group $\Gamma=<g_1,\ldots,g_N>$, the Woronowicz algebra $A=C^*(\Gamma)$ has a Haar functional, given on the standard generators $g\in\Gamma$ by:*

$$\int_{\widehat{\Gamma}}g=\delta_{g,1}$$

*This functional is faithful on the image on $C^*(\Gamma)$ in the regular representation. Also, in the abelian case, we obtain in this way the counit of $C(\widehat{\Gamma})$.*

*Proof.* Consider indeed the left regular representation $\pi:C^*(\Gamma)\to B(l^2(\Gamma))$, given by $\pi(g)(h)=gh$, that we met in section 1. By composing with the functional $T\to<T1,1>$, the functional $\int_{\widehat{\Gamma}}$ that we obtain is given by the following formula:

$$\int_{\widehat{\Gamma}}g=<g1,1>=\delta_{g,1}$$

But this gives all the assertions in the statement, namely the existence, traciality, left and right invariance properties, and faithfulness on the reduced algebra. As for the last assertion, this is clear from the Pontrjagin duality isomorphism. □

In order to discuss now the general case, let us define the convolution operation for linear forms by $\phi * \psi = (\phi \otimes \psi)\Delta$. We have then the following result, from [148]:

**Proposition 3.16.** *Given an arbitrary unital linear form $\varphi \in A^*$, the limit*

$$\int_\varphi a = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n \varphi^{*k}(a)$$

*exists, and for a coefficient of a corepresentation $a = (\tau \otimes id)v$, we have*

$$\int_\varphi a = \tau(P)$$

*where $P$ is the orthogonal projection onto the 1-eigenspace of $(id \otimes \varphi)v$.*

*Proof.* By linearity, it is enough to prove the first assertion for elements of the following type, where $v$ is one of the Peter-Weyl corepresentations, and $\tau$ is a linear form:

$$a = (\tau \otimes id)v$$

Thus we are led into the second assertion, and more precisely we can have the whole result proved if we can establish the following formula, with $a = (\tau \otimes id)v$:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n \varphi^{*k}(a) = \tau(P)$$

In order to prove this latter formula, observe that we have:

$$\varphi^{*k}(a) = (\tau \otimes \varphi^{*k})v = \tau((id \otimes \varphi^{*k})v)$$

Consider now the following scalar matrix:

$$M = (id \otimes \varphi)v$$

In terms of this matrix, we have the following formula:

$$((id \otimes \varphi^{*k})v)_{i_0 i_{k+1}} = \sum_{i_1 \ldots i_k} M_{i_0 i_1} \ldots M_{i_k i_{k+1}} = (M^k)_{i_0 i_{k+1}}$$

Thus for any $k \in \mathbb{N}$ we have the following formula:

$$(id \otimes \varphi^{*k})v = M^k$$

It follows that our Cesàro limit is given by:

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n \varphi^{*k}(a) &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n \tau(M^k) \\
&= \tau\left(\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n M^k\right)
\end{aligned}$$

Now since $v$ is unitary we have $||v||=1$, and we conclude that we have $||M||\leq 1$. Thus, by standard calculus, the above Cesàro limit on the right exists, and equals the orthogonal projection onto the 1-eigenspace of $M$:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n M^k=P$$

Thus our initial Cesàro limit converges as well, to $\tau(P)$, as desired. □

When $\varphi$ is chosen faithful, we have the following finer result, also from [148]:

**Proposition 3.17.** *Given a faithful unital linear form $\varphi\in A^*$, the limit*

$$\int_\varphi a=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\varphi^{*k}(a)$$

*exists, and is independent of $\varphi$, given on coefficients of corepresentations by*

$$\left(id\otimes\int_\varphi\right)v=P$$

*where $P$ is the orthogonal projection onto $Fix(v)=\{\xi\in\mathbb C^n|v\xi=\xi\}$.*

*Proof.* In view of Proposition 3.16, it remains to prove that when $\varphi$ is faithful, the 1-eigenspace of $M=(id\otimes\varphi)v$ equals $Fix(v)$.

"$\supset$" This is clear, and for any $\varphi$, because we have $v\xi=\xi\implies M\xi=\xi$.

"$\subset$" Here we must prove that, when $\varphi$ is faithful, we have $M\xi=\xi\implies v\xi=\xi$. For this purpose, we use a standard trick. Assume that we have $M\xi=\xi$, and set:

$$a=\sum_i\left(\sum_j v_{ij}\xi_j-\xi_i\right)\left(\sum_k v_{ik}\xi_k-\xi_i\right)^*$$

We must prove that we have $a=0$. Since $v$ is biunitary, we have:

$$\begin{aligned}
a&=\sum_i\left(\sum_j\left(v_{ij}\xi_j-\frac{1}{N}\xi_i\right)\right)\left(\sum_k\left(v_{ik}^*\bar\xi_k-\frac{1}{N}\bar\xi_i\right)\right)\\
&=\sum_{ijk}v_{ij}v_{ik}^*\xi_j\bar\xi_k-\frac{1}{N}v_{ij}\xi_j\bar\xi_i-\frac{1}{N}v_{ik}^*\xi_i\bar\xi_k+\frac{1}{N^2}\xi_i\bar\xi_i\\
&=\sum_j|\xi_j|^2-\sum_{ij}v_{ij}\xi_j\bar\xi_i-\sum_{ik}v_{ik}^*\xi_i\bar\xi_k+\sum_i|\xi_i|^2\\
&=||\xi||^2-<v\xi,\xi>-\overline{<v\xi,\xi>}+||\xi||^2\\
&=2(||\xi||^2-Re(<v\xi,\xi>))
\end{aligned}$$

By using now our assumption $M\xi=\xi$, we obtain from this:

$$\begin{aligned}\varphi(a) &= 2\varphi(\|\xi\|^2 - Re(<v\xi,\xi>))\\ &= 2(\|\xi\|^2 - Re(<M\xi,\xi>))\\ &= 2(\|\xi\|^2 - \|\xi\|^2)\\ &= 0\end{aligned}$$

Now since $\varphi$ is faithful, this gives $a=0$, and so $v\xi=\xi$, as claimed. $\square$

We can now formulate the general Haar measure result, due to Woronowicz [148]:

**Theorem 3.18.** *Any Woronowicz algebra has a unique Haar integration, which can be constructed by starting with any faithful positive unital state $\varphi\in A^*$, and setting*

$$\int_G = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n \varphi^{*k}$$

*where $\phi*\psi=(\phi\otimes\psi)\Delta$. Moreover, for any corepresentation $v$ we have*

$$\left(id\otimes\int_G\right)v = P$$

*where $P$ is the orthogonal projection onto $Fix(v)=\{\xi\in\mathbb{C}^n|v\xi=\xi\}$.*

*Proof.* Let us first go back to the general context of Proposition 3.16 above. Since convolving one more time with $\varphi$ will not change the Cesàro limit appearing there, the functional $\int_\varphi\in A^*$ constructed there has the following invariance property:

$$\int_\varphi *\varphi = \varphi * \int_\varphi = \int_\varphi$$

In the case where $\varphi$ is assumed to be faithful, as in Proposition 3.17 above, our claim is that we have the following formula, valid this time for any $\psi\in A^*$:

$$\int_\varphi *\psi = \psi * \int_\varphi = \psi(1)\int_\varphi$$

It is enough to prove this formula on a coefficient of a corepresentation, $a=(\tau\otimes id)v$. In order to do so, consider the following matrices:

$$P=\left(id\otimes\int_\varphi\right)v \quad , \quad Q=(id\otimes\psi)v$$

In terms of these matrices, we have:

$$\left(\int_\varphi *\psi\right)a = \left(\tau\otimes\int_\varphi\otimes\psi\right)(v_{12}v_{13}) = \tau(PQ)$$

Similarly, we have the following computation:

$$\left(\psi*\int_\varphi\right)a=\left(\tau\otimes\psi\otimes\int_\varphi\right)(v_{12}v_{13})=\tau(QP)$$

Finally, regarding the term on the right, this is given by:

$$\psi(1)\int_\varphi a=\psi(1)\tau(P)$$

Thus, our claim is equivalent to the following equality:

$$PQ=QP=\psi(1)P$$

But this latter equality follows from the fact, coming from Proposition 3.17 above, that $P=(id\otimes\int_\varphi)v$ equals the orthogonal projection onto $Fix(v)$. Thus, we have proved our claim. Now observe that our formula can be written as:

$$\psi\left(\int_\varphi\otimes id\right)\Delta=\psi\left(id\otimes\int_\varphi\right)\Delta=\psi\int_\varphi(.)1$$

This formula being true for any $\psi\in A^*$, we can simply delete $\psi$, and we conclude that the invariance formula in Definition 3.14 holds indeed, with $\int_G=\int_\varphi$.

Finally, assuming that we have two invariant integrals $\int_G,\int_G'$, we have:

$$\left(\int_G\otimes\int_G'\right)\Delta=\left(\int_G'\otimes\int_G\right)\Delta=\int_G(.)1=\int_G'(.)1$$

Thus we have $\int_G=\int_G'$, and this finishes the proof. See [148]. □

As an illustration, for the basic product operations, we have:

**Proposition 3.19.** *We have the following results:*

1. *For a product $G\times H$, we have $\int_{G\times H}=\int_G\otimes\int_H$.*
2. *For a dual free product $G\,\hat{*}\,H$, we have $\int_{G\,\hat{*}\,H}=\int_G*\int_H$.*
3. *For a quotient $G\to H$, we have $\int_H=\left(\int_G\right)_{|C(H)}$.*
4. *For a projective version $G\to PG$, we have $\int_{PG}=\left(\int_G\right)_{|C(PG)}$.*

*Proof.* These formulae all follow from the invariance property, as follows:

(1) Here the tensor product form $\int_G\otimes\int_H$ satisfies the left and right invariance properties of the Haar functional $\int_{G\times H}$, and so by uniqueness, it is equal to it.

(2) Here the situation is similar, with the free product of linear forms being defined with some inspiration from the discrete group case, where $\int_{\widehat{\Gamma}}g=\delta_{g,1}$.

(3) Here the restriction $\left(\int_G\right)_{|C(H)}$ satisfies by definition the required left and right invariance properties, so once again we can conclude by uniqueness.

(4) Here we simply have a particular case of (3) above. □

In practice, the last assertion in Theorem 3.18 is the most useful one. By applying it to the Peter-Weyl corepresentations, we obtain the following alternative statement:

**Theorem 3.20.** *The Haar integration of a Woronowicz algebra is given, on the coefficients of the Peter-Weyl corepresentations, by the Weingarten formula*

$$\int_G u_{i_1j_1}^{e_1}\ldots u_{i_kj_k}^{e_k}=\sum_{\pi,\sigma\in D_k}\delta_\pi(i)\delta_\sigma(j)W_k(\pi,\sigma)$$

*valid for any colored integer $k=e_1\ldots e_k$ and any multi-indices $i,j$, where:*

(1) *$D_k$ is a linear basis of $Fix(u^{\otimes k})$.*
(2) *$\delta_\pi(i)=<\pi,e_{i_1}\otimes\ldots\otimes e_{i_k}>$.*
(3) *$W_k=G_k^{-1}$, with $G_k(\pi,\sigma)=<\pi,\sigma>$.*

*Proof.* We know from Theorem 3.18 that the integrals in the statement form altogether the orthogonal projection $P$ onto the following space:

$$Fix(u^{\otimes k})=span(D_k)$$

Consider now the following linear map:

$$E(x)=\sum_{\pi\in D_k}<x,\pi>\pi$$

By a standard linear algebra computation, it follows that we have $P=WE$, where $W$ is the inverse on $span(D_k)$ of the restriction of $E$. But this restriction is the linear map given by $G_k$, and so $W$ is the linear map given by $W_k$, and this gives the result. $\square$

Let us go back now to algebra, and establish two more Peter-Weyl theorems. We will need the following result, which is very useful, and is of independent interest:

**Theorem 3.21.** *We have a Frobenius type isomorphism*

$$Hom(v,w)\simeq Fix(\bar{v}\otimes w)$$

*valid for any two corepresentations $v,w$.*

*Proof.* According to the definitions, we have the following equivalence:

$$\begin{aligned}T\in Hom(v,w)&\iff Tv=wT\\&\iff \sum_jT_{aj}v_{ji}=\sum_bw_{ab}T_{bi}\end{aligned}$$

On the other hand, we have as well the following equivalence:

$$\begin{aligned}T\in Fix(\bar{v}\otimes w)&\iff (\bar{v}\otimes w)T=T\\&\iff \sum_{kb}v_{ik}^*w_{ab}T_{bk}=T_{ai}\end{aligned}$$

With these formulae in hand, we must prove that we have:

$$\sum_j T_{aj}v_{ji}=\sum_b w_{ab}T_{bi}\iff\sum_{kb}v_{ik}^*w_{ab}T_{bk}=T_{ai}$$

(1) In one sense, the computation is as follows, using the unitarity of $v^t$:

$$\begin{aligned}\sum_{kb}v_{ik}^*w_{ab}T_{bk}&=\sum_k v_{ik}^*\sum_b w_{ab}T_{bk}\\&=\sum_k v_{ik}^*\sum_j T_{aj}v_{jk}\\&=\sum_j(\bar{v}v^t)_{ij}T_{aj}\\&=T_{ai}\end{aligned}$$

(2) In the other sense we have, once again by using the unitarity of $v^t$:

$$\begin{aligned}\sum_j T_{aj}v_{ji}&=\sum_j v_{ji}\sum_{kb}v_{jk}^*w_{ab}T_{bk}\\&=\sum_{kb}(v^t\bar{v})_{ik}w_{ab}T_{bk}\\&=\sum_b w_{ab}T_{bi}\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

With these ingredients, namely first two Peter-Weyl theorems, Haar measure and Frobenius duality, we can establish a third Peter-Weyl theorem, from [148], as follows:

**Theorem 3.22** (PW3)**.** *The dense subalgebra $\mathcal{A}\subset A$ decomposes as a direct sum*

$$\mathcal{A}=\bigoplus_{v\in Irr(A)}M_{\dim(v)}(\mathbb{C})$$

*with this being an isomorphism of $*$-coalgebras, and with the summands being pairwise orthogonal with respect to the scalar product given by*

$$<a,b>=\int_G ab^*$$

*where $\int_G$ is the Haar integration over $G$.*

*Proof.* By combining the previous Peter-Weyl results, from Theorem 3.9 and Theorem 3.12 above, we deduce that we have a linear space decomposition as follows:

$$\mathcal{A}=\sum_{v\in Irr(A)}C(v)=\sum_{v\in Irr(A)}M_{\dim(v)}(\mathbb{C})$$

Thus, in order to conclude, it is enough to prove that for any two irreducible corepresentations $v, w \in Irr(A)$, the corresponding spaces of coefficients are orthogonal:

$$v \not\sim w \implies C(v) \perp C(w)$$

But this follows from Theorem 3.18, via Theorem 3.21. Let us set indeed:

$$P_{ia,jb} = \int_G v_{ij} w_{ab}^*$$

Then $P$ is the orthogonal projection onto the following vector space:

$$Fix(v \otimes \bar{w}) \simeq Hom(\bar{v}, \bar{w}) = \{0\}$$

Thus we have $P = 0$, and this gives the result. □

We can obtain further results by using characters, which are defined as follows:

**Proposition 3.23.** *The characters of the corepresentations, given by*

$$\chi_v = \sum_i v_{ii}$$

*behave as follows, in respect to the various operations:*

$$\chi_{v+w} = \chi_v + \chi_w \quad , \quad \chi_{v \otimes w} = \chi_v \chi_w \quad , \quad \chi_{\bar{v}} = \chi_v^*$$

*In addition, given two equivalent corepresentations, $v \sim w$, we have $\chi_v = \chi_w$.*

*Proof.* The three formulae in the statement are all clear from definitions. Regarding now the last assertion, assuming that we have $v = T^{-1} w T$, we obtain:

$$\chi_v = Tr(v) = Tr(T^{-1} w T) = Tr(w) = \chi_w$$

We conclude that $v \sim w$ implies $\chi_v = \chi_w$, as claimed. □

We have the following result, also from [148], completing the Peter-Weyl theory:

**Theorem 3.24** (PW4). *The characters of irreducible corepresentations belong to the algebra*

$$\mathcal{A}_{central} = \left\{ a \in \mathcal{A} \middle| \Sigma\Delta(a) = \Delta(a) \right\}$$

*of "smooth central functions" on $G$, and form an orthonormal basis of it.*

*Proof.* As a first remark, the linear space $\mathcal{A}_{central}$ defined above is indeed an algebra. In the classical case, we obtain the usual algebra of smooth central functions. Also, in the group dual case, where we have $\Sigma\Delta = \Delta$, we obtain the whole convolution algebra. Regarding now the proof, in general, this goes as follows:

(1) The algebra $\mathcal{A}_{central}$ contains indeed all the characters, because we have:

$$\Sigma\Delta(\chi_v) = \sum_{ij} v_{ji} \otimes v_{ij} = \Delta(\chi_v)$$

(2) Conversely, consider an element $a \in \mathcal{A}$, written as follows:

$$a = \sum_{v \in Irr(A)} a_v$$

The condition $a \in \mathcal{A}_{central}$ is then equivalent to the following conditions:

$$a_v \in \mathcal{A}_{central} \quad , \forall v \in Irr(A)$$

But each condition $a_v \in \mathcal{A}_{central}$ means that $a_v$ must be a scalar multiple of the corresponding character $\chi_v$, and so the characters form a basis of $\mathcal{A}_{central}$, as stated.

(3) The fact that we have an orthogonal basis follows from Theorem 3.22.

(4) Finally, regarding the norm 1 assertion, consider the following integrals:

$$P_{ik,jl} = \int_G v_{ij} v_{kl}^*$$

We know from Theorem 3.18 that these integrals form the orthogonal projection onto the following vector space, computed via Theorem 3.21:

$$Fix(v \otimes \bar{v}) \simeq End(\bar{v}) = \mathbb{C}1$$

By using this fact, we obtain the following formula:

$$\int_G \chi_v \chi_v^* = \sum_{ij} \int_G v_{ii} v_{jj}^* = \sum_i \frac{1}{N} = 1$$

Thus the characters have indeed norm 1, and we are done. □

As a first application of the Peter-Weyl theory, and more specifically of the last result from the series, Theorem 3.24, we can clarify a question left open in section 2 above, regarding the cocommutative case. Once again following [148], we have:

**Theorem 3.25.** *For a Woronowicz algebra $A$, the following are equivalent:*

(1) *$A$ is cocommutative, $\Sigma\Delta = \Delta$.*
(2) *The irreducible corepresentations of $A$ are all 1-dimensional.*
(3) *$A = C^*(\Gamma)$, for some group $\Gamma = < g_1, \ldots, g_N >$, up to equivalence.*

*Proof.* This follows from the Peter-Weyl theory, as follows:

(1) $\implies$ (2) The assumption $\Sigma\Delta = \Delta$ tells us that the inclusion $\mathcal{A}_{central} \subset \mathcal{A}$ is an isomorphism, and by using Theorem 3.24 we conclude that any irreducible corepresentation of $A$ must be equal to its character, and so must be 1-dimensional.

(2) $\implies$ (3) This follows once again from Peter-Weyl, because if we denote by $\Gamma$ the group formed by the 1-dimensional corepresentations, then we have $\mathcal{A} = \mathbb{C}[\Gamma]$, and so $A = C^*(\Gamma)$ up to the standard equivalence relation for Woronowicz algebras.

(3) $\implies$ (1) This is something trivial, that we already know from section 2. □

At the level of the examples coming from operations, we have, following [139]:

**Proposition 3.26.** *We have the following results:*

1. *The irreducible corepresentations of $C(G\times H)$ are the tensor products of the form $v\otimes w$, with $v,w$ being irreducible corepresentations of $C(G),C(H)$.*
2. *The irreducible corepresentations of $C(G\,\hat{*}\,H)$ appear as alternating tensor products of irreducible corepresentations of $C(G)$ and of $C(H)$.*
3. *The irreducible corepresentations of $C(H)\subset C(G)$ are the irreducible corepresentations of $C(G)$ whose coefficients belong to $C(H)$.*
4. *The irreducible corepresentations of $C(PG)\subset C(G)$ are the irreducible corepresentations of $C(G)$ which appear by decomposing the tensor powers of $u\otimes\bar{u}$.*

*Proof.* This is routine, the idea being as follows:

(1) Here we can integrate characters, by using Proposition 3.19 (1), and we conclude that if $v,w$ are irreducible corepresentations of $C(G),C(H)$, then $v\otimes w$ is an irreducible corepresentation of $C(G\times H)$. Now since the coefficients of these latter corepresentations span $\mathcal{C}(G\times H)$, by Peter-Weyl these are all the irreducible corepresentations.

(2) Here we can use a similar method. By using Proposition 3.19 (2) we conclude that if $v_1,v_2,\ldots$ are irreducible corepresentations of $C(G)$ and $w_1,w_2,\ldots$ are irreducible corepresentations of $C(H)$, then $v_1\otimes w_1\otimes v_2\otimes w_2\otimes\ldots$ is an irreducible corepresentation of $C(G\,\hat{*}\,H)$, and then we can conclude by using the Peter-Weyl theory.

(3) This is clear from definitions, and from the Peter-Weyl theory.

(4) This is a particular case of the result (3) above. □

Finally, let us discuss the notion of amenability. The basic result here is as follows:

**Theorem 3.27.** *Let $A_{full}$ be the enveloping $C^*$-algebra of $\mathcal{A}$, and let $A_{red}$ be the quotient of $A$ by the null ideal of the Haar integration. The following are then equivalent:*

1. *The Haar functional of $A_{full}$ is faithful.*
2. *The projection map $A_{full}\to A_{red}$ is an isomorphism.*
3. *The counit map $\varepsilon:A\to\mathbb{C}$ factorizes through $A_{red}$.*
4. *We have $N\in\sigma(Re(\chi_u))$, the spectrum being taken inside $A_{red}$.*

*If this is the case, we say that the underlying discrete quantum group $\Gamma$ is amenable.*

*Proof.* This is well-known in the group dual case, $A=C^*(\Gamma)$, with $\Gamma$ being a usual discrete group. In general, the result follows by adapting the group dual case proof:

(1) $\Longrightarrow$ (2) This follows from the fact that the GNS construction for the algebra $A_{full}$ with respect to the Haar functional produces the algebra $A_{red}$.

(2) $\Longrightarrow$ (3) This is trivial, because we have quotient maps $A_{full}\to A\to A_{red}$, and so our assumption $A_{full}=A_{red}$ implies that we have $A=A_{red}$.

(3) $\implies$ (4) This implication is clear too, because we have:

$$\begin{aligned}\varepsilon(Re(\chi_u)) &= \frac{1}{2}\left(\sum_{i=1}^N \varepsilon(u_{ii}) + \sum_{i=1}^N \varepsilon(u_{ii}^*)\right)\\ &= \frac{1}{2}(N+N)\\ &= N\end{aligned}$$

Thus the element $N - Re(\chi_u)$ is not invertible in $A_{red}$, as claimed.

(4) $\implies$ (1) In terms of the corepresentation $v = u + \bar{u}$, whose dimension is $2N$ and whose character is $2Re(\chi_u)$, our assumption $N \in \sigma(Re(\chi_u))$ reads:

$$\dim v \in \sigma(\chi_v)$$

By functional calculus the same must hold for $w = v + 1$, and then once again by functional calculus, the same must hold for any tensor power of $w$:

$$w_k = w^{\otimes k}$$

Now choose for each $k \in \mathbb{N}$ a state $\varepsilon_k \in A_{red}^*$ having the following property:

$$\varepsilon_k(w_k) = \dim w_k$$

By Peter-Weyl we must have $\varepsilon_k(r) = \dim r$ for any $r \leq w_k$, and since any irreducible corepresentation appears in this way, the sequence $\varepsilon_k$ converges to a counit map:

$$\varepsilon : A_{red} \to \mathbb{C}$$

In order to finish, we can use the right regular corepresentation. Indeed, as explained in [114], we can define such a corepresentation by the following formula:

$$W(a \otimes x) = \Delta(a)(1 \otimes x)$$

This corepresentation is unitary, so we can define a morphism as follows:

$$\Delta' : A_{red} \to A_{red} \otimes A_{full}$$

$$a \to W(a \otimes 1)W^*$$

Now by composing with $\varepsilon \otimes id$, we obtain a morphism as follows:

$$(\varepsilon \otimes id)\Delta' : A_{red} \to A_{full}$$

$$u_{ij} \to u_{ij}$$

Thus, we have our inverse map for the projection $A_{full} \to A_{red}$, as desired. $\square$

All the above was of course quite short, but we will be back to this, with full details, and with a systematic study of the notion of amenability, in section 14 below.

## 4. Tannakian duality

In order to have more insight into the structure of the compact quantum groups, in general and for the concrete examples too, and to effectively compute their representations, we can use algebraic geometry methods, and more precisely Tannakian duality.

Tannakian duality rests on the basic principle in any kind of mathematics, algebra, geometry or analysis, "linearize". In the present setting, where we do not have a Lie algebra, this will be in fact our only possible linearization method. Let us start with:

**Theorem 4.1.** *Given a Woronowicz algebra $(A,u)$, the Hom spaces for its corepresentations form a tensor $*$-category, in the sense that:*

(1) $T\in Hom(u,v), S\in Hom(v,w) \implies ST\in Hom(u,w)$.
(2) $S\in Hom(p,q), T\in Hom(v,w) \implies S\otimes T\in Hom(p\otimes v,q\otimes w)$.
(3) $T\in Hom(v,w) \implies T^*\in Hom(w,v)$.

*Proof.* This is something that we already know, from section 3 above. □

Generally speaking, Tannakian duality amounts in recovering $(A,u)$ from the tensor category constructed in Theorem 4.1. In what follows we will present a "soft form" of this duality, coming from [106], [149], which uses the following smaller category:

**Definition 4.2.** *The Tannakian category associated to a Woronowicz algebra $(A,u)$ is the collection $C=(C(k,l))$ of vector spaces*

$$C(k,l)=Hom(u^{\otimes k},u^{\otimes l})$$

*where $u^{\otimes k}$ with $k=\circ\bullet\bullet\circ\ldots$ colored integer are the Peter-Weyl corepresentations.*

We know from Theorem 4.1 above that $C$ is a tensor $*$-category. To be more precise, if we denote by $H=\mathbb{C}^N$ the Hilbert space where $u\in M_N(A)$ coacts, then $C$ is a tensor $*$-subcategory of the tensor $*$-category formed by the following linear spaces:

$$E(k,l)=\mathcal{L}(H^{\otimes k},H^{\otimes l})$$

Here the tensor powers $H^{\otimes k}$ with $k=\circ\bullet\bullet\circ\ldots$ colored integer are those where the corepresentations $u^{\otimes k}$ act, defined by the following formulae, and multiplicativity:

$$H^{\otimes\emptyset}=\mathbb{C}\quad,\quad H^{\otimes\circ}=H\quad,\quad H^{\otimes\bullet}=\bar{H}\simeq H$$

Our purpose in what follows will be that of reconstructing $(A,u)$ in terms of the category $C=(C(k,l))$. We will see afterwards that this method has many applications.

As a first, elementary result on the subject, we have:

**Proposition 4.3.** *Given a morphism $\pi:(A,u)\to(B,v)$ we have inclusions*

$$Hom(u^{\otimes k},u^{\otimes l})\subset Hom(v^{\otimes k},v^{\otimes l})$$

*for any $k,l$, and if these inclusions are all equalities, $\pi$ is an isomorphism.*

*Proof.* The fact that we have indeed inclusions as in the statement is clear from definitions. As for the last assertion, this follows from the Peter-Weyl theory. □

The Tannakian duality result that we want to prove states, in a simplified form, that in what concerns the last conclusion in the above statement, the assumption that we have a morphism $\pi:(A,u)\to(B,v)$ is not needed. In other words, if we know that the Tannakian categories of $A,B$ are different, then $A,B$ themselves must be different.

In order to get started now, our first goal will be that of gaining some familiarity with the notion of Tannakian category. As a starting point here, we have:

**Proposition 4.4.** *An abstract matrix $u\in M_N(A)$ is biunitary if and only if*

$$R\in Hom(1,u\otimes\bar{u})\quad,\quad R\in Hom(1,\bar{u}\otimes u)$$

$$R^*\in Hom(u\otimes\bar{u},1)\quad,\quad R^*\in Hom(\bar{u}\otimes u,1)$$

*where $R:\mathbb{C}\to\mathbb{C}^N\otimes\mathbb{C}^N$ is the linear operator given by:*

$$R(1)=\sum_ie_i\otimes e_i$$

*Proof.* With $R$ being as in the statement, we have the following computation:

$$\begin{aligned}(u\otimes\bar{u})(R(1)\otimes1)&=\sum_{ijk}e_i\otimes e_k\otimes u_{ij}u_{kj}^*\\&=\sum_{ik}e_i\otimes e_k\otimes(uu^*)_{ik}\end{aligned}$$

We conclude from this that we have the following equivalence:

$$R\in Hom(1,u\otimes\bar{u})\iff uu^*=1$$

Consider now the adjoint operator $R^*:\mathbb{C}^N\otimes\mathbb{C}^N\to\mathbb{C}$, which is given by:

$$R^*(e_i\otimes e_j)=\delta_{ij}$$

We have then the following computation:

$$(R^*\otimes id)(u\otimes\bar{u})(e_j\otimes e_l\otimes1)=\sum_iu_{ij}u_{il}^*=(u^t\bar{u})_{jl}$$

We conclude from this that we have the following equivalence:

$$R^*\in Hom(u\otimes\bar{u},1)\iff u^t\bar{u}=1$$

Similarly, or simply by replacing $u$ in the above two conclusions with its conjugate $\bar{u}$, which is a corepresentation too, we have as well the following two equivalences:

$$R\in Hom(1,\bar{u}\otimes u)\iff\bar{u}u^t=1$$

$$R^*\in Hom(\bar{u}\otimes u,1)\iff u^*u=1$$

Thus, we are led to the biunitarity conditions, and we are done. □

As a consequence of this computation, we have the following result:

**Proposition 4.5.** *The Tannakian category $C = (C(k,l))$ associated to a Woronowicz algebra $(A,u)$ must contain the operators*

$$R : 1 \to \sum_i e_i \otimes e_i$$

$$R^*(e_i \otimes e_j) = \delta_{ij}$$

*in the sense that we must have:*

$$R \in C(\emptyset, \circ\bullet) \quad , \quad R \in C(\emptyset, \bullet\circ)$$

$$R^* \in C(\circ\bullet, \emptyset) \quad , \quad R^* \in C(\bullet\circ, \emptyset)$$

*In fact, $C$ must contain the whole tensor category $< R, R^* >$ generated by $R, R^*$.*

*Proof.* The first assertion is clear from the above result. As for the second assertion, this is clear from definitions, because $C = (C(k,l))$ is indeed a tensor category. □

Let us formulate now the following key definition:

**Definition 4.6.** *Let $H$ be a finite dimensional Hilbert space. A tensor category over $H$ is a collection $C = (C(k,l))$ of subspaces*

$$C(k,l) \subset \mathcal{L}(H^{\otimes k}, H^{\otimes l})$$

*satisfying the following conditions:*

1. *$S, T \in C$ implies $S \otimes T \in C$.*
2. *If $S, T \in C$ are composable, then $ST \in C$.*
3. *$T \in C$ implies $T^* \in C$.*
4. *Each $C(k,k)$ contains the identity operator.*
5. *$C(\emptyset, \circ\bullet)$ and $C(\emptyset, \bullet\circ)$ contain the operator $R : 1 \to \sum_i e_i \otimes e_i$.*

In relation with the quantum groups, this formalism generalizes the Tannakian category formalism from Definition 4.2 above, because we have the following result:

**Proposition 4.7.** *Let $(A,u)$ be a Woronowicz algebra, with fundamental corepresentation $u \in M_N(A)$. The associated Tannakian category $C = (C(k,l))$, given by*

$$C(k,l) = Hom(u^{\otimes k}, u^{\otimes l})$$

*is then a tensor category over the Hilbert space $H = \mathbb{C}^N$.*

*Proof.* The fact that the above axioms (1-5) are indeed satisfied is clear, as follows:

(1) This follows from Theorem 4.1.

(2) Once again, this follows from Theorem 4.1.

(3) This once again follows from Theorem 4.1.

(4) This is clear from definitions.

(5) This follows from Proposition 4.5 above. □

Our main purpose in what follows will be that of proving that the converse of the above statement holds. In other words, we would like to prove that any tensor category in the sense of Definition 4.6 must appear as a Tannakian category. We first have:

**Proposition 4.8.** *Given a tensor category $C = (C(k,l))$, the following algebra, with $u$ being the fundamental corepresentation of $C(U_N^+)$, is a Woronowicz algebra:*

$$A_C = C(U_N^+)\Big/\left\langle T \in Hom(u^{\otimes k}, u^{\otimes l})\Big|\forall k,l,\forall T \in C(k,l)\right\rangle$$

*In the case where $C$ comes from a Woronowicz algebra $(A,v)$, we have a quotient map:*

$$A_C \to A$$

*Moreover, this map is an isomorphism in the discrete group algebra case.*

*Proof.* Given colored integers $k,l$ and an arbitrary linear operator $T \in \mathcal{L}(H^{\otimes k}, H^{\otimes l})$, consider the following $*$-ideal of the algebra $C(U_N^+)$:

$$I = \left\langle T \in Hom(u^{\otimes k}, u^{\otimes l})\right\rangle$$

Our claim is that $I$ is a Hopf ideal. Indeed, let us set:

$$U = \sum_k u_{ik} \otimes u_{kj}$$

It is elementary to check that we have the following implication, which proves our claim:

$$T \in Hom(u^{\otimes k}, u^{\otimes l}) \implies T \in Hom(U^{\otimes k}, U^{\otimes l})$$

With this claim in hand, $A_C$ appears from $C(U_N^+)$ by dividing by a collection of Hopf ideals, and is therefore a Woronowicz algebra. Since the relations defining $A_C$ are satisfied in $A$, we have a quotient map as in the statement:

$$A_C \to A$$

Regarding now the last assertion, assume that we are in the case $A = C^*(\Gamma)$, with $\Gamma =< g_1, \ldots, g_N >$ being a finitely generated discrete group. If we denote by $\mathcal{R}$ the complete collection of relations between the generators, then we have:

$$\Gamma = F_N/\mathcal{R}$$

By using now the basic functoriality properties of the group algebra construction, we deduce from this that we have:

$$A_C = C^*\left(F_N\Big/\left\langle\mathcal{R}\right\rangle\right)$$

Thus the quotient map $A_C \to A$ is indeed an isomorphism, as claimed. □

With the above construction in hand, the theorem that we want to prove states that the operations $A\to A_C$ and $C\to C_A$ are inverse to each other. We first have:

**Proposition 4.9.** *Consider the following conditions:*

(1) $C=C_{A_C}$, *for any Tannakian category* $C$.
(2) $A=A_{C_A}$, *for any Woronowicz algebra* $(A,u)$.

*We have then* $(1)\implies(2)$. *Also,* $C\subset C_{A_C}$ *is automatic.*

*Proof.* Given a Woronowicz algebra $(A,u)$, let us set $C=C_A$. By using (1) we have then:

$$C_A=C_{A_{C_A}}$$

On the other hand, by Proposition 4.8 above we have an arrow:

$$A_{C_A}\to A$$

Thus, we are in the general situation from Proposition 4.3 above, with a surjective arrow of Woronowicz algebras, which becomes an isomorphism at the level of the associated Tannakian categories. We conclude that Proposition 4.3 can be applied, and this gives the isomorphism of the associated Woronowicz algebras, $A_{C_A}=A$, as desired.

Finally, the fact that we have an inclusion $C\subset C_{A_C}$ is clear from definitions. □

Summarizing, we would like to prove that we have $C_{A_C}\subset C$, for any Tannakian category $C$. Let us begin with some abstract constructions. Following [106], we have:

**Proposition 4.10.** *Given a tensor category* $C=C((k,l))$ *over a Hilbert space* $H$,

$$E_C^{(s)}=\bigoplus_{|k|,|l|\leq s}C(k,l)\subset\bigoplus_{|k|,|l|\leq s}B(H^{\otimes k},H^{\otimes l})=B\left(\bigoplus_{|k|\leq s}H^{\otimes k}\right)$$

*is a finite dimensional* $C^*$*-subalgebra. Also,*

$$E_C=\bigoplus_{k,l}C(k,l)\subset\bigoplus_{k,l}B(H^{\otimes k},H^{\otimes l})\subset B\left(\bigoplus_k H^{\otimes k}\right)$$

*is a closed* $*$*-subalgebra.*

*Proof.* This is clear indeed from the categorical axioms from Definition 4.6. □

Now back to our reconstruction question, given a tensor category $C=(C(k,l))$, we want to prove that we have $C=C_{A_C}$, which is the same as proving that we have:

$$E_C=E_{C_{A_C}}$$

Equivalently, we want to prove that we have isomorphisms as follows, for any $s\in\mathbb{N}$:

$$E_C^{(s)}=E_{C_{A_C}}^{(s)}$$

The problem, however, is that these isomorphims are not easy to establish directly. In order to solve this question, we will use a standard commutant trick, as follows:

**Theorem 4.11.** *For any $C^*$-algebra $B \subset M_n(\mathbb{C})$ we have the formula*

$$B = B''$$

*where prime denotes the commutant, $A' = \{T \in M_n(\mathbb{C}) | Tx = xT, \forall x \in A\}$.*

*Proof.* This is a particular case of von Neumann's bicommutant theorem [138], which follows as well from the explicit description of $B$ given in section 3 above. To be more precise, let us decompose $B$ as there, as a direct sum of matrix algebras:

$$B = M_{r_1}(\mathbb{C}) \oplus \ldots \oplus M_{r_k}(\mathbb{C})$$

The center of each matrix algebra being reduced to the scalars, the commutant of this algebra is then as follows, with each copy of $\mathbb{C}$ corresponding to a matrix block:

$$B' = \mathbb{C} \oplus \ldots \oplus \mathbb{C}$$

By taking once again the commutant we obtain $B$ itself, and we are done. □

Now back to our questions, we recall that we want to prove that we have $C = C_{A_C}$, for any Tannakian category $C$. By using the bicommutant theorem, we have:

**Proposition 4.12.** *Given a Tannakian category $C$, the following are equivalent:*

1. $C = C_{A_C}$.
2. $E_C = E_{C_{A_C}}$.
3. $E_C^{(s)} = E_{C_{A_C}}^{(s)}$, *for any $s \in \mathbb{N}$.*
4. $E_C^{(s)'} = E_{C_{A_C}}^{(s)'}$, *for any $s \in \mathbb{N}$.*

*In addition, the inclusions $\subset$, $\subset$, $\subset$, $\supset$ are automatically satisfied.*

*Proof.* This follows from the above results, as follows:

(1) $\iff$ (2) This is clear from definitions.

(2) $\iff$ (3) This is clear from definitions as well.

(3) $\iff$ (4) This comes from the bicommutant theorem. As for the last assertion, we have indeed $C \subset C_{A_C}$ from Proposition 4.9, and so $E_C \subset E_{C_{A_C}}$. We therefore obtain $E_C^{(s)} \subset E_{C_{A_C}}^{(s)}$, and by taking the commutants, this gives $E_C^{(s)} \supset E_{C_{A_C}}^{(s)}$, as desired. □

Summarizing, in order to finish, given a tensor category $C = (C(k,l))$, we would like to prove that we have inclusions as follows, for any $s \in \mathbb{N}$:

$$E_C^{(s)'} \subset E_{C_{A_C}}^{(s)'}$$

Let us first study the commutant on the right. As a first observation, we have:

**Proposition 4.13.** *Given a Woronowicz algebra $(A,u)$, we have*

$$E_{C_A}^{(s)}=End\left(\bigoplus_{|k|\leq s}u^{\otimes k}\right)$$

*as subalgebras of the algebra $B\left(\bigoplus_{|k|\leq s}H^{\otimes k}\right)$.*

*Proof.* The category $C_A$ is by definition given by:

$$C_A(k,l)=Hom(u^{\otimes k},u^{\otimes l})$$

Thus, according to the various identifications in Proposition 4.10 above, the corresponding algebra $E_{C_A}^{(s)}$ appears as follows:

$$E_{C_A}^{(s)}=\bigoplus_{|k|,|l|\leq s}Hom(u^{\otimes k},u^{\otimes l})\subset\bigoplus_{|k|,|l|\leq s}B(H^{\otimes k},H^{\otimes l})=B\left(\bigoplus_{|k|\leq s}H^{\otimes k}\right)$$

On the other hand, the algebra of intertwiners of $\bigoplus_{|k|\leq s}u^{\otimes k}$ is given by:

$$End\left(\bigoplus_{|k|\leq s}u^{\otimes k}\right)=\bigoplus_{|k|,|l|\leq s}Hom(u^{\otimes k},u^{\otimes l})\subset\bigoplus_{|k|,|l|\leq s}B(H^{\otimes k},H^{\otimes l})=B\left(\bigoplus_{|k|\leq s}H^{\otimes k}\right)$$

Thus we have indeed the same algebra, and we are done. □

In practice now, we have to compute the commutant of the above algebra. For this purpose, we can use the following general result:

**Proposition 4.14.** *Given a corepresentation $v\in M_n(A)$, we have a representation*

$$\pi_v:A^*\to M_n(\mathbb{C})$$

$$\varphi\to(\varphi(v_{ij}))_{ij}$$

*whose image is given by:*

$$Im(\pi_v)=End(v)'$$

*Proof.* The first assertion is clear, with the multiplicativity claim coming from:

$$\begin{aligned}(\pi_v(\varphi*\psi))_{ij}&=(\varphi\otimes\psi)\Delta(v_{ij})\\&=\sum_k\varphi(v_{ik})\psi(v_{kj})\\&=\sum_k(\pi_v(\varphi))_{ik}(\pi_v(\psi))_{kj}\\&=(\pi_v(\varphi)\pi_v(\psi))_{ij}\end{aligned}$$

Let us first prove the inclusion $\subset$. Given $\varphi\in A^*$ and $T\in End(v)$, we have:

$$\begin{aligned}[\pi_v(\varphi),T]=0 &\iff \sum_k\varphi(v_{ik})T_{kj}=\sum_kT_{ik}\varphi(v_{kj}),\forall i,j\\ &\iff \varphi\left(\sum_kv_{ik}T_{kj}\right)=\varphi\left(\sum_kT_{ik}v_{kj}\right),\forall i,j\\ &\iff \varphi((vT)_{ij})=\varphi((Tv)_{ij}),\forall i,j\end{aligned}$$

But this latter formula is true, because $T\in End(v)$ means that we have:

$$vT=Tv$$

As for the converse inclusion $\supset$, the proof is quite similar. Indeed, by using the bicommutant theorem, this is the same as proving that we have:

$$Im(\pi_v)'\subset End(v)$$

But, by using the above equivalences, we have the following computation:

$$\begin{aligned}T\in Im(\pi_v)' &\iff [\pi_v(\varphi),T]=0,\forall\varphi\\ &\iff \varphi((vT)_{ij})=\varphi((Tv)_{ij}),\forall\varphi,i,j\\ &\iff vT=Tv\end{aligned}$$

Thus, we have obtained the desired inclusion, and we are done. □

By combining now the above results, we obtain:

**Theorem 4.15.** *Given a Woronowicz algebra $(A,u)$, we have*

$$E_{C_A}^{(s)'}=Im(\pi_v)$$

*as subalgebras of $B\left(\bigoplus_{|k|\leq s}H^{\otimes k}\right)$, where the corepresentation $v$ is the sum*

$$v=\bigoplus_{|k|\leq s}u^{\otimes k}$$

*and where $\pi_v:A^*\to M_n(\mathbb{C})$ is given by $\varphi\to(\varphi(v_{ij}))_{ij}$.*

*Proof.* This follows indeed from Proposition 4.13 and Proposition 4.14. □

We recall that we want to prove that we have $E_C^{(s)'}\subset E_{C_{A_C}}^{(s)'}$, for any $s\in\mathbb{N}$. For this purpose, we must first refine Theorem 4.15, in the case $A=A_C$. In order to do so, we will use an explicit model for $A_C$. In order to construct such a model, let $<u_{ij}>$ be the free $*$-algebra over $\dim(H)^2$ variables, with comultiplication and counit as follows:

$$\Delta(u_{ij})=\sum_ku_{ik}\otimes u_{kj}$$

$$\varepsilon(u_{ij})=\delta_{ij}$$

Following [106], we can model this $*$-bialgebra, in the following way:

**Proposition 4.16.** *Consider the following pair of dual vector spaces,*

$$F=\bigoplus_k B\left(H^{\otimes k}\right)$$

$$F^*=\bigoplus_k B\left(H^{\otimes k}\right)^*$$

*and let $f_{ij},f_{ij}^*\in F^*$ be the standard generators of $B(H)^*,B(\bar{H})^*$.*

1. *$F^*$ is a $*$-algebra, with multiplication $\otimes$ and involution $f_{ij}\leftrightarrow f_{ij}^*$.*
2. *$F^*$ is a $*$-bialgebra, with $\Delta(f_{ij})=\sum_k f_{ik}\otimes f_{kj}$ and $\varepsilon(f_{ij})=\delta_{ij}$.*
3. *We have a $*$-bialgebra isomorphism $<u_{ij}>\simeq F^*$, given by $u_{ij}\to f_{ij}$.*

*Proof.* Since $F^*$ is spanned by the various tensor products between the variables $f_{ij},f_{ij}^*$, we have a vector space isomorphism as follows, given by $u_{ij}\to f_{ij},u_{ij}^*\to f_{ij}^*$:

$$<u_{ij}>\simeq F^*$$

The corresponding $*$-bialgebra structure induced on $F^*$ is the one in the statement. $\square$

Now back to our algebra $A_C$, we have the following modelling result for it:

**Proposition 4.17.** *The smooth part of the algebra $A_C$ is given by*

$$\mathcal{A}_C\simeq F^*/J$$

*where $J\subset F^*$ is the ideal coming from the following relations,*

$$\begin{aligned}&\sum_{p_1,\ldots,p_k}T_{i_1\ldots i_l,p_1\ldots p_k}f_{p_1j_1}\otimes\ldots\otimes f_{p_kj_k}\\ =\ &\sum_{q_1,\ldots,q_l}T_{q_1\ldots q_l,j_1\ldots j_k}f_{i_1q_1}\otimes\ldots\otimes f_{i_lq_l}\quad,\quad\forall i,j\end{aligned}$$

*one for each pair of colored integers $k,l$, and each $T\in C(k,l)$.*

*Proof.* Our first claim is that $A_C$ appears as enveloping $C^*$-algebra of the following universal $*$-algebra, where $u=(u_{ij})$ is regarded as a formal corepresentation:

$$\mathcal{A}_C=\left\langle(u_{ij})_{i,j=1,\ldots,N}\Big|T\in Hom(u^{\otimes k},u^{\otimes l}),\forall k,l,\forall T\in C(k,l)\right\rangle$$

Indeed, this follows from Proposition 4.4 above, because according to the result there, the relations defining $C(U_N^+)$ are included into those that we impose.

With this claim in hand, the conclusion is that we have a formula as follows, where $I$ is the ideal coming from the relations $T\in Hom(u^{\otimes k},u^{\otimes l})$, with $T\in C(k,l)$:

$$\mathcal{A}_C=<u_{ij}>/I$$

Now if we denote by $J \subset F^*$ the image of the ideal $I$ via the $*$-algebra isomorphism $< u_{ij} >\simeq F^*$ from Proposition 4.16, we obtain an identification as follows:

$$\mathcal{A}_C \simeq F^*/J$$

In order to compute $J$, let us go back to $I$. With standard multi-index notations, and by assuming that $k, l \in \mathbb{N}$ are usual integers, for simplifying, a relation of type $T \in Hom(u^{\otimes k}, u^{\otimes l})$ inside $< u_{ij} >$ is equivalent to the following conditions:

$$\begin{aligned} & \sum_{p_1,\ldots,p_k} T_{i_1\ldots i_l,p_1\ldots p_k} u_{p_1 j_1} \ldots u_{p_k j_k} \\ = & \sum_{q_1,\ldots,q_l} T_{q_1\ldots q_l,j_1\ldots j_k} u_{i_1 q_1} \ldots u_{i_l q_l} \quad , \quad \forall i, j \end{aligned}$$

Now by recalling that the isomorphism of $*$-algebras $< u_{ij} >\to F^*$ is given by $u_{ij} \to f_{ij}$, and that the multiplication operation of $F^*$ corresponds to the tensor product operation $\otimes$, we conclude that $J \subset F^*$ is the ideal from the statement. □

With the above result in hand, let us go back to Theorem 4.15. We have:

**Proposition 4.18.** *The linear space $\mathcal{A}_C^*$ is given by the formula*

$$\mathcal{A}_C^* = \left\{ a \in F \Big| T a_k = a_l T, \forall T \in C(k, l) \right\}$$

*and the representation*

$$\pi_v : \mathcal{A}_C^* \to B\left( \bigoplus_{|k| \leq s} H^{\otimes k} \right)$$

*appears diagonally, by truncating:*

$$\pi_v : a \to (a_k)_{kk}$$

*Proof.* We know from Proposition 4.17 that we have:

$$\mathcal{A}_C \simeq F^*/J$$

But this gives a quotient map $F^* \to \mathcal{A}_C$, and so an inclusion as follows:

$$\mathcal{A}_C^* \subset F$$

To be more precise, we have the following formula:

$$\mathcal{A}_C^* = \left\{ a \in F \Big| f(a) = 0, \forall f \in J \right\}$$

Now since $J =< f_T >$, where $f_T$ are the relations in Proposition 4.17, we obtain:

$$\mathcal{A}_C^* = \left\{ a \in F \Big| f_T(a) = 0, \forall T \in C \right\}$$

Given $T\in C(k,l)$, for an arbitrary element $a=(a_k)$, we have:

$$\begin{aligned}
& f_T(a)=0\\
\iff\ & \sum_{p_1,\ldots,p_k} T_{i_1\ldots i_l,p_1\ldots p_k}(a_k)_{p_1\ldots p_k,j_1\ldots j_k}=\sum_{q_1,\ldots,q_l} T_{q_1\ldots q_l,j_1\ldots j_k}(a_l)_{i_1\ldots i_l,q_1\ldots q_l},\forall i,j\\
\iff\ & (Ta_k)_{i_1\ldots i_l,j_1\ldots j_k}=(a_lT)_{i_1\ldots i_l,j_1\ldots j_k},\forall i,j\\
\iff\ & Ta_k=a_lT
\end{aligned}$$

Thus, the dual space $\mathcal{A}_C^*$ is given by the formula in the statement.

It remains to compute the representation $\pi_v$, which appears as follows:

$$\pi_v:\mathcal{A}_C^*\to B\left(\bigoplus_{|k|\leq s}H^{\otimes k}\right)$$

With $a=(a_k)$, we have the following computation:

$$\begin{aligned}
\pi_v(a)_{i_1\ldots i_k,j_1\ldots j_k} &= a(v_{i_1\ldots i_k,j_1\ldots j_k})\\
&= (f_{i_1j_1}\otimes\ldots\otimes f_{i_kj_k})(a)\\
&= (a_k)_{i_1\ldots i_k,j_1\ldots j_k}
\end{aligned}$$

Thus, our representation $\pi_v$ appears diagonally, by truncating, as claimed. □

In order to further advance, consider the following vector spaces:

$$F_s=\bigoplus_{|k|\leq s}B\left(H^{\otimes k}\right)\quad,\quad F_s^*=\bigoplus_{|k|\leq s}B\left(H^{\otimes k}\right)^*$$

We denote by $a\to a_s$ the truncation operation $F\to F_s$. We have:

**Proposition 4.19.** *The following hold:*

(1) $E_C^{(s)'}\subset F_s$.
(2) $E_C'\subset F$.
(3) $\mathcal{A}_C^*=E_C'$.
(4) $Im(\pi_v)=(E_C')_s$.

*Proof.* These results basically follow from what we have, as follows:

(1) We have an inclusion as follows, as a diagonal subalgebra:

$$F_s\subset B\left(\bigoplus_{|k|\leq s}H^{\otimes k}\right)$$

The commutant of this algebra is given by:

$$F_s'=\left\{b\in F_s\Big|b=(b_k),b_k\in\mathbb{C},\forall k\right\}$$

On the other hand, we know from the identity axiom for $C$ that this algebra is contained inside $E_C^{(s)}$:

$$F_s' \subset E_C^{(s)}$$

Thus, our result follows from the bicommutant theorem, as follows:

$$F_s' \subset E_C^{(s)} \implies F_s \supset E_C^{(s)'}$$

(2) This follows from (1), by taking inductive limits.

(3) With the present notations, the formula of $\mathcal{A}_C^*$ from Proposition 4.18 reads:

$$\mathcal{A}_C^* = F \cap E_C'$$

Now since by (2) we have $E_C' \subset F$, we obtain from this $\mathcal{A}_C^* = E_C'$.

(4) This follows from (3), and from the formula of $\pi_v$ in Proposition 4.18. □

Following [106], we can now state and prove our main result, as follows:

**Theorem 4.20.** *The Tannakian duality constructions*

$$C \to A_C$$

$$A \to C_A$$

*are inverse to each other, modulo identifying full and reduced versions.*

*Proof.* According to Proposition 4.9, Proposition 4.12, Theorem 5.15 and Proposition 4.19, we have to prove that, for any Tannakian category $C$, and any $s \in \mathbb{N}$:

$$E_C^{(s)'} \subset (E_C')_s$$

By taking duals, this is the same as proving that we have:

$$\left\{f \in F_s^* \middle| f_{|(E_C')_s} = 0\right\} \subset \left\{f \in F_s^* \middle| f_{|E_C^{(s)'}} = 0\right\}$$

For this purpose, we use the following formula, coming from Proposition 4.19:

$$\mathcal{A}_C^* = E_C'$$

We know that we have:

$$\mathcal{A}_C = F^*/J$$

We conclude that the ideal $J$ is given by:

$$J = \left\{f \in F^* \middle| f_{|E_C'} = 0\right\}$$

Our claim is that we have the following formula, for any $s \in \mathbb{N}$:

$$J \cap F_s^* = \left\{f \in F_s^* \middle| f_{|E_C^{(s)'}} = 0\right\}$$

Indeed, let us denote by $X_s$ the spaces on the right. The categorical axioms for $C$ show that these spaces are increasing, that their union $X = \cup_s X_s$ is an ideal, and that:

$$X_s = X \cap F_s^*$$

We must prove that we have $J = X$, and this can be done as follows:

"$\subset$" This follows from the following fact, for any $T \in C(k,l)$ with $|k|, |l| \leq s$:

$$\begin{aligned} (f_T)_{|\{T\}'} = 0 \quad &\implies \quad (f_T)_{|E_C^{(s)\prime}} = 0 \\ &\implies \quad f_T \in X_s \end{aligned}$$

"$\supset$" This follows from our description of $J$, because from $E_C^{(s)} \subset E_C$ we obtain:

$$f_{|E_C^{(s)\prime}} = 0 \implies f_{|E_C'} = 0$$

Summarizing, we have proved our claim. On the other hand, we have:

$$\begin{aligned} J \cap F_s^* &= \left\{ f \in F^* \Big| f_{|E_C'} = 0 \right\} \cap F_s^* \\ &= \left\{ f \in F_s^* \Big| f_{|E_C'} = 0 \right\} \\ &= \left\{ f \in F_s^* \Big| f_{|(E_C')_s} = 0 \right\} \end{aligned}$$

Thus, our claim is exactly the inclusion that we wanted to prove, and we are done. □

As a first application, let us record the following theoretical fact, from [16]:

**Theorem 4.21.** *Each closed subgroup $G \subset U_N^+$ appears as an algebraic manifold of the free complex sphere,*

$$G \subset S^{N^2-1}_{\mathbb{C},+}$$

*the embedding being given by:*

$$x_{ij} = \frac{u_{ij}}{\sqrt{N}}$$

*Proof.* This follows from Theorem 4.20, by using the following inclusions:

$$G \subset U_N^+ \subset S^{N^2-1}_{\mathbb{C},+}$$

Indeed, both these inclusions are algebraic, and this gives the result. □

As a second application of the above results, let us study now in detail the quantum groups $O_N^+, U_N^+$. In order to get started, let us get back to the operators $R, R^*$, discussed in the beginning of this section. We know that these two operators must be present in any Tannakian category, and in what concerns $U_N^+$, which is the biggest $N \times N$ compact quantum group, a converse of this fact holds, by contravariant functoriality, as follows:

**Proposition 4.22.** *The tensor category $< R, R^* >$ generated by the operators*

$$R : 1 \to \sum_i e_i \otimes e_i$$

$$R^*(e_i \otimes e_j) = \delta_{ij}$$

*produces via Tannakian duality the algebra $C(U_N^+)$.*

*Proof.* By Proposition 4.5 the intertwining relations coming from $R, R^*$, and so from any element of the tensor category $< R, R^* >$, hold automatically, so the quotient operation in Proposition 4.8 is trivial, and we obtain the algebra $C(U_N^+)$ itself, as stated. $\square$

As a conclusion, in order to compute the Tannakian category of $U_N^+$, we must simply solve a linear algebra question, namely computing the category $< R, R^* >$.

Regarding now $O_N^+$, the result here is similar, as follows:

**Proposition 4.23.** *The tensor category $< R, R^* >$ generated by the operators*

$$R : 1 \to \sum_i e_i \otimes e_i$$

$$R^*(e_i \otimes e_j) = \delta_{ij}$$

*with identifying the colors, $\circ = \bullet$, produces via Tannakian duality the algebra $C(O_N^+)$.*

*Proof.* By Proposition 4.5 the intertwining relations coming from $R, R^*$, and so from any element of the tensor category $< R, R^* >$, hold automatically, so the quotient operation in Proposition 4.8 is trivial, and we obtain the algebra $C(O_N^+)$ itself, as stated. $\square$

Our goal now will be that of reaching to a better understanding of $R, R^*$. In order to do so, we use a diagrammatic formalism, as follows:

**Definition 4.24.** *Let $k, l$ be two colored integers, having lengths $|k|, |l| \in \mathbb{N}$.*

1. *$P_2(k, l)$ is the set of pairings between an upper row of $|k|$ points, and a lower row of $|l|$ points, with these two rows of points colored by $k, l$.*
2. *$\mathcal{P}_2(k, l) \subset P_2(k, l)$ is the set of matching pairings, whose horizontal strings connect $\circ - \circ$ or $\bullet - \bullet$, and whose vertical strings connect $\circ - \bullet$.*
3. *$NC_2(k, l) \subset P_2(k, l)$ is the set of pairings which are noncrossing, in the sense that we can draw the pairing as for the strings to be noncrossing.*
4. *$\mathcal{NC}_2(k, l) \subset P_2(k, l)$ is the subset of noncrossing matching pairings, obtained as an intersection, $\mathcal{NC}_2(k, l) = NC_2(k, l) \cap \mathcal{P}_2(k, l)$.*

The relation with the Tannakian categories of linear maps comes from the fact that we can associate linear maps to the pairings, as in [38], as follows:

**Definition 4.25.** *Associated to any pairing $\pi\in P_2(k,l)$ and any $N\in\mathbb N$ is the linear map*
$$T_\pi:(\mathbb C^N)^{\otimes k}\to(\mathbb C^N)^{\otimes l}$$
*given by the following formula, with $\{e_1,\ldots,e_N\}$ being the standard basis of $\mathbb C^N$,*
$$T_\pi(e_{i_1}\otimes\ldots\otimes e_{i_k})=\sum_{j_1\ldots j_l}\delta_\pi\begin{pmatrix}i_1&\ldots&i_k\\ j_1&\ldots&j_l\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_l}$$
*and with the Kronecker symbols $\delta_\pi\in\{0,1\}$ depending on whether the indices fit or not.*

To be more precise here, in the definition of the Kronecker symbols, we agree to put the two multi-indices on the two rows of points of the pairing, in the obvious way. The Kronecker symbols are then defined by $\delta_\pi=1$ when all the strings of $\pi$ join equal indices, and by $\delta_\pi=0$ otherwise. Observe that all this is independent of the coloring.

Here are a few basic examples of such linear maps:

**Proposition 4.26.** *The correspondence $\pi\to T_\pi$ has the following properties:*
(1) $T_\cap=R$.
(2) $T_\cup=R^*$.
(3) $T_{||\ldots||}=id$.
(4) $T_{\chi}=\Sigma$.

*Proof.* We can assume if we want that all the upper and lower legs of $\pi$ are colored $\circ$. With this assumption made, the proof goes as follows:

(1) We have $\cap\in P_2(\emptyset,\circ\circ)$, and so the corresponding operator is a certain linear map $T_\cap:\mathbb C\to\mathbb C^N\otimes\mathbb C^N$. The formula of this map is as follows:
$$\begin{aligned}T_\cap(1)&=\sum_{ij}\delta_\cap(i\ j)e_i\otimes e_j\\&=\sum_{ij}\delta_{ij}e_i\otimes e_j\\&=\sum_i e_i\otimes e_i\end{aligned}$$

We recognize here the formula of $R(1)$, and so we have $T_\cap=R$, as claimed.

(2) Here we have $\cup\in P_2(\circ\circ,\emptyset)$, and so the corresponding operator is a certain linear form $T_\cap:\mathbb C^N\otimes\mathbb C^N\to\mathbb C$. The formula of this linear form is as follows:
$$\begin{aligned}T_\cap(e_i\otimes e_j)&=\delta_\cap(i\ j)\\&=\delta_{ij}\end{aligned}$$

Since this is the same as $R^*(e_i\otimes e_j)$, we have $T_\cup=R^*$, as claimed.

(3) Consider indeed the "identity" pairing $||\ldots|| \in P_2(k,k)$, with $k = \circ\circ\ldots\circ\circ$. The corresponding linear map is then the identity, because we have:

$$\begin{aligned}
T_{||\ldots||}(e_{i_1}\otimes\ldots\otimes e_{i_k}) &= \sum_{j_1\ldots j_k}\delta_{||\ldots||}\begin{pmatrix}i_1&\ldots&i_k\\ j_1&\ldots&j_k\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_k}\\
&= \sum_{j_1\ldots j_k}\delta_{i_1j_1}\ldots\delta_{i_kj_k}e_{j_1}\otimes\ldots\otimes e_{j_k}\\
&= e_{i_1}\otimes\ldots\otimes e_{i_k}
\end{aligned}$$

(4) In the case of the basic crossing $\slashed{\times} \in P_2(\circ\circ,\circ\circ)$, the corresponding linear map $T_{\slashed{\times}}:\mathbb{C}^N\otimes\mathbb{C}^N\to\mathbb{C}^N\otimes\mathbb{C}^N$ can be computed as follows:

$$\begin{aligned}
T_{\slashed{\times}}(e_i\otimes e_j) &= \sum_{kl}\delta_{\slashed{\times}}\begin{pmatrix}i&j\\ k&l\end{pmatrix}e_k\otimes e_l\\
&= \sum_{kl}\delta_{il}\delta_{jk}e_k\otimes e_l\\
&= e_j\otimes e_i
\end{aligned}$$

Thus we obtain the flip operator $\Sigma(a\otimes b)=b\otimes a$, as claimed. $\square$

We have the following key result, from [38]:

**Proposition 4.27.** *The assignement $\pi\to T_\pi$ is categorical, in the sense that we have*

$$T_\pi\otimes T_\sigma = T_{[\pi\sigma]}$$

$$T_\pi T_\sigma = N^{c(\pi,\sigma)}T_{[^\sigma_\pi]}$$

$$T_\pi^* = T_{\pi^*}$$

*where $c(\pi,\sigma)$ are certain integers, coming from the erased components in the middle.*

*Proof.* The formulae in the statement are all elementary, as follows:

(1) The concatenation axiom follows from the following computation:

$$\begin{aligned}
&(T_\pi\otimes T_\sigma)(e_{i_1}\otimes\ldots\otimes e_{i_p}\otimes e_{k_1}\otimes\ldots\otimes e_{k_r})\\
=&\sum_{j_1\ldots j_q}\sum_{l_1\ldots l_s}\delta_\pi\begin{pmatrix}i_1&\ldots&i_p\\ j_1&\ldots&j_q\end{pmatrix}\delta_\sigma\begin{pmatrix}k_1&\ldots&k_r\\ l_1&\ldots&l_s\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_q}\otimes e_{l_1}\otimes\ldots\otimes e_{l_s}\\
=&\sum_{j_1\ldots j_q}\sum_{l_1\ldots l_s}\delta_{[\pi\sigma]}\begin{pmatrix}i_1&\ldots&i_p&k_1&\ldots&k_r\\ j_1&\ldots&j_q&l_1&\ldots&l_s\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_q}\otimes e_{l_1}\otimes\ldots\otimes e_{l_s}\\
=&\ T_{[\pi\sigma]}(e_{i_1}\otimes\ldots\otimes e_{i_p}\otimes e_{k_1}\otimes\ldots\otimes e_{k_r})
\end{aligned}$$

(2) The composition axiom follows from the following computation:

$$\begin{aligned}
& T_\pi T_\sigma(e_{i_1}\otimes\ldots\otimes e_{i_p})\\
= & \sum_{j_1\ldots j_q}\delta_\sigma\begin{pmatrix}i_1&\ldots&i_p\\ j_1&\ldots&j_q\end{pmatrix}\sum_{k_1\ldots k_r}\delta_\pi\begin{pmatrix}j_1&\ldots&j_q\\ k_1&\ldots&k_r\end{pmatrix}e_{k_1}\otimes\ldots\otimes e_{k_r}\\
= & \sum_{k_1\ldots k_r}N^{c(\pi,\sigma)}\delta_{[^\sigma_\pi]}\begin{pmatrix}i_1&\ldots&i_p\\ k_1&\ldots&k_r\end{pmatrix}e_{k_1}\otimes\ldots\otimes e_{k_r}\\
= & N^{c(\pi,\sigma)}T_{[^\sigma_\pi]}(e_{i_1}\otimes\ldots\otimes e_{i_p})
\end{aligned}$$

(3) Finally, the involution axiom follows from the following computation:

$$\begin{aligned}
& T_\pi^*(e_{j_1}\otimes\ldots\otimes e_{j_q})\\
= & \sum_{i_1\ldots i_p}< T_\pi^*(e_{j_1}\otimes\ldots\otimes e_{j_q}),e_{i_1}\otimes\ldots\otimes e_{i_p}> e_{i_1}\otimes\ldots\otimes e_{i_p}\\
= & \sum_{i_1\ldots i_p}\delta_\pi\begin{pmatrix}i_1&\ldots&i_p\\ j_1&\ldots&j_q\end{pmatrix}e_{i_1}\otimes\ldots\otimes e_{i_p}\\
= & T_{\pi^*}(e_{j_1}\otimes\ldots\otimes e_{j_q})
\end{aligned}$$

Summarizing, our correspondence is indeed categorical. □

We can now formulate a first non-trivial result regarding $O_N^+,U_N^+$, which is a Brauer type theorem for these quantum groups, as follows:

**Theorem 4.28.** *For the quantum groups $O_N^+,U_N^+$ we have*

$$Hom(u^{\otimes k},u^{\otimes l})=span\left(T_\pi\Big|\pi\in D(k,l)\right)$$

*with the sets on the right being respectively as follows,*

$$D=NC_2,\mathcal{NC}_2$$

*and with the correspondence $\pi\to T_\pi$ being constructed as above.*

*Proof.* We know from Proposition 4.22 that $U_N^+$ corresponds via Tannakian duality to the category $C=<R,R^*>$. On the other hand, it follows from the above categorical considerations that this latter category is given by the following formula:

$$C=span\left(T_\pi\Big|\pi\in\mathcal{NC}_2\right)$$

To be more precise, consider the following collection of vector spaces:

$$C'=span\left(T_\pi\Big|\pi\in\mathcal{NC}_2\right)$$

According to the various formulae in Proposition 4.27, these vector spaces form a tensor category. But since the two matching semicircles generate the whole collection of matching pairings, via the operations in Proposition 4.27, we obtain from this $C = C'$. As for the result for $O_N^+$, this follows by adding to the picture the self-adjointness condition $u = \bar{u}$, which corresponds, at the level of pairings, to removing the colors. $\square$

By using the same methods, namely the general Tannakian duality result established above, we can recover as well the classical Brauer theorem [54], as follows:

**Theorem 4.29.** *For the groups $O_N, U_N$ we have*

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(T_\pi\Big|\pi \in D(k,l)\right)$$

*with $D = P_2, \mathcal{P}_2$ respectively, and with $\pi \to T_\pi$ being constructed as above.*

*Proof.* As already mentioned, this result is due to Brauer [54], and is closely related to the Schur-Weyl duality [144]. There are several proofs of this result, one classical proof being via classical Tannakian duality, for the usual closed subgroups $G \subset U_N$.

In the present context, we can deduce this result from the one that we already have, for $O_N^+, U_N^+$. The idea is very simple, namely that of "adding crossings", as follows:

(1) The group $U_N \subset U_N^+$ is defined via the following relations:

$$[u_{ij}, u_{kl}] = 0$$

$$[u_{ij}, \bar{u}_{kl}] = 0$$

But these relations which tell us that the following operators must be in the associated Tannakian category $C$:

$$T_\pi \quad , \quad \pi = \slash\!\!\!\backslash$$

$$T_\pi \quad , \quad \pi = \slash\!\!\!\backslash$$

Thus the associated Tannakian category is $C = span(T_\pi|\pi \in D)$, with:

$$D = < \mathcal{NC}_2, \slash\!\!\!\backslash, \slash\!\!\!\backslash > = \mathcal{P}_2$$

Thus, we are led to the conclusion in the statement.

(2) In order to deal now with $O_N$, we can simply use the following formula:

$$O_N = O_N^+ \cap U_N$$

At the categorical level, this tells us that the associated Tannakian category is given by $C = span(T_\pi|\pi \in D)$, with:

$$D = < NC_2, \mathcal{P}_2 > = P_2$$

Thus, we are led to the conclusion in the statement. $\square$

The above material was just an introduction to the compact quantum groups and their representation categories. For more, including various generalizations, we refer to [76], [77], [78], [79], [89], [128], [130], [146], [147], [148], [149], [150].

## 5. Free rotations

Let us begin with a summary of the Brauer type results established in the previous section. The statement here, collecting what we have so far, is as follows:

**Theorem 5.1.** *For the basic unitary quantum groups, namely*

$$\begin{tikzcd} U_N \arrow[r] & U_N^+ \\ O_N \arrow[r] \arrow[u] & O_N^+ \arrow[u] \end{tikzcd}$$

*the intertwiners between the Peter-Weyl representations are given by*

$$Hom(u^{\otimes k},u^{\otimes l})=span\left(T_\pi\Big|\pi\in D(k,l)\right)$$

*with the linear maps $T_\pi$ associated to the pairings $\pi$ being given by*

$$T_\pi(e_{i_1}\otimes\ldots\otimes e_{i_k})=\sum_{j_1\ldots j_l}\delta_\pi\begin{pmatrix}i_1&\ldots&i_k\\ j_1&\ldots&j_l\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_l}$$

*and with the corresponding sets of pairings $D$ being as follows,*

$$\begin{tikzcd} \mathcal{P}_2 \arrow[d] & \mathcal{NC}_2 \arrow[l] \arrow[d] \\ P_2 & NC_2 \arrow[l] \end{tikzcd}$$

*with calligraphic standing for matching, and with NC standing for noncrossing.*

*Proof.* This is indeed a summary of the results that we have, established in the previous section, and coming from Tannakian duality, via some combinatorics. □

In order to work out some concrete applications, we must understand if the above linear maps $T_\pi$ are linearly independent or not. Let us start with:

**Proposition 5.2.** *To any partition $\pi\in P(k)$ let us associate the vector*

$$\xi_\pi=\sum_{i_1\ldots i_k}\delta_\pi(i_1,\ldots,i_k)\,e_{i_1}\otimes\ldots\otimes e_{i_k}$$

*with the Kronecker symbols being defined as usual, according to whether the indices fit or not. The Gram matrix of these vectors is then given by*

$$G_k(\pi,\sigma)=N^{|\pi\vee\sigma|}$$

*where $\pi\vee\sigma\in P(k)$ is obtained by superposing $\pi,\sigma$, and $|.|$ is the number of blocks.*

*Proof.* According to the formula of the vectors $\xi_\pi$, we have:

$$\begin{aligned}
<\xi_\pi,\xi_\sigma> &= \sum_{i_1\ldots i_k}\delta_\pi(i_1,\ldots,i_k)\delta_\sigma(i_1,\ldots,i_k)\\
&= \sum_{i_1\ldots i_k}\delta_{\pi\vee\sigma}(i_1,\ldots,i_k)\\
&= N^{|\pi\vee\sigma|}
\end{aligned}$$

Thus, we have obtained the formula in the statement. □

As an illustration, at $k=2$ we have $P(2)=\{||,\sqcap\}$, and the Gram matrix is:

$$G_2=\begin{pmatrix}N^2&N\\N&N\end{pmatrix}$$

At $k=3$ now, we have $P(3)=\{|||,\sqcap|,\sqcap\hskip-2.3mm{\scriptstyle |}\hskip1.3mm,|\sqcap,\sqcap\hskip-0.7mm\sqcap\}$, and the Gram matrix is:

$$G_3=\begin{pmatrix}N^3&N^2&N^2&N^2&N\\N^2&N^2&N&N&N\\N^2&N&N^2&N&N\\N^2&N&N&N^2&N\\N&N&N&N&N\end{pmatrix}$$

In what follows we will compute the determinant of $G_k$, which will solve the linear independence problem for the vectors $\xi_\pi$. Let us start with:

**Definition 5.3.** *Given two partitions $\pi,\sigma\in P(k)$, we write $\pi\leq\sigma$ if each block of $\pi$ is contained in a block of $\sigma$.*

Observe that this order is compatible with the previous convention for $\pi\vee\sigma$, in the sense that the $\vee$ operation is the supremum operation with respect to $\leq$. At the level of examples, at $k=2$ we have $P(2)=\{||,\sqcap\}$, and the order relation is as follows:

$$||\leq\sqcap$$

At $k=3$ now, we have $P(3)=\{|||,\sqcap|,\sqcap\hskip-2.3mm{\scriptstyle |}\hskip1.3mm,|\sqcap,\sqcap\hskip-0.7mm\sqcap\}$, and the order relation is:

$$|||\leq\sqcap|,\sqcap\hskip-2.3mm{\scriptstyle |}\hskip1.3mm,|\sqcap\leq\sqcap\hskip-0.7mm\sqcap$$

Summarizing, this order is very intuitive, and simple to compute. By using now this order, we can talk about the Möbius function of $P(k)$, as follows:

**Definition 5.4.** *The Möbius function of any lattice, and so of $P(k)$, is given by*

$$\mu(\pi,\sigma)=\begin{cases}1&\text{if }\pi=\sigma\\-\sum_{\pi\leq\tau<\sigma}\mu(\pi,\tau)&\text{if }\pi<\sigma\\0&\text{if }\pi\not\leq\sigma\end{cases}$$

*with this construction being performed by recurrence.*

This is something standard in combinatorics. As an illustration here, let us go back to the set of 2-point partitions, $P(2)=\{||,\sqcap\}$. We have by definition:

$$\mu(||,||)=\mu(\sqcap,\sqcap)=1$$

Next in line, we know that we have $||<\sqcap$, with no intermediate partition in between, and so the above recurrence procedure gives:

$$\mu(||,\sqcap)=-\mu(||,||)=-1$$

Finally, we have $\sqcap\not\leq||$, and so the last value of the Möbius function is:

$$\mu(\sqcap,||)=0$$

Thus, as a conclusion, we have computed the Möbius matrix $M_2(\pi,\sigma)=\mu(\pi,\sigma)$ of the lattice $P(2)=\{||,\sqcap\}$, the formula of this matrix being as follows:

$$M_2=\begin{pmatrix}1&-1\\0&1\end{pmatrix}$$

The computation for $P(3)=\{|||,\sqcap|,\sqcap\!\!\!\!\!\;|\,,|\sqcap,\sqcap\!\sqcap\}$ is similar, and leads to the following formula for the associated Möbius matrix:

$$M_3=\begin{pmatrix}1&-1&-1&-1&2\\0&1&0&0&-1\\0&0&1&0&-1\\0&0&0&1&-1\\0&0&0&0&1\end{pmatrix}$$

Back to the general case now, the main interest in the Möbius function comes from the Möbius inversion formula, which states that the following happens:

$$f(\sigma)=\sum_{\pi\leq\sigma}g(\pi)\quad\Longrightarrow\quad g(\sigma)=\sum_{\pi\leq\sigma}\mu(\pi,\sigma)f(\pi)$$

In linear algebra terms, the statement and proof of this formula are as follows:

**Theorem 5.5.** *The inverse of the adjacency matrix of $P(k)$, given by*

$$A_k(\pi,\sigma)=\begin{cases}1&\text{if }\pi\leq\sigma\\0&\text{if }\pi\not\leq\sigma\end{cases}$$

*is the Möbius matrix of $P$, given by $M_k(\pi,\sigma)=\mu(\pi,\sigma)$.*

*Proof.* This is well-known, coming for instance from the fact that $A_k$ is upper triangular. Indeed, when inverting, we are led into the recurrence from Definition 5.4. $\square$

As an illustration, for $P(2)=\{||,\sqcap\}$ the formula $M_2=A_2^{-1}$ appears as follows:

$$\begin{pmatrix}1&-1\\0&1\end{pmatrix}=\begin{pmatrix}1&1\\0&1\end{pmatrix}^{-1}$$

Also, for $P(3)=\{|||,\sqcap|,\sqcap\!\!\!\!\!|\ ,|\sqcap,\sqcap\!\!\sqcap\}$ the formula $M_3=A_3^{-1}$ reads:

$$\begin{pmatrix}1&-1&-1&-1&2\\0&1&0&0&-1\\0&0&1&0&-1\\0&0&0&1&-1\\0&0&0&0&1\end{pmatrix}=\begin{pmatrix}1&1&1&1&1\\0&1&0&0&1\\0&0&1&0&1\\0&0&0&1&1\\0&0&0&0&1\end{pmatrix}^{-1}$$

Now back to our Gram matrix considerations, we have the following key result, based on this technology, which basically solves our determinant question:

**Proposition 5.6.** *The Gram matrix is given by $G_k=A_kL_k$, where*

$$L_k(\pi,\sigma)=\begin{cases}N(N-1)\ldots(N-|\pi|+1)&\text{if }\sigma\leq\pi\\0&\text{otherwise}\end{cases}$$

*and where $A_k=M_k^{-1}$ is the adjacency matrix of $P(k)$.*

*Proof.* We have the following computation, using Proposition 5.2:

$$\begin{aligned}G_k(\pi,\sigma)&=N^{|\pi\vee\sigma|}\\&=\#\left\{i_1,\ldots,i_k\in\{1,\ldots,N\}\Big|\ker i\geq\pi\vee\sigma\right\}\\&=\sum_{\tau\geq\pi\vee\sigma}\#\left\{i_1,\ldots,i_k\in\{1,\ldots,N\}\Big|\ker i=\tau\right\}\\&=\sum_{\tau\geq\pi\vee\sigma}N(N-1)\ldots(N-|\tau|+1)\end{aligned}$$

According now to the definition of $A_k,L_k$, this formula reads:

$$\begin{aligned}G_k(\pi,\sigma)&=\sum_{\tau\geq\pi}L_k(\tau,\sigma)\\&=\sum_\tau A_k(\pi,\tau)L_k(\tau,\sigma)\\&=(A_kL_k)(\pi,\sigma)\end{aligned}$$

Thus, we are led to the formula in the statement. $\square$

As an illustration for the above result, at $k=2$ we have $P(2)=\{||,\sqcap\}$, and the above decomposition $G_2=A_2L_2$ appears as follows:

$$\begin{pmatrix}N^2&N\\N&N\end{pmatrix}=\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}N^2-N&0\\N&N\end{pmatrix}$$

At $k=3$ now, we have $P(3)=\{|||,\sqcap|,\sqcap\!\!\!\!|\ ,|\sqcap,\sqcap\!\!\sqcap\}$, and the Gram matrix is:

$$G_3=\begin{pmatrix}N^3&N^2&N^2&N^2&N\\N^2&N^2&N&N&N\\N^2&N&N^2&N&N\\N^2&N&N&N^2&N\\N&N&N&N&N\end{pmatrix}$$

Regarding $L_3$, this can be computed by writing down the matrix $E_3(\pi,\sigma)=\delta_{\sigma\leq\pi}|\pi|$, and then replacing each entry by the corresponding polynomial in $N$. We reach to the conclusion that the product $A_3L_3$ is as follows, producing the above matrix $G_3$:

$$A_3L_3=\begin{pmatrix}1&1&1&1&1\\0&1&0&0&1\\0&0&1&0&1\\0&0&0&1&1\\0&0&0&0&1\end{pmatrix}\begin{pmatrix}N^3-3N^2+2N&0&0&0&0\\N^2-N&N^2-N&0&0&0\\N^2-N&0&N^2-N&0&0\\N^2-N&0&0&N^2-N&0\\N&N&N&N&N\end{pmatrix}$$

In general, the formula $G_k=A_kL_k$ appears a bit in the same way, with $A_k$ being binary and upper triangular, and with $L_k$ depending on $N$, and being lower triangular.

We are led in this way to the following formula, due to Lindstöm [103]:

**Theorem 5.7.** *The determinant of the Gram matrix $G_k$ is given by*

$$\det(G_k)=\prod_{\pi\in P(k)}\frac{N!}{(N-|\pi|)!}$$

*with the convention that in the case $N<k$ we obtain $0$.*

*Proof.* If we order $P(k)$ as usual, with respect to the number of blocks, and then lexicographically, then $A_k$ is upper triangular, and $L_k$ is lower triangular. Thus, we have:

$$\begin{aligned}\det(G_k)&=\det(A_k)\det(L_k)\\&=\det(L_k)\\&=\prod_\pi L_k(\pi,\pi)\\&=\prod_\pi N(N-1)\ldots(N-|\pi|+1)\end{aligned}$$

Thus, we are led to the formula in the statement. □

Getting back now to quantum groups, or rather to the corresponding Tannakian categories, written as spans of diagrams, we have the following result:

**Theorem 5.8.** *The vectors associated to the partitions, namely*

$$\left\{\xi_\pi \in (\mathbb{C}^N)^{\otimes k}\Big|\pi \in P(k)\right\}$$

*are linearly independent for $N \geq k$.*

*Proof.* Here the first assertion follows from Theorem 5.7, the Gram determinant computed there being nonzero for $N \geq k$, and the second assertion follows from it. □

As a first application, we can study the laws of characters. First, we have:

**Proposition 5.9.** *For the basic unitary quantum groups, namely*

$$\begin{array}{ccc}U_N & \longrightarrow & U_N^+\\ \uparrow & & \uparrow\\ O_N & \longrightarrow & O_N^+\end{array}$$

*the moments of the main character, which are the numbers*

$$M_k = \int_G \chi^k$$

*depending on a colored integer $k$, are smaller than the following numbers,*

$$\begin{array}{ccc}|\mathcal{P}_2(k)| & - & |\mathcal{NC}_2(k)|\\ | & & |\\ |P_2(k)| & - & |NC_2(k)|\end{array}$$

*and with equality happening in each case at $N \geq k$.*

*Proof.* We have the following computation, based on Theorem 5.1, and on the character formulae from Peter-Weyl theory, for each of our quantum groups:

$$\begin{aligned}\int_G \chi^k &= \dim(Fix(u^{\otimes k}))\\ &= \dim\left(span\left(\xi_\pi\Big|\pi \in D(k)\right)\right)\\ &\leq |D(k)|\end{aligned}$$

Thus, we have the inequalities in the statement, coming from easiness and Peter-Weyl. As for the last assertion, this follows from Theorem 5.8. □

In order to advance now, we must do some combinatorics and probability, first by counting the numbers in Proposition 5.9, and then by recovering the measures having these numbers as moments. We will restrict the attention to the orthogonal case, which is simpler, and leave the unitary case, which is more complicated, for later. Since there are no pairings when $k$ is odd, we can assume that $k$ is even, and with the change $k \to 2k$, the partition count in the orthogonal case is as follows:

**Proposition 5.10.** *We have the following formulae for pairings,*

$$|P_2(2k)| = (2k)!!$$

$$|NC_2(2k)| = C_k$$

*with the numbers involved, double factorials and Catalan numbers, being as follows:*

$$(2k)!! = (2k-1)(2k-3)(2k-5)\dots$$

$$C_k = \frac{1}{k+1}\binom{2k}{k}$$

*Proof.* We have two assertions here, the idea being as follows:

(1) We must count the pairings of $\{1,\dots,2k\}$. Now observe that such a pairing appears by pairing 1 to a certain number, and there are $2k-1$ choices here, then pairing the next number, 2 if free or 3 if 2 was taken, to another number, and there are $2k-3$ choices here, and so on. Thus, we are led to the formula in the statement, namely:

$$|P_2(2k)| = (2k-1)(2k-3)(2k-5)\dots$$

(2) We must count the noncrossing pairings of $\{1,\dots,2k\}$. Now observe that such a pairing appears by pairing 1 to an odd number, $2a+1$, and then inserting a noncrossing pairing of $\{2,\dots,2a\}$, and a noncrossing pairing of $\{2a+2,\dots,2k\}$. We conclude from this that we have the following recurrence for the numbers $C_k = |NC_2(2k)|$:

$$C_k = \sum_{a+b=k-1} C_a C_b$$

In terms of the generating series $f(z) = \sum_{k\geq 0} C_k z^k$, this recurrence gives:

$$\begin{aligned} zf^2 &= \sum_{a,b\geq 0} C_a C_b z^{a+b+1} \\ &= \sum_{k\geq 1}\sum_{a+b=k-1} C_a C_b z^k \\ &= \sum_{k\geq 1} C_k z^k \\ &= f-1 \end{aligned}$$

Thus the generating series satisfies the following degree 2 equation:

$$zf^2 - f + 1 = 0$$

Now by solving this equation, using the usual degree 2 formula, and choosing the solution which is bounded at $z = 0$, we obtain:

$$f(z) = \frac{1 - \sqrt{1-4z}}{2z}$$

By using now the Taylor formula for $\sqrt{x}$, we obtain the following formula:

$$f(z) = \sum_{k\geq 0} \frac{1}{k+1}\binom{2k}{k} z^k$$

Thus, we are led to the conclusion in the statement. $\square$

Let us do now the second computation, which is probabilistic. We must find the real probability measures having the above numbers as moments, and we have here:

**Theorem 5.11.** *The standard Gaussian law, and standard Wigner semicircle law*

$$g_1 = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

$$\gamma_1 = \frac{1}{2\pi}\sqrt{4 - x^2} dx$$

*have as $2k$-th moments the numbers $(2k)!!$ and $C_k$, and their odd moments vanish.*

*Proof.* There are several proofs here, the simplest being as follows:

(1) The moments of the normal law $g_1$ in the statement are given by:

$$\begin{aligned}
M_k &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x^k e^{-x^2/2} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (x^{k-1}) \left(-e^{-x^2/2}\right)' dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (k-1) x^{k-2} e^{-x^2/2} dx \\
&= (k-1) \times \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x^{k-2} e^{-x^2/2} dx \\
&= (k-1) M_{k-2}
\end{aligned}$$

Thus by recurrence we have $M_{2k} = (2k)!!$, and we are done.

(2) The moments of the Wigner law $\gamma_1$ in the statement are given by:

$$\begin{aligned}
N_k &= \frac{1}{2\pi}\int_{-2}^2\sqrt{4-x^2}\,x^{2k}dx\\
&= \frac{1}{2\pi}\int_0^\pi\sqrt{4-4\cos^2 t}\,(2\cos t)^{2k}(2\sin t)dt\\
&= \frac{2^{2k+1}}{\pi}\int_0^\pi\cos^{2k}t\sin^2 tdt\\
&= \frac{2^{2k+1}}{\pi}\cdot\frac{(2k)!!2!!}{(2k+3)!!}\cdot\pi\\
&= 2^{2k+1}\cdot\frac{3\cdot 5\cdot 7\ldots(2k-1)}{2\cdot 4\cdot 6\ldots(2k+2)}\\
&= 2^{2k+1}\cdot\frac{(2k)!}{2^k k!2^{k+1}(k+1)!}\\
&= \frac{(2k)!}{k!(k+1)!}
\end{aligned}$$

Here we have used an advanced calculus formula, but a routine computation based on partial integration works as well. Thus we have $N_k = C_k$, and we are done. □

Now back to our orthogonal quantum groups, by using the above we can formulate a concrete result regarding them, as follows:

**Theorem 5.12.** *For the quantum groups $O_N, O_N^+$, the main character*

$$\chi = \sum_i u_{ii}$$

*follows respectively the standard Gaussian, and the Wigner semicircle law*

$$g_1 = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx \quad , \quad \gamma_1 = \frac{1}{2\pi}\sqrt{4-x^2}dx$$

*in the $N\to\infty$ limit.*

*Proof.* This follows by putting together the results that we have, namely Proposition 5.9 applied with $N > k$, and then Proposition 5.10 and Theorem 5.11. □

In the case of $O_N$ the above result cannot really be improved, the fixed $N\in\mathbb{N}$ laws being fairly complicated objects, related to Young tableaux and their combinatorics. In the case of $O_N^+$, however, we will see that some miracles happen, and the convergence in the above result is in fact stationary, starting from $N = 2$. Following [1], we have:

**Theorem 5.13.** *For the quantum group $O_N^+$, the main character follows the standard Wigner semicircle law, and this regardless of the value of $N\geq 2$:*

$$\chi\sim\frac{1}{2\pi}\sqrt{4-x^2}dx$$

*The irreducible representations of $O_N^+$ are all self-adjoint, and can be labelled by positive integers, with their fusion rules being the Clebsch-Gordan ones,*

$$r_k\otimes r_l=r_{|k-l|}+r_{|k-l|+2}+\ldots+r_{k+l}$$

*as for the group $SU_2$. The dimensions of these representations are given by*

$$\dim r_k=\frac{q^{k+1}-q^{-k-1}}{q-q^{-1}}$$

*where $q,q^{-1}$ are the solutions of $X^2-NX+1=0$.*

*Proof.* There are several proofs for this fact, the simplest one being via purely algebraic methods, based on the easiness property of $O_N^+$ from Theorem 5.1 alone:

(1) In order to get started, let us first work out the first few values of the representations $r_k$ that we want to construct, computed by recurrence, according to the Clebsch-Gordan rules in the statement, which will be useful for various illustrations:

$$r_0=1$$

$$r_1=u$$

$$r_2=u^{\otimes 2}-1$$

$$r_3=u^{\otimes 3}-2u$$

$$r_4=u^{\otimes 4}-3u^{\otimes 2}+1$$

$$r_5=u^{\otimes 5}-4u^{\otimes 3}+3u$$

$$\vdots$$

(2) We can see that what we want to do is to split the Peter-Weyl representations $u^{\otimes k}$ into irreducibles, because the above formulae can be written as well as follows:

$$u^{\otimes 0}=r_0$$

$$u^{\otimes 1}=r_1$$

$$u^{\otimes 2}=r_2+r_0$$

$$u^{\otimes 3}=r_3+2r_1$$

$$u^{\otimes 4}=r_4+3r_2+2r_0$$

$$u^{\otimes 5}=r_5+4r_3+5r_1$$

$$\vdots$$

(3) In order to get fully started now, our claim, which will basically prove the theorem, is that we can define, by recurrence on $k \in \mathbb{N}$, a sequence $r_0, r_1, r_2, \ldots$ of irreducible, self-adjoint and distinct representations of $O_N^+$, satisfying:

$$r_0 = 1$$

$$r_1 = u$$

$$r_k + r_{k-2} = r_{k-1} \otimes r_1$$

(4) Indeed, at $k = 0$ this is clear, and at $k = 1$ this is clear as well, with the irreducibility of $r_1 = u$ coming from the embedding $O_N \subset O_N^+$. So assume now that $r_0, \ldots, r_{k-1}$ as above are constructed, and let us construct $r_k$. We have, by recurrence:

$$r_{k-1} + r_{k-3} = r_{k-2} \otimes r_1$$

In particular we have an inclusion of representations, as follows:

$$r_{k-1} \subset r_{k-2} \otimes r_1$$

Now since $r_{k-2}$ is irreducible, by Frobenius reciprocity we have:

$$r_{k-2} \subset r_{k-1} \otimes r_1$$

Thus, there exists a certain representation $r_k$ such that:

$$r_k + r_{k-2} = r_{k-1} \otimes r_1$$

(5) As a first observation, this representation $r_k$ is self-adjoint. Indeed, our recurrence formula $r_k + r_{k-2} = r_{k-1} \otimes r_1$ for the representations $r_0, r_1, r_2, \ldots$ shows that the characters of these representations are polynomials in $\chi_u$. Now since $\chi_u$ is self-adjoint, all the characters that we can obtain via our recurrence are self-adjoint as well.

(6) It remains to prove that $r_k$ is irreducible, and non-equivalent to $r_0, \ldots, r_{k-1}$. For this purpose, observe that according to our recurrence formula, $r_k + r_{k-2} = r_{k-1} \otimes r_1$, we can now split $u^{\otimes k}$, as a sum of the following type, with positive coefficients:

$$u^{\otimes k} = c_k r_k + c_{k-2} r_{k-2} + \ldots$$

We conclude by Peter-Weyl that we have an inequality as follows, with equality precisely when $r_k$ is irreducible, and non-equivalent to the other summands $r_i$:

$$\sum_i c_i^2 \leq \dim(End(u^{\otimes k}))$$

(7) Now let us use the easiness property of $O_N^+$. This gives us an upper bound for the number on the right, that we can add to our inequality, as follows:

$$\sum_i c_i^2 \leq \dim(End(u^{\otimes k})) \leq C_k$$

The point now is that the coefficients $c_i$ come straight from the Clebsch-Gordan rules, and their combinatorics shows that $\sum_i c_i^2$ equals the Catalan number $C_k$, with the remark that this follows as well from the known theory of $SU_2$. Thus, we have global equality in the above estimate, and in particular we have equality at left, as desired.

(8) In order to finish the proof of our claim, it still remains to prove that $r_k$ is non-equivalent to $r_{k-1}, r_{k-3}, \ldots$ But these latter representations appear inside $u^{\otimes k-1}$, and the result follows by using the embedding $O_N \subset O_N^+$, which shows that the even and odd tensor powers of $u$ cannot have common irreducible components.

(9) Summarizing, we have proved our claim, made in step (3) above.

(10) In order now to finish, since by the Peter-Weyl theory any irreducible representation of $O_N^+$ must appear in some tensor power of $u$, and we have a formula for decomposing each $u^{\otimes k}$ into sums of representations $r_k$, as explained above, we conclude that these representations $r_k$ are all the irreducible representations of $O_N^+$.

(11) In what regards now the law of the main character, we obtain here the Wigner law $\gamma_1$, as stated, due to the fact that the equality in (7) gives us the even moments of this law, and that the observation in (8) tells us that the odd moments vanish.

(12) Finally, from the Clebsch-Gordan rules we have in particular:

$$r_k r_1 = r_{k-1} + r_{k+1}$$

We obtain from this, by recurrence, with $q^2 - Nq + 1 = 0$:

$$\dim r_k = q^k + q^{k-2} + \ldots + q^{-k+2} + q^{-k}$$

But this gives the dimension formula in the statement, and we are done. □

Let us discuss now the relation with $SU_2$. This group is the most well-known group in mathematics, and there is an enormous quantity of things known about it. For our purposes, we need a functional analytic approach to it. This can be done as follows:

**Theorem 5.14.** *The algebra of continuous functions on $SU_2$ appears as*

$$C(SU_2) = C^*\left((u_{ij})_{i,j=1,2}\Big| u = F\bar{u}F^{-1} = \text{unitary}\right)$$

*where $F$ is the following matrix,*

$$F = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

*called super-identity matrix.*

*Proof.* This can be done in several steps, as follows:

(1) Let us first compute $SU_2$. Consider an arbitrary $2 \times 2$ complex matrix:

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Assuming $\det U = 1$, the unitarity condition $U^{-1} = U^*$ reads:

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}$$

Thus we must have $d = \bar{a}$, $c = -\bar{b}$, and we obtain the following formula:

$$SU_2 = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \Big| \, |a|^2 + |b|^2 = 1 \right\}$$

(2) With the above formula in hand, the fundamental corepresentation of $SU_2$ is:

$$u = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

Now observe that we have the following equality:

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -b & a \\ -\bar{a} & -\bar{b} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{b} \\ -b & a \end{pmatrix}$$

Thus, with $F$ being as in the statement, we have $uF = F\bar{u}$, and so:

$$u = F\bar{u}F^{-1}$$

We conclude that, if $A$ is the universal algebra in the statement, we have:

$$A \to C(SU_2)$$

(3) Conversely now, let us compute the universal algebra $A$ in the statement. For this purpose, let us write its fundamental corepresentation as follows:

$$u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

We have $uF = F\bar{u}$, with these quantities being respectively given by:

$$uF = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -b & a \\ -d & c \end{pmatrix}$$

$$F\bar{u} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} = \begin{pmatrix} c^* & d^* \\ -a^* & -b^* \end{pmatrix}$$

Thus we must have $d = a^*$, $c = -b^*$, and we obtain the following formula:

$$u = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$$

We also know that this matrix must be unitary, and we have:

$$uu^* = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix} = \begin{pmatrix} aa^* + bb^* & ba - ab \\ a^*b^* - b^*a^* & a^*a + b^*b \end{pmatrix}$$

$$u^*u = \begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix} \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} = \begin{pmatrix} a^*a + bb^* & a^*b - ba^* \\ b^*a - ab^* & aa^* + b^*b \end{pmatrix}$$

Thus, the unitarity equations for $u$ are as follows:

$$aa^* = a^*a = 1 - bb^* = 1 - b^*b$$

$$ab = ba, a^*b = ba^*, ab^* = a^*b, a^*b^* = b^*a^*$$

It follows that $a, b, a^*, b^*$ commute, so our algebra is commutative. Now since this algebra is commutative, the involution $*$ becomes the usual conjugation $-$, and so:

$$u = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

But this tells us that we have $A = C(X)$ with $X \subset SU_2$, and so we have a quotient map $C(SU_2) \to A$, which is inverse to the map constructed in (2), as desired. $\square$

Now with the above result in hand, we can see right away the relation with $O_N^+$, and more specifically with $O_2^+$. Indeed, this latter quantum group appears as follows:

$$C(O_2^+) = C^*\left((u_{ij})_{i,j=1,2}\Big| u = \bar{u} = \text{unitary}\right)$$

Thus, $SU_2$ appears from $O_2^+$ by replacing the identity with the super-identity, or perhaps vice versa, $O_2^+$ appears from $SU_2$ by replacing the super-identity with the identity. In any case, these two quantum groups are definitely related by some "twisting" operation, so they should have similar representation theory. This is indeed the case:

**Theorem 5.15.** *For the group $SU_2$, the main character follows the standard Wigner semicircle law:*

$$\chi \sim \frac{1}{2\pi}\sqrt{4 - x^2}dx$$

*The irreducible representations of $SU_2$ are all self-adjoint, and can be labelled by positive integers, with their fusion rules being the Clebsch-Gordan ones,*

$$r_k \otimes r_l = r_{|k-l|} + r_{|k-l|+2} + \ldots + r_{k+l}$$

*as for the quantum group $O_N^+$. The dimensions of these representations are given by*

$$\dim r_k = k + 1$$

*exactly as for the quantum group $O_2^+$.*

*Proof.* This result is as old as modern mathematics, with many proofs available, all being instructive. One proof, which is straightforward but rather long, is by taking everything that has been said so far about $O_N^+$, starting from the middle of section 4 above, setting $N = 2$, and then twisting everything with the help of the super-identity matrix:

$$F = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

What happens then is that a Brauer theorem for $SU_2$ holds, involving the set $D = NC_2$ as before, but with the implementation of the partitions $\pi \to T_\pi$ being twisted by $F$. In particular, we obtain in this way, as before, inequalities as follows:

$$\dim(End(u^{\otimes k})) \leq C_k$$

But with such inequalities in hand, the proof of Theorem 5.13 applies virtually unchanged, and gives the result, with of course $q = 1$ in the dimension formula. $\square$

Let us discuss now the unification of the $O_N^+$ and $SU_2$ results. In view of Theorem 5.14, and of the comments made afterwards, the idea is clear, namely that of looking at compact quantum groups appearing via relations of the following type:

$$u = F\bar{u}F^{-1} = \text{unitary}$$

In order to clarify what exact matrices $F \in GL_N(\mathbb{C})$ we can use, we must do some computations. Following [1], [48], we first have the following result:

**Proposition 5.16.** *Given a closed subgroup $G \subset U_N^+$, with irreducible fundamental corepresentation $u = (u_{ij})$, this corepresentation is self-adjoint, $u \sim \bar{u}$, precisely when*

$$u = F\bar{u}F^{-1}$$

*for some unitary matrix $F \in U_N$, satisfying the following condition:*

$$F\bar{F} = \pm 1$$

*Moreover, when $N$ is odd we must have $F\bar{F} = 1$.*

*Proof.* Since $u$ is self-adjoint, $u \sim \bar{u}$, we must have $u = F\bar{u}F^{-1}$, for a certain matrix $F \in GL_N(\mathbb{C})$. We obtain from this, by using our assumption that $u$ is irreducible:

$$\begin{aligned} u = F\bar{u}F^{-1} \quad &\implies \quad \bar{u} = \bar{F}u\bar{F}^{-1} \\ &\implies \quad u = (F\bar{F})u(F\bar{F})^{-1} \\ &\implies \quad F\bar{F} = c1 \\ &\implies \quad \bar{F}F = \bar{c}1 \\ &\implies \quad c \in \mathbb{R} \end{aligned}$$

Now by rescaling we can assume $c = \pm 1$, so we have proved so far that:

$$F\bar{F} = \pm 1$$

In order to establish now the formula $FF^*=1$, we can proceed as follows:

$$\begin{aligned}(id\otimes S)u=u^* &\implies (id\otimes S)\bar{u}=u^t\\ &\implies (id\otimes S)(F\bar{u}F^{-1})=Fu^tF^{-1}\\ &\implies u^*=Fu^tF^{-1}\\ &\implies u=(F^*)^{-1}\bar{u}F^*\\ &\implies \bar{u}=F^*u(F^*)^{-1}\\ &\implies \bar{u}=F^*F\bar{u}F^{-1}(F^*)^{-1}\\ &\implies FF^*=d1\end{aligned}$$

We have $FF^*>0$, so $d>0$. On the other hand, from $F\bar{F}=\pm1$, $FF^*=d1$ we get:

$$|\det F|^2=\det(F\bar{F})=(\pm1)^N$$

$$|\det F|^2=\det(FF^*)=d^N$$

Since $d>0$ we obtain from this $d=1$, and so $FF^*=1$ as claimed. We obtain as well that when $N$ is odd the sign must be 1, and so $F\bar{F}=1$, as claimed. □

It is convenient to diagonalize the matrices $F$ that we found. Once again following [48], up to an orthogonal base change, we can assume that our matrix is as follows, where $N=2p+q$ and $\varepsilon=\pm1$, with the $1_q$ block at right disappearing if $\varepsilon=-1$:

$$F=\begin{pmatrix}0&1\\ \varepsilon1&0_{(0)}\\ &&\ddots\\ &&&0&1\\ &&&\varepsilon1&0_{(p)}\\ &&&&&1_{(1)}\\ &&&&&&\ddots\\ &&&&&&&1_{(q)}\end{pmatrix}$$

We are therefore led into the following definition, from [35]:

**Definition 5.17.** *The "super-space" $\mathbb{C}^N_F$ is the usual space $\mathbb{C}^N$, with its standard basis $\{e_1,\ldots,e_N\}$, with a chosen sign $\varepsilon=\pm1$, and a chosen involution on the set of indices,*

$$i\to\bar{i}$$

*with $F$ being the "super-identity" matrix, $F_{ij}=\delta_{i\bar{j}}$ for $i\leq j$ and $F_{ij}=\varepsilon\delta_{i\bar{j}}$ for $i\geq j$.*

In what follows we will usually assume that $F$ is the explicit matrix appearing above. Indeed, up to a permutation of the indices, we have a decomposition $n=2p+q$ such that the involution is, in standard permutation notation:

$$(12)\ldots(2p-1,2p)(2p+1)\ldots(q)$$

Let us construct now some basic compact quantum groups, in our "super" setting. Once again following [35], let us formulate:

**Definition 5.18.** *Associated to the super-space $\mathbb{C}^N_F$ are the following objects:*

(1) *The super-orthogonal group, given by:*

$$O_F = \left\{ U \in U_N \Big| U = F\bar{U}F^{-1} \right\}$$

(2) *The super-orthogonal quantum group, given by:*

$$C(O_F^+) = C^*\left( (u_{ij})_{i,j=1,\ldots,n} \Big| u = F\bar{u}F^{-1} = \text{unitary} \right)$$

As explained in [35], [36], it it possible to considerably extend this list, but for our purposes here, this is what we need for the moment. We have indeed the following result, from [35], making the connection with our unification problem for $O_N^+$ and $SU_2$:

**Theorem 5.19.** *The basic orthogonal groups and quantum groups are as follows:*

(1) *At $\varepsilon = -1$ we have $O_F = Sp_N$ and $O_F^+ = Sp_N^+$.*
(2) *At $\varepsilon = -1$ and $N = 2$ we have $O_F = O_F^+ = SU_2$.*
(3) *At $\varepsilon = 1$ we have $O_F = O_N$ and $O_F^+ = O_N^+$.*

*Proof.* These results are all elementary, as follows:

(1) At $\varepsilon = -1$ this follows from definitions, because the symplectic group $Sp_N \subset U_N$ is by definition the following group:

$$Sp_N = \left\{ U \in U_N \Big| U = F\bar{U}F^{-1} \right\}$$

(2) Still at $\varepsilon = -1$, the equation $U = F\bar{U}F^{-1}$ tells us that the symplectic matrices $U \in Sp_N$ are exactly the unitaries $U \in U_N$ which are patterned as follows:

$$U = \begin{pmatrix} a & b & \ldots \\ -\bar{b} & \bar{a} & \\ \vdots & & \ddots \end{pmatrix}$$

In particular, the symplectic matrices at $N = 2$ are as follows:

$$U = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

Thus we have $Sp_2 = U_2$, and the formula $Sp_2^+ = Sp_2$ is elementary as well, via an analysis similar to the one in the proof of Theorem 5.14 above.

(3) At $\varepsilon = 1$ now, consider the root of unity $\rho = e^{\pi i/4}$, and set:

$$J = \frac{1}{\sqrt{2}} \begin{pmatrix} \rho & \rho^7 \\ \rho^3 & \rho^5 \end{pmatrix}$$

This matrix $J$ is then unitary, and we have:

$$J\begin{pmatrix}0&1\\1&0\end{pmatrix}J^t=1$$

Thus the following matrix is unitary as well, and satisfies $KFK^t=1$:

$$K=\begin{pmatrix}J^{(1)}\\&\ddots\\&&J^{(p)}\\&&&1_q\end{pmatrix}$$

Thus in terms of the matrix $V=KUK^*$ we have:

$$U=F\bar{U}F^{-1}=\text{unitary}\quad\iff\quad V=\bar{V}=\text{unitary}$$

We obtain in this way an isomorphism $O_F^+=O_N^+$ as in the statement, and by passing to classical versions, we obtain as well $O_F=O_N$, as desired. $\square$

With the above formalism and results in hand, we can now formulate the unification result for $O_N^+$ and $SU_2$, which in complete form is as follows:

**Theorem 5.20.** *For the quantum group $O_F^+\in\{O_N^+,Sp_N^+\}$ with $N\geq 2$, the main character follows the standard Wigner semicircle law,*

$$\chi\sim\frac{1}{2\pi}\sqrt{4-x^2}dx$$

*the irreducible representations are all self-adjoint, and can be labelled by positive integers, with their fusion rules being the Clebsch-Gordan ones,*

$$r_k\otimes r_l=r_{|k-l|}+r_{|k-l|+2}+\ldots+r_{k+l}$$

*and the dimensions of these representations are given by*

$$\dim r_k=\frac{q^{k+1}-q^{-k-1}}{q-q^{-1}}$$

*where $q,q^{-1}$ are the solutions of $X^2-NX+1=0$. Also, we have $Sp_2^+=SU_2$.*

*Proof.* This is a straightforward unification of the results that we already have for $O_N^+$ and $SU_2$, the technical details being all standard. See [1]. $\square$

We will be back to $O_N^+$ and $O_F^+$ later on, first in section 7 below, with a number of more advanced algebraic considerations, in relation with super-structures and twists, and then in section 8 below, with a number of advanced probabilistic computations.

## 6. Unitary groups

We have seen in the previous section that the Brauer type results for $O_N,O_N^+,U_N,U_N^+$ lead to concrete and interesting consequences regarding $O_N,O_N^+$. In this section we discuss similar results for $U_N,U_N^+$. The situation here is a bit more complicated than for $O_N,O_N^+$, and we will only do a part of the work here, namely algebra and basic probability, with the other part, advanced probability, being left for later, in section 8 below.

Let us start with a summary of what we know so far about $U_N,U_N^+$:

**Theorem 6.1.** *For the basic unitary quantum groups, namely*

$$U_N\subset U_N^+$$

*the intertwiners between the Peter-Weyl representations are given by*

$$Hom(u^{\otimes k},u^{\otimes l})=span\left(T_\pi\Big|\pi\in D(k,l)\right)$$

*with the linear maps $T_\pi$ associated to the pairings $\pi$ being given by*

$$T_\pi(e_{i_1}\otimes\ldots\otimes e_{i_k})=\sum_{j_1\ldots j_l}\delta_\pi\begin{pmatrix}i_1&\ldots&i_k\\ j_1&\ldots&j_l\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_l}$$

*and with the pairings $D$ being as follows, with calligraphic standing for matching:*

$$\mathcal{P}_2\supset\mathcal{NC}_2$$

*At the level of the moments of the main character, we have in both cases*

$$\int\chi^k\leq|D(k)|$$

*with $D$ being the above sets of pairings, with equality happening at $N\geq k$.*

*Proof.* This is indeed a summary of the results that we have, established in the previous sections, and coming from Tannakian duality, via some combinatorics. To be more precise, the Brauer type results are from section 4, the estimates for the moments follows from this and from Peter-Weyl, as explained in section 5, and finally the last assertion, regarding the equality at $N\geq k$, is something more subtle, explained in section 5 above. □

Let us first investigate the unitary group $U_N$. As it was the case for the orthogonal group $O_N$, in section 5 above, the representation theory here is something quite complicated, related to Young tableaux, and we will not get into this subject. However, once again in analogy with $O_N$, there is one straightforward thing to be done, namely the computation of the law of the main character, in the $N\to\infty$ limit.

In order to do this, we will need a basic probability result, as follows:

**Theorem 6.2.** *The moments of the complex Gaussian law, given by*

$$G_1 \sim \frac{1}{\sqrt{2}}(a+ib)$$

*with $a,b$ being independent, each following the real Gaussian law $g_1$, are given by*

$$M_k = |\mathcal{P}_2(k)|$$

*for any colored integer $k = \circ\bullet\bullet\circ\ldots$*

*Proof.* This is something well-known, which can be done in several steps, as follows:

(1) We recall from section 5 above that the moments of the real Gaussian law $g_1$, with respect to integer exponents $k \in \mathbb{N}$, are the following numbers:

$$m_k = |P_2(k)|$$

Numerically, we have the following formula, explained as well in section 5:

$$m_k = \begin{cases} k!! & (k \text{ even}) \\ 0 & (k \text{ odd}) \end{cases}$$

(2) We will show here that in what concerns the complex Gaussian law $G_1$, similar results hold. Numerically, we will prove that we have the following formula, where a colored integer $k = \circ\bullet\bullet\circ\ldots$ is called uniform when it contains the same number of $\circ$ and $\bullet$, and where $|k| \in \mathbb{N}$ is the length of such a colored integer:

$$M_k = \begin{cases} (|k|/2)! & (k \text{ uniform}) \\ 0 & (k \text{ not uniform}) \end{cases}$$

Now since the matching partitions $\pi \in \mathcal{P}_2(k)$ are counted by exactly the same numbers, and this for trivial reasons, we will obtain the formula in the statement, namely:

$$M_k = |\mathcal{P}_2(k)|$$

(3) This was for the plan. In practice now, we must compute the moments, with respect to colored integer exponents $k = \circ\bullet\bullet\circ\ldots$, of the variable in the statement:

$$c = \frac{1}{\sqrt{2}}(a+ib)$$

As a first observation, in the case where such an exponent $k = \circ\bullet\bullet\circ\ldots$ is not uniform in $\circ,\bullet$, a rotation argument shows that the corresponding moment of $c$ vanishes. To be more precise, the variable $c' = wc$ can be shown to be complex Gaussian too, for any $w \in \mathbb{C}$, and from $M_k(c) = M_k(c')$ we obtain $M_k(c) = 0$, in this case.

(4) In the uniform case now, where $k=\circ\bullet\bullet\circ\ldots$ consists of $p$ copies of $\circ$ and $p$ copies of $\bullet$, the corresponding moment can be computed as follows:

$$\begin{aligned}
M_k &= \int(c\bar{c})^p\\
&= \frac{1}{2^p}\int(a^2+b^2)^p\\
&= \frac{1}{2^p}\sum_s\binom{p}{s}\int a^{2s}\int b^{2p-2s}\\
&= \frac{1}{2^p}\sum_s\binom{p}{s}(2s)!!(2p-2s)!!\\
&= \frac{1}{2^p}\sum_s\frac{p!}{s!(p-s)!}\cdot\frac{(2s)!}{2^s s!}\cdot\frac{(2p-2s)!}{2^{p-s}(p-s)!}\\
&= \frac{p!}{4^p}\sum_s\binom{2s}{s}\binom{2p-2s}{p-s}
\end{aligned}$$

(5) In order to finish now the computation, let us recall that we have the following formula, coming from the generalized binomial formula, or from the Taylor formula:

$$\frac{1}{\sqrt{1+t}}=\sum_{k=0}^\infty\binom{2k}{k}\left(\frac{-t}{4}\right)^k$$

By taking the square of this series, we obtain the following formula:

$$\begin{aligned}
\frac{1}{1+t} &= \sum_{ks}\binom{2k}{k}\binom{2s}{s}\left(\frac{-t}{4}\right)^{k+s}\\
&= \sum_p\left(\frac{-t}{4}\right)^p\sum_s\binom{2s}{s}\binom{2p-2s}{p-s}
\end{aligned}$$

Now by looking at the coefficient of $t^p$ on both sides, we conclude that the sum on the right equals $4^p$. Thus, we can finish the moment computation in (4), as follows:

$$M_p=\frac{p!}{4^p}\times 4^p=p!$$

(6) As a conclusion, if we denote by $|k|$ the length of a colored integer $k=\circ\bullet\bullet\circ\ldots$, the moments of the variable $c$ in the statement are given by:

$$M_k=\begin{cases}(|k|/2)! & (k\text{ uniform})\\ 0 & (k\text{ not uniform})\end{cases}$$

On the other hand, the numbers $|\mathcal{P}_2(k)|$ are given by exactly the same formula. Indeed, in order to have matching pairings of $k$, our exponent $k = \circ\bullet\bullet\circ\ldots$ must be uniform, consisting of $p$ copies of $\circ$ and $p$ copies of $\bullet$, with $p = |k|/2$. But then the matching pairings of $k$ correspond to the permutations of the $\bullet$ symbols, as to be matched with $\circ$ symbols, and so we have $p!$ such matching pairings. Thus, we have the same formula as for the moments of $c$, and we are led to the conclusion in the statement. $\square$

We should mention that the above proof is just one proof among others, designed for a reader having 0 background or almost in probability. There is a lot of interesting mathematics behind the complex Gaussian variables, whose knowledge can avoid some of the above computations, and we recommend here any good probability book.

By getting back now to the unitary group $U_N$, with the above results in hand we can formulate our first concrete result about it, as follows:

**Theorem 6.3.** *For the unitary group $U_N$, the main character*

$$\chi = \sum_i u_{ii}$$

*follows the standard complex Gaussian law*

$$\chi \sim G_1$$

*in the $N \to \infty$ limit.*

*Proof.* This follows by putting together the results that we have, namely Theorem 6.1 applied with $N > k$, and then Theorem 6.2. $\square$

As already mentioned above, as it was the case for the orthogonal group $O_N$, in section 5, the representation theory for $U_N$ at fixed $N \in \mathbb{N}$ is something quite complicated, related to the combinatorics of Young tableaux, and we will not get into this subject here.

There is, however, one more interesting topic regarding $U_N$ to be discussed, namely its precise relation with $O_N$, and more specifically the passage $O_N \to U_N$.

Contrary to the passage $\mathbb{R}^N \to \mathbb{C}^N$, or to the passage $S^{N-1}_\mathbb{R} \to S^{N-1}_\mathbb{C}$, which are both elementary, the passage $O_N \to U_N$ cannot be understood directly. In order to understand this passage we must pass through the corresponding Lie algebras, a follows:

**Theorem 6.4.** *The passage $O_N \to U_N$ appears via Lie algebra complexification,*

$$O_N \to \mathfrak{o}_N \to \mathfrak{u}_n \to U_N$$

*with the Lie algebra $\mathfrak{u}_N$ being a complexification of the Lie algebra $\mathfrak{o}_N$.*

*Proof.* This is something rather philosophical, and advanced as well, that we will not really need here, the idea being as follows:

(1) The unitary and orthogonal groups $U_N, O_N$ are both Lie groups, in the sense that they are smooth manifolds, and the corresponding Lie algebras $\mathfrak{u}_N, \mathfrak{o}_N$, which are by definition the respective tangent spaces at 1, can be computed by differentiating the equations defining $U_N, O_N$, with the conclusion being as follows:

$$\mathfrak{u}_N = \left\{ A \in M_N(\mathbb{C}) \Big| A^* = -A \right\}$$

$$\mathfrak{o}_N = \left\{ B \in M_N(\mathbb{R}) \Big| B^t = -B \right\}$$

(2) This was for the correspondences $U_N \to \mathfrak{u}_N$ and $O_N \to \mathfrak{o}_N$. In the other sense, the correspondences $\mathfrak{u}_N \to U_N$ and $\mathfrak{o}_N \to O_N$ appear by exponentiation, the result here stating that, around 1, the unitary matrices can be written as $U = e^A$, with $A \in \mathfrak{u}_N$, and the orthogonal matrices can be written as $U = e^B$, with $B \in \mathfrak{o}_N$.

(3) In view of all this, in order to understand the passage $O_N \to U_N$ it is enough to understand the passage $\mathfrak{o}_N \to \mathfrak{u}_N$. But, in view of the above explicit formulae for $\mathfrak{o}_N, \mathfrak{u}_N$, this is basically an elementary linear algebra problem. Indeed, let us pick an arbitrary matrix $A \in M_N(\mathbb{C})$, and write it as follows, with $B, C \in M_N(\mathbb{R})$:

$$A = B + iC$$

In terms of $B, C$, the equation $A^* = -A$ defining the Lie algebra $\mathfrak{u}_N$ reads:

$$B^t = -B$$

$$C^t = C$$

(4) As a first observation, we must have $B \in \mathfrak{o}_N$. Regarding now $C$, let us decompose it as follows, with $D$ being its diagonal, and $C'$ being the reminder:

$$C = D + C'$$

The reminder $C'$ being symmetric with 0 on the diagonal, by swithcing all the signs below the main diagonal we obtain a certain matrix $C'_- \in \mathfrak{o}_N$. Thus, we have decomposed $A \in \mathfrak{u}_N$ as follows, with $B, C' \in \mathfrak{o}_N$, and with $D \in M_N(\mathbb{R})$ being diagonal:

$$A = B + iD + iC'_-$$

(5) As a conclusion now, we have shown that we have a direct sum decomposition of real linear spaces as follows, with $\Delta \subset M_N(\mathbb{R})$ being the diagonal matrices:

$$\mathfrak{u}_N \simeq \mathfrak{o}_N \oplus \Delta \oplus \mathfrak{o}_N$$

Thus, we can stop our study here, and say that we have reached the conclusion in the statement, namely that $\mathfrak{u}_N$ appears as a "complexification" of $\mathfrak{o}_N$. $\square$

As before with many other things, that we will not really need in what follows, this was just an introduction to the subject. More can be found in any Lie group book.

Let us discuss now the unitary quantum group $U_N^+$. We have 3 main topics to be discussed, namely the character law with $N\to\infty$, the representation theory at fixed $N\in\mathbb N$, and complexification, and the situation with respect to $U_N$ is as follows:

(1) The asymptotic character law appears as a "free complexification" of the Wigner law, with the combinatorics being similar to the classical case one.

(2) The representation theory is definitely simpler, with the fusion rules being given by a "free complexification" of the Clebsch-Gordan rules, at any $N\geq 2$.

(3) As for the complexification aspects, here the situation is extremely simple, with the passage $O_N^+\to U_N^+$ being a usual free complexification.

More in detail now, let us first discuss the character problematics for $U_N^+$, or rather the difficulties that appear here. We have the following theoretical result, to start with, coming from the general $C^*$-algebra theory developed in section 1 above:

**Theorem 6.5.** *Given a $C^*$-algebra with a faithful trace $(A,tr)$, any normal variable,*

$$aa^*=a^*a$$

*has a "law", which is by definition a complex probability measure $\mu\in\mathcal P(\mathbb C)$ satisfying:*

$$tr(a^k)=\int_{\mathbb C}z^kd\mu(z)$$

*This law is unique, and is supported by the spectrum $\sigma(a)\subset\mathbb C$. In the non-normal case, $aa^*\neq a^*a$, such a law does not exist.*

*Proof.* We have two assertions here, the idea being as follows:

(1) In the normal case, $aa^*=a^*a$, the Gelfand theorem, or rather the subsequent continuous functional calculus theorem, tells us that we have:

$$<a>=C(\sigma(a))$$

Thus the functional $f(a)\to tr(f(a))$ can be regarded as an integration functional on the algebra $C(\sigma(a))$, and by the Riesz theorem, this latter functional must come from a probability measure $\mu$ on the spectrum $\sigma(a)$, in the sense that we must have:

$$tr(f(a))=\int_{\sigma(a)}f(z)d\mu(z)$$

We are therefore led to the conclusions in the statement, with the uniqueness assertion coming from the fact that the elements $a^k$, taken as usual with respect to colored integer exponents, $k=\circ\bullet\bullet\circ\ldots$, generate the whole $C^*$-algebra $C(\sigma(a))$.

(2) In the non-normal case now, $aa^* \neq a^*a$, we must show that such a law does not exist. For this purpose, we can use a positivity trick, as follows:

$$\begin{aligned}
aa^* - a^*a \neq 0 &\implies (aa^* - a^*a)^2 > 0 \\
&\implies aa^*aa^* - aa^*a^*a - a^*aaa^* + a^*aa^*a > 0 \\
&\implies tr(aa^*aa^* - aa^*a^*a - a^*aaa^* + a^*aa^*a) > 0 \\
&\implies tr(aa^*aa^* + a^*aa^*a) > tr(aa^*a^*a + a^*aaa^*) \\
&\implies tr(aa^*aa^*) > tr(aaa^*a^*)
\end{aligned}$$

Now assuming that $a$ has a law $\mu \in \mathcal{P}(\mathbb{C})$, in the sense that the moment formula in the statement holds, the above two different numbers would have to both appear by integrating $|z|^2$ with respect to this law $\mu$, which is contradictory, as desired. □

All the above might look a bit abstract, so as an illustration here, consider the following matrix, which is the simplest example of a non-normal matrix:

$$Z = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

We have then the following formulae, which show that $Z$ has no law, indeed:

$$tr(ZZZ^*Z^*) = tr\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

$$tr(ZZ^*ZZ^*) = tr\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}$$

Getting back now to $U_N^+$, its main character is not normal, so it does not have a law $\mu \in \mathcal{P}(\mathbb{C})$. Here is a concrete illustration for this phenomenon:

**Proposition 6.6.** *The main character of $U_N^+$ satisfies, at $N \geq 4$,*

$$\int_{U_N^+} \chi\chi\chi^*\chi^* = 1$$

$$\int_{U_N^+} \chi\chi^*\chi\chi^* = 2$$

*and so this main character $\chi$ does not have a law $\mu \in \mathcal{P}(\mathbb{C})$.*

*Proof.* This follows from the last assertion in Theorem 6.1, which tells us that the moments of $\chi$ are given by the following formula, valid at any $N \geq k$:

$$\int_{U_N^+} \chi^k = |\mathcal{NC}_2(k)|$$

Indeed, we obtain from this the following formula, valid at any $N\geq 4$:

$$\begin{aligned}\int_{U_N^+}\chi\chi\chi^*\chi^* &= |\mathcal{NC}_2(\circ\circ\bullet\bullet)|\\ &= |\Cap|\\ &= 1\end{aligned}$$

On the other hand, we obtain as well the following formula, once again at $N\geq 4$:

$$\begin{aligned}\int_{U_N^+}\chi\chi^*\chi\chi^* &= |\mathcal{NC}_2(\circ\bullet\circ\bullet)|\\ &= |\cap\cap,\Cap|\\ &= 2\end{aligned}$$

Thus, we have the formulae in the statement. Now since we cannot obtain both 1 and 2 by integrating $|z|^2$ with respect to a measure, our variable has no law $\mu\in\mathcal{P}(\mathbb{C})$. $\square$

Summarizing, we are a bit in trouble here, but we can nevertheless advance, in connection with our questions, in the following rather formal way:

**Definition 6.7.** *Given a $C^*$-algebra with a faithful trace $(A,tr)$, we call a variable $a\in A$ circular when its moments are given by:*

$$tr(a^k)=|\mathcal{NC}_2(k)|$$

*In this case we also write $a\sim\Gamma_1$, and call $\Gamma_1$ the Voiculescu circular law.*

In other words, what we are doing here is calling $\Gamma_1$ the "formal law" having as moments the numbers $M_k=|\mathcal{NC}_2(k)|$. We will see later, in section 8 below, some theory here. We will see there as well that the passage $\gamma_1\to\Gamma_1$ is similar to the passage $g_1\to G_1$, in the sense that, with suitable definitions for everything, $\Gamma_1$ can be alternatively defined as follows, with $\alpha,\beta$ being "free", each following the Wigner semicircle law $\gamma_1$:

$$\Gamma_1\sim\frac{1}{\sqrt{2}}(\alpha+i\beta)$$

In what follows, in order to do our quantum group work for $U_N^+$, we will use the above definition as it is. We will also need the following result, which is the standard illustration for the above-mentioned freeness decomposition of the circular variables:

**Theorem 6.8.** *Let $H$ be the Hilbert space having as basis the colored integers $k=\circ\bullet\bullet\circ\ldots$, and consider the shift operators $S:k\to\circ k$ and $T:k\to\bullet k$. We have then*

$$S+S^*\sim\gamma_1$$

$$S+T^*\sim\Gamma_1$$

*with respect to the state $\varphi(T)=<Te,e>$, where $e$ is the empty word.*

*Proof.* This is standard free probability, the idea being as follows:

(1) We must compute the moments of the shift $S : k \to \circ k$ with respect to the state $\varphi(T) =< Te, e >$. Our claim is that these moments are given by:

$$< (S + S^*)^k e, e >= |NC_2(k)|$$

Indeed, when expanding $(S + S^*)^k$ and computing the value of $\varphi : T \to< Te, e >$, the only contributions will come via the formula $S^*S = 1$, which must succesively apply, as to collapse the whole product of $S, S^*$ variables into a 1 quantity. But these applications of $S^*S = 1$ must appear in a non-crossing manner, and so the contributions, which are each worth 1, are parametrized by the partitions $\pi \in NC_2(k)$. Thus, we obtain the above moment formula, which shows that we have $S + S^* \sim \gamma_1$, as claimed.

(2) The proof of the second formula is similar. With $S : k \to \circ k$ and $T : k \to \bullet k$, our claim is that we have the following moment formula:

$$< (S + T^*)^k e, e >= |\mathcal{NC}_2(k)|$$

Indeed, let us expand the quantity $(S + T^*)^k$, and apply the state $\varphi$. This time the contributions will come via the formulae $S^*S = 1$, $T^*T = 1$, which must succesively apply, as to collapse the whole product of $S, S^*, T, T^*$ variables into a 1 quantity. As before, these applications of $S^*S = 1$, $T^*T = 1$ must appear in a non-crossing manner, but what happens now, in contrast with (1) above, where $S + S^*$ was self-adjoint, is that at each point where the exponent $k$ has a $\circ$ entry we must use $T^*T = 1$, and at each point where the exponent $k$ has a $\bullet$ entry we must use $S^*S = 1$. Thus the contributions, which are each worth 1, are parametrized by the partitions $\pi \in \mathcal{NC}_2(k)$. Thus, we obtain the above moment formula, which shows that we have $S + T^* \sim \Gamma_1$, as claimed. □

We will be back with more explanations on all this in section 8 below. For our purposes now, the above definition and theorem are all we need.

Getting back now to the quantum group $U_N^+$, we can reformulate the main result that we have so far about it, by using the above notions, as follows:

**Theorem 6.9.** *For the quantum group $U_N^+$ with $N \geq 2$ we have*

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(T_\pi \middle| \pi \in D(k, l)\right)$$

*and at the level of the moments of the main character we have*

$$\int_{U_N^+} \chi^k \leq |\mathcal{NC}_2(k)|$$

*with equality at $N \geq k$, the numbers on the right being the moments of $\Gamma_1$.*

*Proof.* This is something that we already know. To be more precise, the Brauer type result is from section 4, the estimate for the moments follows from this and from Peter-Weyl, as explained in section 5, the equality at $N \geq k$ is something more subtle, explained in section 5, and the last statement comes from the above discussion. □

With the above result in hand, we can now go ahead and do with $U_N^+$ exactly what we did with $O_N^+$ in section 5, with modifications where needed, namely constructing the irreducible representations by recurrence, using a Frobenius duality trick, computing the fusion rules, and concluding as well that we have $\chi \sim \Gamma_1$, at any $N \geq 2$.

In practice, all this will be more complicated than for $O_N^+$, mainly because the fusion rules will be something new, in need of some preliminary combinatorial study. These fusion rules will be a kind of "free Clebsch-Gordan rules", as follows:

$$r_k \otimes r_l = \sum_{k=xy,l=\bar{y}z} r_{xz}$$

Let $W$ be the set of colored integers $k = \circ\bullet\bullet\circ\ldots$, and consider the complex algebra $E$ spanned by $W$. We have then an isomorphism, as follows:

$$(\mathbb{C} < X, X^* >, +, \cdot) \simeq (E, +, \cdot)$$

$$X \to \circ \quad , \quad X^* \to \bullet$$

We define an involution on our algebra $E$, by antilinearity and antimultiplicativity, according to the following formulae, with $e$ being as usual the empty word:

$$\bar{e} = e \quad , \quad \bar{\circ} = \bullet \quad , \quad \bar{\bullet} = \circ$$

With these conventions, we have the following result:

**Proposition 6.10.** *The map $\times : W \times W \to E$ given by*

$$x \times y = \sum_{x=ag,y=\bar{g}b} ab$$

*extends by linearity into an associative multiplication of $E$.*

*Proof.* Observe first that $\times$ is well-defined, the sum being finite. Let us prove now that $\times$ is associative. Let $x, y, z \in W$. Then:

$$\begin{aligned}(x \times y) \times z &= \sum_{x=a\bar{g},y=gb} ab \times z\\ &= \sum_{x=a\bar{g},y=gb,ab=ch,z=\bar{h}d} cd\end{aligned}$$

Now observe that for $a,b,c,h\in W$ the equality $ab=ch$ is equivalent to $b=uh, c=au$ with $u\in W$, or to $a=cv, h=vb$ with $v\in W$. Thus, we have:

$$\begin{aligned}(x\times y)\times z &= \sum_{x=a\bar{g},y=guh,z=\bar{h}d} aud\\ &+ \sum_{x=cv\bar{g},y=gb,z=\bar{b}\bar{v}d} cd\end{aligned}$$

A similar computation shows that $x\times(y\times z)$ is given by the same formula. □

Next, we have the following result:

**Proposition 6.11.** *Consider the following morphism, with $S,T$ being the shifts,*

$$P:(E,+,\cdot)\to(B(l^2(W)),+,\circ)$$

$$\alpha\to S+T^*$$

*and let $E_n\subset E$ be the linear space generated by the words of $W$ having length $\leq n$.*

(1) *If $J:E\to E$ is the map $f\to P(f)e$, then $(J-Id)E_n\subset E_{n-1}$ for any $n$.*
(2) *$J$ is an isomorphism of $*$-algebras $(E,+,\cdot)\simeq(E,+,\times)$.*

*Proof.* We have several assertions here, the idea being as follows:

(1) Let $f\in E$. We have then the following formula:

$$P(\alpha)f=(S+T^*)f=\circ\times f$$

Thus, for any $g\in E$, we have the following formula:

$$\begin{aligned}J(\circ g) &= P(\circ)J(g)\\ &= \circ\times J(g)\\ &= J(\circ)\times J(g)\end{aligned}$$

The same argument shows that we have, for any $g\in E$:

$$J(\bullet g)=J(\bullet)\times J(g)$$

Now the algebra $(E,+,\cdot)$ being generated by $\circ$ and $\bullet$, we conclude that $J$ is a morphism of algebras, as follows:

$$J:(E,+,\cdot)\to(E,+,\times)$$

We prove now by recurrence on $n\geq 1$ that we have:

$$(J-Id)E_n\subset E_{n-1}$$

At $n=1$ we have $J(\circ)=\circ$, $J(\bullet)=\bullet$ and $J(e)=e$, and since $E_1$ is generated by $e,\circ,\bullet$, we have $J=Id$ on $E_1$. Now assume that this is true for $n$, and let $k\in E_{n+1}$. We write

$k=\circ f+\bullet g+h$ with $f,g,h\in E_n$, and we have:

$$\begin{aligned}
&(J-Id)k\\
=\ &J(\circ f+\bullet g+h)-(\circ f+\bullet g+h)\\
=\ &[(S+T^*)J(f)+(S^*+T)J(g)+J(h)]-[Sf+Tg+h]\\
=\ &S(J(f)-f)+T(J(g)-g)+T^*J(f)+S^*J(g)+(J(h)-h)
\end{aligned}$$

By using the recurrence assumption, applied to $f,g,h$ we find that $E_n$ contains all the terms of the above sum, and so contains $(J-Id)k$, and we are done.

(2) Here we have to prove that $J$ preserves the involution $*$, and that it is bijective. We have $J*=*J$ on the generators $\{e,\circ,\bullet\}$ of $E$, so $J$ preserves the involution. Also, by (1), the restriction of $J-Id$ to $E_n$ is nilpotent, so $J$ is bijective. $\square$

Following [1], we can now formulate a result about $U_N^+$, which is quite similar to the result for $O_N^+$ from section 5 above, as follows:

**Theorem 6.12.** *For the quantum group $U_N^+$, with $N\geq 2$, the main character follows the Voiculescu circular law,*

$$\chi\sim\Gamma_1$$

*and the irreducible representations can be labelled by the colored integers, $k=\circ\bullet\bullet\circ\ldots$, with $r_e=1$, $r_\circ=u$, $r_\bullet=\bar{u}$, and with the involution and the fusion rules being*

$$\bar{r}_k=r_{\bar{k}}$$

$$r_k\otimes r_l=\sum_{k=xy,l=\bar{y}z}r_{xz}$$

*where $k\to\bar{k}$ is obtained by reversing the word, and switching the colors.*

*Proof.* This is similar to the proof for $O_N^+$, as follows:

(1) In order to get familiar with the fusion rules, let us first work out a few values of the representations $r_k$, computed according to the formula in the statement:

$$r_e=1$$

$$r_\circ=u$$

$$r_\bullet=\bar{u}$$

$$r_{\circ\circ}=u\otimes u$$

$$r_{\circ\bullet}=u\otimes\bar{u}-1$$

$$r_{\bullet\circ}=\bar{u}\otimes u-1$$

$$r_{\bullet\bullet}=\bar{u}\otimes\bar{u}$$

$$\vdots$$

(2) Equivalently, we want to decompose into irreducibles the Peter-Weyl representations, because the above formulae can be written as follows:

$$u^{\otimes e} = r_e$$
$$u^{\otimes \circ} = r_\circ$$
$$u^{\otimes \bullet} = r_\bullet$$
$$u^{\otimes \circ\circ} = r_{\circ\circ}$$
$$u^{\otimes \circ\bullet} = r_{\circ\bullet} + r_e$$
$$u^{\bullet\circ} = r_{\bullet\circ} + r_e$$
$$u^{\bullet\bullet} = r_{\bullet\bullet}$$
$$\vdots$$

(3) In order to prove the fusion rule assertion, let us construct a morphism as follows, by using the polynomiality of the algebra on the left:

$$\Psi : (E, +, \times) \to C(U_N^+)$$
$$\circ \to \chi(u) \quad , \quad \bullet \to \chi(\bar{u})$$

Our claim is that, given an integer $n \geq 1$, assuming that $\Psi(x)$ is the character of an irreducible representation $r_x$ of $U_N^+$, for any $x \in W$ having length $\leq n$, then $\Psi(x)$ is the character of a non-null representation of $U_N^+$, for any $x \in W$ of length $n+1$.

(4) At $n = 1$ this is clear. Assume $n \geq 2$, and let $x \in W$ of length $n+1$. If $x$ contains a $\geq 2$ power of $\circ$ or of $\bullet$, for instance if $x = z \circ \circ y$, then we can set:

$$r_x = r_{z\circ} \otimes r_{\circ y}$$

Assume now that $x$ is an alternating product of $\circ$ and $\bullet$. We can assume that $x$ begins with $\circ$. Then $x = \circ \bullet \circ y$, with $y \in W$ being of length $n-2$. Observe that $\Psi(\bar{z}) = \Psi(z)^*$ holds on the generators $\{e, \circ, \bullet\}$ of $W$, so it holds for any $z \in W$. Thus, we have:

$$\begin{aligned}
< \chi(r_\circ \otimes r_{\bullet\circ y}), \chi(r_{\circ y}) > &= < \chi(r_{\bullet\circ y}), \chi(r_\bullet \otimes r_{\circ y}) > \\
&= < \chi(r_{\bullet\circ y}), \Psi(\bullet \times \circ y) > \\
&= < \chi(r_{\bullet\circ y}), \Psi(\bullet \circ y) + \Psi(y) > \\
&= < \chi(r_{\bullet\circ y}), \chi(r_{\bullet\circ y}) + \chi(r_y) > \\
&\geq 1
\end{aligned}$$

Now since $r_{\circ y}$ is by assumption irreducible, we have $r_{\circ y} \subset r_\circ \otimes r_{\bullet\circ y}$. Consider now the following quantity:

$$\begin{aligned}
\chi(r_\circ \otimes r_{\bullet\circ y}) - \chi(r_{\circ y}) &= \Psi(\circ \times \bullet \circ y - \circ y) \\
&= \Psi(x)
\end{aligned}$$

This is then the character of a representation, as desired.

(5) We know from easiness that we have the following estimate:

$$\dim(Fix(u^{\otimes k})) \leq |\mathcal{NC}_2(k)|$$

By identifying as usual $(\mathbb{C} < X, X^* >, +, \cdot) = (E, +, \cdot)$, the noncommutative monomials in $X, X^*$ correspond to the elements of $W \subset E$. Thus, we have, on $W$:

$$h\Psi J \leq \tau J$$

(6) We prove now by recurrence on $n \geq 0$ that for any $z \in W$ having length $n$, $\Psi(z)$ is the character of an irreducible representation $r_z$.

(7) At $n = 0$ we have $\Psi_G(e) = 1$. So, assume that our claim holds at $n \geq 0$, and let $x \in W$ having length $n+1$. By Proposition 6.11 (1) we have, with $z \in E_n$:

$$J(x) = x + z$$

Let $E^N \subset E$ be the set of functions $f$ such that $f(x) \in \mathbb{N}$ for any $x \in W$. Then $J(\alpha), J(\beta) \in E^N$, so by multiplicativity $J(W) \subset E^N$. In particular, $J(x) \in E^N$. Thus there exist numbers $m(z) \in \mathbb{N}$ such that:

$$J(x) = x + \sum_{l(z) \leq n} m(z)z$$

(8) It is clear that for $a, b \in W$ we have $\tau(a \times \bar{b}) = \delta_{a,b}$. Thus:

$$\begin{aligned} \tau J(x\bar{x}) &= \tau\left(\left(x + \sum m(z)z\right) \times \left(\bar{x} + \sum m(z)\bar{z}\right)\right) \\ &= 1 + \sum m(z)^2 \end{aligned}$$

(9) By recurrence and by (3), $\Psi(x)$ is the character of a representation $r_x$. Thus $\Psi J(x)$ is the character of $r_x + \sum_{l(z) \leq n} m(z) r_z$, and we obtain from this:

$$h\Psi J(x\bar{x}) \geq h(\chi(r_x)\chi(r_x)^*) + \sum m(z)^2$$

(10) By using (5), (8), (9) we conclude that $r_x$ is irreducible, which proves (6).

(11) The fact that the $r_x$ are distinct comes from (5). Indeed, $W$ being an orthonormal basis of $((E, +, \times), \tau)$, for any $x, y \in W$, $x \neq y$ we have $\tau(x \times \bar{y}) = 0$, and so:

$$\begin{aligned} h(\chi(r_x \otimes \bar{r_y})) &= h\Psi J(x\bar{y}) \\ &\leq \tau J(x\bar{y}) \\ &= \tau(x \times \bar{y}) \\ &= 0 \end{aligned}$$

(12) The fact that we obtain all the irreducible representations is clear too, because we can now decompose all the tensor powers $u^{\otimes k}$ into irreducibles.

(13) Finally, since $W$ is an orthonormal system in $((E,+,\times),\tau)$, the set $\Psi(W)=\{\chi(r_x)|x\in W\}$ is an orthonormal system in $C(U_N^+)$, and so we have:

$$h\Psi J=\tau_0 P$$

Now since the distribution of $\chi(u)\in(C(G),h)$ is the functional $h\Psi_G J$, and the distribution of $S+T^*\in(B(l^2(\mathbb{N}*\mathbb{N})),\tau_0)$ is the functional $\tau_0P$, we have $\chi\sim\Gamma_1$, as claimed. $\square$

Let us discuss now the relation with $O_N^+$. As mentioned earlier in this section, in the classical case the passage $O_N\to U_N$ is something not trivial, requiring a passage via the associated Lie algebras. In the free case the situation is very simple, as follows:

**Theorem 6.13.** *We have an identification as follows,*

$$U_N^+=\widetilde{O_N^+}$$

*modulo the usual equivalence relation for compact quantum groups.*

*Proof.* We recall from section 2 above that the free complexification operation $G\to\widetilde{G}$ is obtained by multiplying the coefficients of the fundamental representation by a unitary free from them. We have embeddings as follows, with the first one coming by using the counit, and with the second one coming from the universality property of $U_N^+$:

$$O_N^+\subset\widetilde{O_N^+}\subset U_N^+$$

We must prove that the embedding on the right is an isomorphism, and there are several ways of doing this, all instructive, as follows:

(1) The original argument, from [1], is something quick and advanced, based on the standard free probability fact, from [135], that when freely multiplying a semicircular variable by a Haar unitary we obtain a circular variable. Thus, the main character of $\widetilde{O_N^+}$ is circular, exactly as for $U_N^+$, and by Peter-Weyl we obtain that the inclusion $\widetilde{O_N^+}\subset U_N^+$ must be an isomorphism, modulo the usual equivalence relation for quantum groups.

(2) A version of this proof, not using any prior free probability knowledge, is by using fusion rules. Indeed, as explained in section 2 above, the representations of the dual free products, and in particular of the free complexifications, can be explicitely computed. Thus the fusion rules for $\widetilde{O_N^+}$ appear as a "free complexification" of the Clebsch-Gordan rules for $O_N^+$, and in practice this leads to the same fusion rules as for $U_N^+$. As before, by Peter-Weyl we obtain from this that the inclusion $\widetilde{O_N^+}\subset U_N^+$ must be an isomorphism, modulo the usual equivalence relation for the compact quantum groups.

(3) A third proof, based on the same idea, and which is perhaps the simplest, makes use of the easiness property of $O_N^+,U_N^+$ only. Indeed, if we denote by $v,zv,u$ the fundamental

representations of the quantum groups $O_N^+\subset\widetilde{O_N^+}\subset U_N^+$, at the level of the associated Hom spaces we obtain reverse inclusions, as follows:

$$Hom(v^{\otimes k},v^{\otimes l})\supset Hom((zv)^{\otimes k},(zv)^{\otimes l})\supset Hom(u^{\otimes k},u^{\otimes l})$$

The spaces on the left and on the right are known from section 4 above, the result there stating that these spaces are as follows:

$$span\left(T_\pi\Big|\pi\in NC_2(k,l)\right)\supset span\left(T_\pi\Big|\pi\in\mathcal{NC}_2(k,l)\right)$$

Regarding the spaces in the middle, these are obtained from those on the left by "coloring", so we obtain the same spaces as those on the right. Thus, by Tannakian duality, our embedding $\widetilde{O_N^+}\subset U_N^+$ is an isomorphism, modulo the usual equivalence relation. □

As a comment here, the proof (3) above, when properly worked out, provides as well an alternative proof for Theorem 6.12. Indeed, once we know that we have $U_N^+=\widetilde{O_N^+}$, it follows that the fusion rules for $U_N^+$ appear as a "free complexification" of the Clebsch-Gordan rules for $O_N^+$, and in practice this leads to the formulae in Theorem 6.12.

As an interesting consequence of the above result, we have:

**Theorem 6.14.** *We have an identification as follows,*

$$PO_N^+=PU_N^+$$

*modulo the usual equivalence relation for compact quantum groups.*

*Proof.* As before, we have several proofs for this result, as follows:

(1) This follows from Theorem 6.13, because we have:

$$PU_N^+=P\widetilde{O_N^+}=PO_N^+$$

(2) We can deduce this as well directly. With notations as before, we have:

$$Hom\left((v\otimes v)^k,(v\otimes v)^l\right)=span\left(T_\pi\Big|\pi\in NC_2((\circ\bullet)^k,(\circ\bullet)^l)\right)$$

$$Hom\left((u\otimes\bar{u})^k,(u\otimes\bar{u})^l\right)=span\left(T_\pi\Big|\pi\in\mathcal{NC}_2((\circ\bullet)^k,(\circ\bullet)^l)\right)$$

The sets on the right being equal, we conclude that the inclusion $PO_N^+\subset PU_N^+$ preserves the corresponding Tannakian categories, and so must be an isomorphism. □

As a conclusion, the passage $O_N^+\to U_N^+$ is something much simpler than the passage $O_N\to U_N$, with this ultimately coming from the fact that the combinatorics of $O_N^+,U_N^+$ is something much simpler than the combinatorics of $O_N,U_N$. In addition, all this leads as well to the interesting conclusion that the free projective geometry does not fall into real and complex, but is rather unique and "scalarless". We will be back to this.

More generally now, once again by following [1], we have similar results obtained by replacing $O_N^+$ with the more general super-orthogonal quantum groups $O_F^+$ from the previous section, which include as well the free symplectic groups $Sp_N^+$. Let us start with:

**Theorem 6.15.** *We have an identification as follows,*

$$U_N^+ = \widetilde{O_F^+}$$

*valid for any super-orthogonal quantum group $O_F^+$.*

*Proof.* This is a straightforward extension of Theorem 6.13 above, with any of the proofs there extending to the case of the quantum groups $O_F^+$. See [1]. □

We have as well a projective version of the above result, as follows:

**Theorem 6.16.** *We have an identification as follows,*

$$PU_N^+ = PO_F^+$$

*valid for any super-orthogonal quantum group $O_F^+$.*

*Proof.* This is a straightforward extension of Theorem 6.14, with any of the proofs there extending to the case of the quantum groups $O_F^+$. Alternatively, the result follows from Theorem 6.15, by taking the projective versions of the quantum groups there. □

The free symplectic result at $N = 2$ is particularly interesting, because here we have $Sp_2^+ = SU_2$, and so we obtain that $U_2^+$ is the free complexification of $SU_2$:

**Theorem 6.17.** *We have an identification as follows,*

$$U_2^+ = \widetilde{SU_2}$$

*modulo the usual equivalence relation for compact quantum groups.*

*Proof.* As explained above, this follows from Theorem 6.15, and from $Sp_2^+ = SU_2$, via the material explained in section 5 above. See [1]. □

Finally, we have a projective version of the above result, as follows:

**Theorem 6.18.** *We have an identification as follows, and this even without using the standard equivalence relation for the compact quantum groups:*

$$PU_2^+ = SO_3$$

*A similar result holds for the "left" projective version of $U_2^+$, constructed by using the corepresentation $\bar{u} \otimes u$ instead of $u \otimes \bar{u}$.*

*Proof.* We have several assertions here, the idea being as follows:

(1) By using Theorem 6.17 we obtain, modulo the equivalence relation:

$$PU_2^+ = P\widetilde{SU_2} = PSU_2 = SO_3$$

(2) Now since $SO_3$ is coamenable, the above formula must hold in fact in a plain way, meaning without using the equivalence relation. This can be checked as well directly, by verifying that the coefficients of $u \otimes \bar{u}$ commute indeed.

(3) Finally, the last assertion can be either deduced from the first one, or proved directly, by using "left" free complexification operations, in all the above. $\square$

We refer to [1] for some further applications of the above $N = 2$ results, for instance with structure results regarding the von Neumann algebra $L^\infty(U_2^+)$.

We will be back to the quantum groups $U_N^+$ in section 8 below, with a number of more advanced probabilistic results about them.

## 7. Easiness, twisting

Our purpose here will be that of extending the main findings about $O_N^+,U_N^+$ from the previous sections to $O_N,U_N$ too, and to other compact quantum groups as well. Let us begin with a general definition, from [38], [127], as follows:

**Definition 7.1.** *Let $P(k,l)$ be the set of partitions between an upper colored integer $k$, and a lower colored integer $l$. A collection of subsets*

$$D=\bigsqcup_{k,l}D(k,l)$$

*with $D(k,l)\subset P(k,l)$ is called a category of partitions when it has the following properties:*

1. *Stability under the horizontal concatenation, $(\pi,\sigma)\to[\pi\sigma]$.*
2. *Stability under vertical concatenation $(\pi,\sigma)\to[^\sigma_\pi]$, with matching middle symbols.*
3. *Stability under the upside-down turning $*$, with switching of colors, $\circ\leftrightarrow\bullet$.*
4. *Each set $P(k,k)$ contains the identity partition $||\ldots||$.*
5. *The sets $P(\emptyset,\circ\bullet)$ and $P(\emptyset,\bullet\circ)$ both contain the semicircle $\cap$.*

We have already met a number of such categories, in section 4 above. There are many other examples of such categories, as for instance $P$ itself, or the category $NC\subset P$ of all noncrossing partitions. The relation with the Tannakian categories comes from:

**Proposition 7.2.** *Each partition $\pi\in P(k,l)$ produces a linear map*

$$T_\pi:(\mathbb C^N)^{\otimes k}\to(\mathbb C^N)^{\otimes l}$$

*given by the following formula, where $e_1,\ldots,e_N$ is the standard basis of $\mathbb C^N$,*

$$T_\pi(e_{i_1}\otimes\ldots\otimes e_{i_k})=\sum_{j_1\ldots j_l}\delta_\pi\begin{pmatrix}i_1&\ldots&i_k\\ j_1&\ldots&j_l\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_l}$$

*and with the Kronecker type symbols $\delta_\pi\in\{0,1\}$ depending on whether the indices fit or not. The assignement $\pi\to T_\pi$ is categorical, in the sense that we have*

$$T_\pi\otimes T_\sigma=T_{[\pi\sigma]}$$

$$T_\pi T_\sigma=N^{c(\pi,\sigma)}T_{[^\sigma_\pi]}$$

$$T_\pi^*=T_{\pi^*}$$

*where $c(\pi,\sigma)$ are certain integers, coming from the erased components in the middle.*

*Proof.* This is something that we already know for the pairings, from section 4 above. In general, the proof is identical. □

In relation with the quantum groups, we have the following result, from [38]:

**Theorem 7.3.** *Each category of partitions $D=(D(k,l))$ produces a family of compact quantum groups $G=(G_N)$, one for each $N\in\mathbb N$, via the formula*

$$Hom(u^{\otimes k},u^{\otimes l})=span\left(T_\pi\Big|\pi\in D(k,l)\right)$$

*which produces a Tannakian category, and the Tannakian duality correspondence.*

*Proof.* This follows indeed from Woronowicz's Tannakian duality, in its "soft" form from [106], as explained in section 4 above. Indeed, let us set:

$$C(k,l)=span\left(T_\pi\Big|\pi\in D(k,l)\right)$$

By using the axioms in Definition 7.1, and the categorical properties of the operation $\pi\to T_\pi$, from Proposition 7.2 above, we deduce that $C=(C(k,l))$ is a Tannakian category. Thus the Tannakian duality applies, and gives the result. □

We already know, from section 4 above, that the quantum groups $O_N^+,U_N^+$ appear in this way, with $D$ being respectively $NC_2,\mathcal{NC}_2$. In general now, let us formulate:

**Definition 7.4.** *A closed subgroup $G\subset U_N^+$ is called easy when we have*

$$Hom(u^{\otimes k},u^{\otimes l})=span\left(T_\pi\Big|\pi\in D(k,l)\right)$$

*for any colored integers $k,l$, for a certain category of partitions $D\subset P$.*

In other words, a compact quantum group is called easy when its Tannakian category appears in the simplest possible way: from a category of partitions. Observe that the category $D$ is not unique, for instance because at $N=1$ all the categories of partitions produce the same easy quantum group, namely $G=\{1\}$. We will be back to this.

In practice now, what we know so far, from section 4 above, is that $U_N,U_N^+,O_N,O_N^+$ are easy. Regarding now the half-liberations, we have here:

**Theorem 7.5.** *We have the following results:*

(1) *$U_N^*$ is easy, coming from the category $\mathcal P_2^*\subset\mathcal P_2$ of pairings having the property that, when the legs are relabelled clockwise $\circ\bullet\circ\bullet\ldots$, each string connects $\circ-\bullet$.*
(2) *$O_N^*$ is easy too, coming from the category $P_2^*\subset P_2$ of pairings having the same property: when legs are labelled clockwise $\circ\bullet\circ\bullet\ldots$, each string connects $\circ-\bullet$.*

*Proof.* We can proceed here as in the proof for $U_N,O_N$, from section 4 above, by replacing the basic crossing by the half-commutation crossing, as follows:

(1) Regarding $U_N^*\subset U_N^+$, the corresponding Tannakian category is generated by the operators $T_\pi$, with $\pi=\cancel{\mid}\!\!\times$, taken with all the possible $2^3=8$ matching colorings. Since these latter 8 partitions generate the category $\mathcal P_2^*$, we obtain the result.

(2) For $O_N^*$ we can proceed similarly, by using the following formula:

$$O_N^* = O_N^+ \cap U_N^*$$

At the categorical level, this tells us that the associated Tannakian category is given by $C = span(T_\pi|\pi \in D)$, with:

$$D = < NC_2, \mathcal{P}_2^* > = P_2^*$$

Thus, we are led to the conclusion in the statement. $\square$

Let us collect now the results that we have so far in a single theorem, as follows:

**Theorem 7.6.** *The basic unitary quantum groups, namely*

$$\begin{array}{ccccc} U_N & \longrightarrow & U_N^* & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow & & \uparrow \\ O_N & \longrightarrow & O_N^* & \longrightarrow & O_N^+ \end{array}$$

*are all easy, the corresponding categories of partitions being:*

$$\begin{array}{ccccc} \mathcal{P}_2 & \longleftarrow & \mathcal{P}_2^* & \longleftarrow & \mathcal{NC}_2 \\ \downarrow & & \downarrow & & \downarrow \\ P_2 & \longleftarrow & P_2^* & \longleftarrow & NC_2 \end{array}$$

*Proof.* This follows indeed from the various results established so far, in section 4 and here. $\square$

We have seen in sections 5-6 above that the easiness property of $O_N^+, U_N^+$ leads to some interesting consequences. Regarding $O_N^*, U_N^*$, as a main consequence, we can now compute their projective versions, as part of the following general result:

**Theorem 7.7.** *The projective versions of the basic quantum groups are as follows,*

$$\begin{array}{ccccc} PU_N & \longrightarrow & PU_N & \longrightarrow & PU_N^+ \\ \uparrow & & \uparrow & & \uparrow \\ PO_N & \longrightarrow & PU_N & \longrightarrow & PU_N^+ \end{array}$$

*when identifying, in the free case, full and reduced version algebras.*

*Proof.* In the classical case, there is nothing to prove. Regarding the half-classical versions, consider the inclusions $O_N^*, U_N \subset U_N^*$. These induce inclusions as follows:

$$PO_N^*, PU_N \subset PU_N^*$$

Our claim is that these inclusions are isomorphisms. Let indeed $u, v, w$ be the fundamental corepresentations of $O_N^*, U_N, U_N^*$. According to Theorem 7.5, we have:

$$Hom\left((u\otimes\bar{u})^k, (u\otimes\bar{u})^l\right) = span\left(T_\pi\Big|\pi\in P_2^*((\circ\bullet)^k, (\circ\bullet)^l)\right)$$

$$Hom\left((u\otimes\bar{u})^k, (u\otimes\bar{u})^l\right) = span\left(T_\pi\Big|\pi\in \mathcal{P}_2((\circ\bullet)^k, (\circ\bullet)^l)\right)$$

$$Hom\left((u\otimes\bar{u})^k, (u\otimes\bar{u})^l\right) = span\left(T_\pi\Big|\pi\in \mathcal{P}_2^*((\circ\bullet)^k, (\circ\bullet)^l)\right)$$

The sets on the right being equal, we conclude that the inclusions $O_N^*, U_N \subset U_N^*$ preserve the corresponding Tannakian categories, and so must be isomorphisms.

Finally, in the free case the result follows either from the free complexification result from section 5, or from Theorem 7.6, by using the same method. □

Let us discuss now composition operations. We will be interested in:

**Proposition 7.8.** *The closed subgroups of $U_N^+$ are subject to operations as follows:*

(1) *Intersection: $H\cap K$ is the biggest quantum subgroup of $H, K$.*
(2) *Generation: $< H, K >$ is the smallest quantum group containing $H, K$.*

*Proof.* We must prove that the universal quantum groups in the statement exist indeed. For this purpose, let us pick writings as follows, with $I, J$ being Hopf ideals:

$$C(H) = C(U_N^+)/I \quad , \quad C(K) = C(U_N^+)/J$$

We can then construct our two universal quantum groups, as follows:

$$C(H\cap K) = C(U_N^+)/ < I, J >$$

$$C(< H, K >) = C(U_N^+)/(I\cap J)$$

Thus, we obtain the result. □

In practice, the operation $\cap$ can be usually computed by using:

**Proposition 7.9.** *Assuming $H, K\subset G$, the intersection $H\cap K$ is given by*

$$C(H\cap K) = C(G)/\{\mathcal{R}, \mathcal{P}\}$$

*whenever*

$$C(H) = C(G)/\mathcal{R} \quad , \quad C(K) = C(G)/\mathcal{P}$$

*with $\mathcal{R}, \mathcal{P}$ being certain sets of polynomial $*$-relations between the coordinates $u_{ij}$.*

*Proof.* This follows from Proposition 7.8 above, or rather from its proof, and from the following trivial fact, regarding relations and ideals:

$$I =<\mathcal{R}>, J=<\mathcal{P}> \implies <I,J>=<\mathcal{R},\mathcal{P}>$$

Thus, we obtain the result. □

In order to discuss the generation operation, let us call Hopf image of a representation $C(G) \to A$ the smallest Hopf algebra quotient $C(L)$ producing a factorization:

$$C(G) \to C(L) \to A$$

The fact that this quotient exists indeed is routine, by dividing by a suitable ideal, and we will be back to this in section 16 below. This notion can be generalized as follows:

**Proposition 7.10.** *Assuming $H, K \subset G$, the quantum group $<H, K>$ is such that*

$$C(G) \to C(H \cap K) \to C(H), C(K)$$

*is the joint Hopf image of the following quotient maps:*

$$C(G) \to C(H), C(K)$$

*Proof.* In the particular case from the statement, the joint Hopf image appears as the smallest Hopf algebra quotient $C(L)$ producing factorizations as follows:

$$C(G) \to C(L) \to C(H), C(K)$$

We conclude from this that we have $L =< H, K >$, as desired. See [56]. □

In the Tannakian setting now, we have the following result:

**Theorem 7.11.** *The intersection and generation operations $\cap$ and $<,>$ can be constructed via the Tannakian correspondence $G \to C_G$, as follows:*

1. *Intersection: defined via $C_{G\cap H} =< C_G, C_H >$.*
2. *Generation: defined via $C_{<G,H>} = C_G \cap C_H$.*

*Proof.* This follows from Proposition 7.8, or rather from its proof, by taking $I, J$ to be the ideals coming from Tannakian duality, in its soft form, from section 4 above. □

In relation now with our easiness questions, we first have the following result:

**Proposition 7.12.** *Assuming that $H, K$ are easy, then so is $H \cap K$, and we have*

$$D_{H\cap K} =< D_H, D_K >$$

*at the level of the corresponding categories of partitions.*

*Proof.* We have indeed the following computation:

$$\begin{aligned} C_{H\cap K} &= < C_H, C_K > \\ &= < span(D_H), span(D_K) > \\ &= span(< D_H, D_K >) \end{aligned}$$

Thus, by Tannakian duality we obtain the result. □

Regarding the generation operation, the situation is more complicated, as follows:

**Proposition 7.13.** *Assuming that $H,K$ are easy, we have an inclusion*

$$< H,K > \subset \{H,K\}$$

*coming from an inclusion of Tannakian categories as follows,*

$$C_H \cap C_K \supset span(D_H \cap D_K)$$

*where $\{H,K\}$ is the easy quantum group having as category of partitions $D_H \cap D_K$.*

*Proof.* This follows from the definition and properties of the generation operation, and from the following computation:

$$\begin{aligned} C_{<H,K>} &= C_H \cap C_K \\ &= span(D_H) \cap span(D_K) \\ &\supset span(D_H \cap D_K) \end{aligned}$$

Indeed, by Tannakian duality we obtain from this all the assertions. □

Summarizing, we have some problems here, and we must proceed as follows:

**Theorem 7.14.** *The intersection and easy generation operations $\cap$ and $\{\,,\}$ can be constructed via the Tannakian correspondence $G \to D_G$, as follows:*

(1) *Intersection: defined via $D_{G\cap H} =< D_G, D_H >$.*
(2) *Easy generation: defined via $D_{\{G,H\}} = D_G \cap D_H$.*

*Proof.* Here the situation is as follows:

(1) This is an result coming from Proposition 7.12.

(2) This is more of an empty statement, coming from Proposition 7.13. □

With the above notions in hand, we can formulate a nice result, which improves our main result so far, namely Theorem 7.6 above, as follows:

**Theorem 7.15.** *The basic unitary quantum groups, namely*

$$\begin{tikzcd} U_N \arrow[r] & U_N^* \arrow[r] & U_N^+ \\ O_N \arrow[r] \arrow[u] & O_N^* \arrow[r] \arrow[u] & O_N^+ \arrow[u] \end{tikzcd}$$

*are all easy, and they form an intersection and easy generation diagram, in the sense that any rectangle $P \subset Q, R \subset S$ of the above diagram satisfies $P = Q \cap R, \{Q,R\} = S$.*

*Proof.* We know from Theorem 7.6 that the quantum groups in the statement are all easy. Since the corresponding categories of partitions form an intersection and generation diagram, by using Theorem 7.14 we obtain the result. □

Let us explore now a number of further examples of easy quantum groups, which appear as "versions" of the basic unitary groups. With the convention that a matrix is called bistochastic when its entries sum up to 1, on each row and each column, we have:

**Proposition 7.16.** *We have the following groups and quantum groups:*

(1) $B_N \subset O_N$, *consisting of the orthogonal matrices which are bistochastic.*
(2) $C_N \subset U_N$, *consisting of the unitary matrices which are bistochastic.*
(3) $B_N^+ \subset O_N^+$, *coming via $u\xi = \xi$, where $\xi$ is the all-one vector.*
(4) $C_N^+ \subset U_N^+$, *coming via $u\xi = \xi$, where $\xi$ is the all-one vector.*

*Also, we have inclusions $B_N \subset B_N^+$ and $C_N \subset C_N^+$, which are both liberations.*

*Proof.* Here the fact that $B_N, C_N$ are indeed groups is clear. As for $B_N^+, C_N^+$, these are quantum groups as well, because the relation $\xi \in Fix(u)$ is categorical.

Finally, observe that for $U \in U_N$ we have:

$$U\xi = \xi \iff U^*\xi = \xi$$

By conjugating, these conditions are equivalent as well to $\bar{U}\xi = \xi$, $U^t\xi = \xi$. Thus $U \in U_N$ is bistochastic precisely when $U\xi = \xi$, and this gives the last assertion. □

The above quantum groups are all easy, and following [38], [127], we have:

**Theorem 7.17.** *The basic orthogonal and unitary quantum groups and their bistochastic versions are all easy, and they form a diagram as follows,*

$$\begin{tikzcd}
 & C_N^+ \arrow[rr] & & U_N^+ \\
B_N^+ \arrow[rr] \arrow[ur] & & O_N^+ \arrow[ur] & \\
 & C_N \arrow[rr] \arrow[uu] & & U_N \arrow[uu] \\
B_N \arrow[rr] \arrow[uu] \arrow[ur] & & O_N \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

*which is an intersection and easy generation diagram, in the sense of Theorem 7.15.*

*Proof.* The first assertion comes from the fact that the all-one vector $\xi$ used in Proposition 7.16 above is the vector associated to the singleton partition:

$$\xi = T_|$$

Indeed, we obtain from this that the quantum groups $B_N,C_N,B_N^+,C_N^+$ are indeed easy, appearing from the categories of partitions for $O_N,U_N,O_N^+,U_N^+$, by adding singletons.

In practice now, according to this observation, and to Theorem 7.15 above, the corresponding categories of partitions are as follows, where the symbol 12 stands for "singletons and pairings", in the same way as the symbol 2 stands for "pairings":

$$\begin{tikzcd}[row sep=small, column sep=small]
& \mathcal{NC}_{12} \arrow[dl] \arrow[dd] & & \mathcal{NC}_2 \arrow[ll] \arrow[dl] \arrow[dd] \\
NC_{12} \arrow[dd] & & NC_2 \arrow[ll, crossing over] & \\
& \mathcal{P}_{12} \arrow[dl] & & \mathcal{P}_2 \arrow[ll] \arrow[dl] \\
P_{12} & & P_2 \arrow[ll] \arrow[from=uu, crossing over] &
\end{tikzcd}$$

Now since both this diagram and the one the statement are intersection diagrams, the quantum groups form an intersection and easy generation diagram, as stated. $\square$

Generally speaking, the above result is quite nice, among others because we are now exiting the world of pairings. However, there are a few problems with it. First, we cannot really merge it with Theorem 7.15, as to obtain a nice cubic diagram, containing all the quantum groups considered so far. Indeed, the half-classical versions of the bistochastic quantum groups collapse, and so cannot be inserted into the cube, as shown by:

**Proposition 7.18.** *The half-classical versions of $B_N^+,C_N^+$ are given by:*

$$B_N^+\cap O_N^*=B_N\quad,\quad C_N^+\cap U_N^*=C_N$$

*In other words, the half-classical versions collapse to the classical versions.*

*Proof.* This follows from Tannakian duality, by using the fact that when capping the half-classical crossing with 2 singletons, we obtain the classical crossing. Alternatively, this follows from a direct computation. $\square$

Yet another problem with the bistochastic groups and quantum groups comes from the fact that these objects are not really "new", because, following [116], we have:

**Proposition 7.19.** *We have isomorphisms as follows:*

(1) $B_N\simeq O_{N-1}$.
(2) $B_N^+\simeq O_{N-1}^+$.
(3) $C_N\simeq U_{N-1}$.
(4) $C_N^+\simeq U_{N-1}^+$.

*Proof.* Let us pick a matrix $F\in U_N$ satisfying the following condition, where $\xi$ is the all-one vector:

$$Fe_0=\frac{1}{\sqrt{N}}\xi$$

Such matrices exist of course, the basic example being the Fourier matrix:

$$F_N=\frac{1}{\sqrt{N}}(w^{ij})_{ij}\quad,\quad w=e^{2\pi i/N}$$

We have then the following computation:

$$\begin{aligned}u\xi=\xi\quad&\iff\quad uFe_0=Fe_0\\&\iff\quad F^*uFe_0=e_0\\&\iff\quad F^*uF=diag(1,w)\end{aligned}$$

Thus we have an isomorphism given by $w_{ij}\to(F^*uF)_{ij}$, as desired. $\square$

Back to generalities now, let us point out the fact that the easy quantum groups are not the only ones "coming from partitions", but are rather the simplest ones having this property. An interesting and important class of compact quantum groups, which appear in relation with many questions, are the $q=-1$ twists of the compact Lie groups. In order to discuss this, the best is to deform first the simplest objects that we have, namely the noncommutative spheres. This can be done as follows:

**Theorem 7.20.** *We have noncommutative spheres as follows, obtained via the twisted commutation relations $ab=\pm ba$, and twisted half-commutation relations $abc=\pm cba$,*

$$\begin{array}{ccccc}\bar{S}^{N-1}_{\mathbb{C}}&\longrightarrow&\bar{S}^{N-1}_{\mathbb{C},*}&\longrightarrow&S^{N-1}_{\mathbb{C},+}\\\uparrow&&\uparrow&&\uparrow\\\bar{S}^{N-1}_{\mathbb{R}}&\longrightarrow&\bar{S}^{N-1}_{\mathbb{R},*}&\longrightarrow&S^{N-1}_{\mathbb{R},+}\end{array}$$

*where the signs at left correspond to the anticommutation of distinct coordinates, and their adjoints, and the other signs come from functoriality.*

*Proof.* For the spheres on the left, if we want to replace some of the commutation relations $z_iz_j=z_jz_i$ by anticommutation relations $z_iz_j=-z_jz_i$, a bit of thinking tells us that the one and only natural choice is:

$$z_iz_j=-z_jz_i\quad,\quad\forall i\neq j$$

In other words, with the notation $\varepsilon_{ij}=1-\delta_{ij}$, we must have:

$$z_iz_j=(-1)^{\varepsilon_{ij}}z_jz_i$$

Regarding now the spheres in the middle, the situation is a priori a bit more tricky, because we have to take into account the various possible collapsings of $\{i,j,k\}$. However, if we want to have embeddings as above, there is only one choice, namely:

$$z_iz_jz_k=(-1)^{\varepsilon_{ij}+\varepsilon_{jk}+\varepsilon_{ik}}z_kz_jz_i$$

Thus, we have constructed our spheres, and embeddings, as needed. □

Let us discuss now the quantum group case. The situation here is considerably more complicated, because the coordinates $u_{ij}$ depend on double indices, and finding for instance the correct signs for $u_{ij}u_{kl}u_{mn}=\pm u_{mn}u_{kl}u_{ij}$ looks nearly impossible. However, we can solve this problem by taking some inspiration from the sphere case, as follows:

**Proposition 7.21.** *We have quantum groups as follows,*

$$\begin{tikzcd}
\bar{U}_N \arrow[r] & U_N^+ \\
\bar{O}_N \arrow[r] \arrow[u] & O_N^+ \arrow[u]
\end{tikzcd}$$

*defined via the following relations,*

$$\alpha\beta=\begin{cases}-\beta\alpha & \text{for } a,b\in\{u_{ij}\}\text{ distinct, on the same row or column}\\ \beta\alpha & \text{otherwise}\end{cases}$$

*with the convention $\alpha=a,a^*$ and $\beta=b,b^*$.*

*Proof.* These quantum groups are well-known, see [17]. The idea indeed is that the existence of $\varepsilon,S$ is clear. Regarding now $\Delta$, set $U_{ij}=\sum_k u_{ik}\otimes u_{kj}$. For $j\neq k$ we have:

$$\begin{aligned}
U_{ij}U_{ik}&=\sum_{s\neq t}u_{is}u_{it}\otimes u_{sj}u_{tk}+\sum_s u_{is}u_{is}\otimes u_{sj}u_{sk}\\
&=\sum_{s\neq t}-u_{it}u_{is}\otimes u_{tk}u_{sj}+\sum_s u_{is}u_{is}\otimes(-u_{sk}u_{sj})\\
&=-U_{ik}U_{ij}
\end{aligned}$$

Also, for $i\neq k,j\neq l$ we have:

$$\begin{aligned}
U_{ij}U_{kl}&=\sum_{s\neq t}u_{is}u_{kt}\otimes u_{sj}u_{tl}+\sum_s u_{is}u_{ks}\otimes u_{sj}u_{sl}\\
&=\sum_{s\neq t}u_{kt}u_{is}\otimes u_{tl}u_{sj}+\sum_s(-u_{ks}u_{is})\otimes(-u_{sl}u_{sj})\\
&=U_{kl}U_{ij}
\end{aligned}$$

This finishes the proof in the real case. In the complex case the remaining relations can be checked in a similar way, by putting $*$ exponents in the middle. □

It remains now to twist $O_N^*, U_N^*$. In order to do so, given three coordinates $a, b, c \in \{u_{ij}\}$, let us set $span(a, b, c) = (r, c)$, where $r, c \in \{1, 2, 3\}$ are the number of rows and columns spanned by $a, b, c$. In other words, if we write $a = u_{ij}, b = u_{kl}, c = u_{pq}$ then $r = \#\{i, k, p\}$ and $l = \#\{j, l, q\}$. With these conventions, we have the following result:

**Proposition 7.22.** *We have intermediate quantum groups as follows,*

$$\begin{array}{ccccc} \bar{U}_N & \longrightarrow & \bar{U}_N^* & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow & & \uparrow \\ \bar{O}_N & \longrightarrow & \bar{O}_N^* & \longrightarrow & O_N^+ \end{array}$$

*defined via the following relations,*

$$\alpha\beta\gamma = \begin{cases} -\gamma\beta\alpha & \text{for } a, b, c \in \{u_{ij}\} \text{ with } span(a, b, c) = (\leq 2, 3) \text{ or } (3, \leq 2) \\ \gamma\beta\alpha & \text{otherwise} \end{cases}$$

*with the conventions $\alpha = a, a^*$, $\beta = b, b^*$ and $\gamma = c, c^*$.*

*Proof.* The rules for the various commutation/anticommutation signs are:

| $r\backslash c$ | 1 | 2 | 3 |
|---|---|---|---|
| 1 | $+$ | $+$ | $-$ |
| 2 | $+$ | $+$ | $-$ |
| 3 | $-$ | $-$ | $+$ |

We first prove the result for $\bar{O}_N^*$. The construction of the counit, $\varepsilon(u_{ij}) = \delta_{ij}$, requires the Kronecker symbols $\delta_{ij}$ to commute/anticommute according to the above table. Equivalently, we must prove that the situation $\delta_{ij}\delta_{kl}\delta_{pq} = 1$ can appear only in a case where the above table indicates "+". But this is clear, because $\delta_{ij}\delta_{kl}\delta_{pq} = 1$ implies $r = c$.

The construction of the antipode $S$ is clear too, because this requires the choice of our $\pm$ signs to be invariant under transposition, and this is true, the table being symmetric.

With $U_{ij} = \sum_k u_{ik} \otimes u_{kj}$, we have the following computation:

$$\begin{aligned} U_{ia}U_{jb}U_{kc} &= \sum_{xyz} u_{ix}u_{jy}u_{kz} \otimes u_{xa}u_{yb}u_{zc} \\ &= \sum_{xyz} \pm u_{kz}u_{jy}u_{ix} \otimes \pm u_{zc}u_{yb}u_{xa} \\ &= \pm U_{kc}U_{jb}U_{ia} \end{aligned}$$

We must prove that, when examining the precise two $\pm$ signs in the middle formula, their product produces the correct $\pm$ sign at the end. The point now is that both these signs depend only on $s = span(x, y, z)$, and for $s = 1, 2, 3$ respectively:

– For a $(3,1)$ span we obtain $+-$, $+-$, $-+$, so a product $-$ as needed.
– For a $(2,1)$ span we obtain $++$, $++$, $--$, so a product $+$ as needed.
– For a $(3,3)$ span we obtain $--$, $--$, $++$, so a product $+$ as needed.
– For a $(3,2)$ span we obtain $+-$, $+-$, $-+$, so a product $-$ as needed.
– For a $(2,2)$ span we obtain $++$, $++$, $--$, so a product $+$ as needed.

Together with the fact that our problem is invariant under $(r, c) \to (c, r)$, and with the fact that for a $(1,1)$ span there is nothing to prove, this finishes the proof. For $\bar{U}_N^*$ the proof is similar, by putting $*$ exponents in the middle. ☐

The above results can be summarized as follows:

**Theorem 7.23.** *We have quantum groups as follows, obtained via the twisted commutation relations $ab = \pm ba$, and twisted half-commutation relations $abc = \pm cba$,*

$$\begin{array}{ccccc} \bar{U}_N & \longrightarrow & \bar{U}_N^* & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow & & \uparrow \\ \bar{O}_N & \longrightarrow & \bar{O}_N^* & \longrightarrow & O_N^+ \end{array}$$

*where the signs at left correspond to anticommutation for distinct entries on rows and columns, and commutation otherwise, and the other signs come from functoriality.*

*Proof.* This follows indeed from Proposition 7.21 and Proposition 7.22. ☐

Our purpose now will be that of showing that the quantum groups constructed above can be in fact defined in a more conceptual way, as "Schur-Weyl twists". Let $P_{even}(k,l) \subset P(k,l)$ be the set of partitions with blocks having even size, and $NC_{even}(k,l) \subset P_{even}(k,l)$ be the subset of noncrossing partitions. Also, we use the standard embedding $S_k \subset P_2(k,k)$, via the pairings having only up-to-down strings. Given a partition $\tau \in P(k,l)$, we call "switch" the operation which consists in switching two neighbors, belonging to different blocks, in the upper row, or in the lower row. With these conventions, we have:

**Proposition 7.24.** *There is a signature map $\varepsilon : P_{even} \to \{-1, 1\}$, given by*

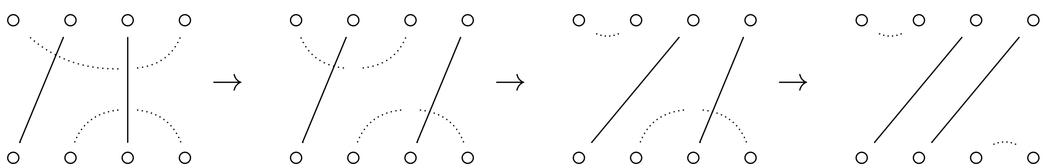
$$\varepsilon(\tau) = (-1)^c$$

*where $c$ is the number of switches needed to make $\tau$ noncrossing. In addition:*

1. *For $\tau \in S_k$, this is the usual signature.*
2. *For $\tau \in P_2$ we have $(-1)^c$, where $c$ is the number of crossings.*
3. *For $\tau \leq \pi \in NC_{even}$, the signature is $1$.*

*Proof.* In order to show that $\varepsilon$ is well-defined, we must prove that the number $c$ in the statement is well-defined modulo 2. It is enough to perform the verification for the noncrossing partitions. More precisely, given $\tau,\tau'\in NC_{even}$ having the same block structure, we must prove that the number of switches $c$ required for the passage $\tau\to\tau'$ is even.

In order to do so, observe that any partition $\tau\in P(k,l)$ can be put in "standard form", by ordering its blocks according to the appearence of the first leg in each block, counting clockwise from top left, and then by performing the switches as for block 1 to be at left, then for block 2 to be at left, and so on. Here the required switches are also uniquely determined, by the order coming from counting clockwise from top left. Here is an example of such an algorithmic switching operation:



The point now is that, under the assumption $\tau\in NC_{even}(k,l)$, each of the moves required for putting a leg at left, and hence for putting a whole block at left, requires an even number of switches. Thus, putting $\tau$ is standard form requires an even number of switches. Now given $\tau,\tau'\in NC_{even}$ having the same block structure, the standard form coincides, so the number of switches $c$ required for the passage $\tau\to\tau'$ is indeed even.

Regarding now the remaining assertions, these are all elementary:

(1) For $\tau\in S_k$ the standard form is $\tau'=id$, and the passage $\tau\to id$ comes by composing with a number of transpositions, which gives the signature.

(2) For a general $\tau\in P_2$, the standard form is of type $\tau'=|\ldots|^{\cup\ldots\cup}_{\cap\ldots\cap}$, and the passage $\tau\to\tau'$ requires $c$ mod 2 switches, where $c$ is the number of crossings.

(3) Assuming that $\tau\in P_{even}$ comes from $\pi\in NC_{even}$ by merging a certain number of blocks, we can prove that the signature is 1 by proceeding by recurrence. $\square$

We can use the above signature map, as follows:

**Definition 7.25.** *Associated to a partition $\pi\in P_{even}(k,l)$ is the linear map*

$$\bar{T}_\pi:(\mathbb{C}^N)^{\otimes k}\to(\mathbb{C}^N)^{\otimes l}$$

*given by the following formula, with $e_1,\ldots,e_N$ being the standard basis of $\mathbb{C}^N$,*

$$\bar{T}_\pi(e_{i_1}\otimes\ldots\otimes e_{i_k})=\sum_{j_1\ldots j_l}\bar{\delta}_\pi\begin{pmatrix}i_1&\ldots&i_k\\ j_1&\ldots&j_l\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_l}$$

*and where $\bar{\delta}_\pi\in\{-1,0,1\}$ is $\bar{\delta}_\pi=\varepsilon(\tau)$ if $\tau\geq\pi$, and $\bar{\delta}_\pi=0$ otherwise, with $\tau=\ker\binom{i}{j}$.*

In other words, what we are doing here is to add signatures to the usual formula of $T_\pi$. Indeed, observe that the usual formula for $T_\pi$ can be written as folllows:

$$T_\pi(e_{i_1}\otimes\ldots\otimes e_{i_k})=\sum_{j:\ker(^i_j)\geq\pi}e_{j_1}\otimes\ldots\otimes e_{j_l}$$

Now by inserting signs, coming from the signature map $\varepsilon:P_{even}\to\{\pm1\}$, we are led to the following formula, which coincides with the one given above:

$$\bar{T}_\pi(e_{i_1}\otimes\ldots\otimes e_{i_k})=\sum_{\tau\geq\pi}\varepsilon(\tau)\sum_{j:\ker(^i_j)=\tau}e_{j_1}\otimes\ldots\otimes e_{j_l}$$

We will be back later to this. For the moment, we must first prove a key categorical result, as follows:

**Proposition 7.26.** *The assignement $\pi\to\bar{T}_\pi$ is categorical, in the sense that*

$$\bar{T}_\pi\otimes\bar{T}_\sigma=\bar{T}_{[\pi\sigma]}\quad,\quad\bar{T}_\pi\bar{T}_\sigma=N^{c(\pi,\sigma)}\bar{T}_{[^\sigma_\pi]}\quad,\quad\bar{T}_\pi^*=\bar{T}_{\pi^*}$$

*where $c(\pi,\sigma)$ are certain positive integers.*

*Proof.* We have to go back to the proof from the untwisted case, from section 4 above, and insert signs. We have to check three conditions, as follows:

1. Concatenation. In the untwisted case, this was based on the following formula:

$$\delta_\pi\begin{pmatrix}i_1\ldots i_p\\ j_1\ldots j_q\end{pmatrix}\delta_\sigma\begin{pmatrix}k_1\ldots k_r\\ l_1\ldots l_s\end{pmatrix}=\delta_{[\pi\sigma]}\begin{pmatrix}i_1\ldots i_p&k_1\ldots k_r\\ j_1\ldots j_q&l_1\ldots l_s\end{pmatrix}$$

In the twisted case, it is enough to check the following formula:

$$\varepsilon\left(\ker\begin{pmatrix}i_1\ldots i_p\\ j_1\ldots j_q\end{pmatrix}\right)\varepsilon\left(\ker\begin{pmatrix}k_1\ldots k_r\\ l_1\ldots l_s\end{pmatrix}\right)=\varepsilon\left(\ker\begin{pmatrix}i_1\ldots i_p&k_1\ldots k_r\\ j_1\ldots j_q&l_1\ldots l_s\end{pmatrix}\right)$$

Let us denote by $\tau,\nu$ the partitions on the left, so that the partition on the right is of the form $\rho\leq[\tau\nu]$. Now by switching to the noncrossing form, $\tau\to\tau'$ and $\nu\to\nu'$, the partition on the right transforms into $\rho\to\rho'\leq[\tau'\nu']$. Now since the partition $[\tau'\nu']$ is noncrossing, we can use Proposition 7.24 (3), and we obtain the result.

2. Composition. In the untwisted case, this was based on the following formula:

$$\sum_{j_1\ldots j_q}\delta_\pi\begin{pmatrix}i_1\ldots i_p\\ j_1\ldots j_q\end{pmatrix}\delta_\sigma\begin{pmatrix}j_1\ldots j_q\\ k_1\ldots k_r\end{pmatrix}=N^{c(\pi,\sigma)}\delta_{[^\pi_\sigma]}\begin{pmatrix}i_1\ldots i_p\\ k_1\ldots k_r\end{pmatrix}$$

In order to prove now the result in the twisted case, it is enough to check that the signs match. More precisely, we must establish the following formula:

$$\varepsilon\left(\ker\begin{pmatrix}i_1\ldots i_p\\ j_1\ldots j_q\end{pmatrix}\right)\varepsilon\left(\ker\begin{pmatrix}j_1\ldots j_q\\ k_1\ldots k_r\end{pmatrix}\right)=\varepsilon\left(\ker\begin{pmatrix}i_1\ldots i_p\\ k_1\ldots k_r\end{pmatrix}\right)$$

Let $\tau,\nu$ be the partitions on the left, so that the partition on the right is of the form $\rho\leq[^\tau_\nu]$. Our claim is that we can jointly switch $\tau,\nu$ to the noncrossing form. Indeed, we can first switch as for $\ker(j_1\ldots j_q)$ to become noncrossing, and then switch the upper legs of $\tau$, and the lower legs of $\nu$, as for both these partitions to become noncrossing. Now observe that when switching in this way to the noncrossing form, $\tau\to\tau'$ and $\nu\to\nu'$, the partition on the right transforms into $\rho\to\rho'\leq[^{\tau'}_{\nu'}]$. Since the partition $[^{\tau'}_{\nu'}]$ is noncrossing, we can apply Proposition 7.24 (3), and we obtain the result.

3. Involution. Here we must prove the following formula:

$$\bar{\delta}_\pi\begin{pmatrix}i_1\ldots i_p\\ j_1\ldots j_q\end{pmatrix}=\bar{\delta}_{\pi^*}\begin{pmatrix}j_1\ldots j_q\\ i_1\ldots i_p\end{pmatrix}$$

But this is clear from the definition of $\bar{\delta}_\pi$, and we are done. □

As a conclusion, our twisted construction $\pi\to\bar{T}_\pi$ has all the needed properties for producing quantum groups, via Tannakian duality. Thus, we can formulate:

**Theorem 7.27.** *Given a category of partitions $D\subset P_{even}$, the construction*

$$Hom(u^{\otimes k},u^{\otimes l})=span\left(\bar{T}_\pi\Big|\pi\in D(k,l)\right)$$

*produces via Tannakian duality a quantum group $\bar{G}_N\subset U_N^+$, for any $N\in\mathbb{N}$.*

*Proof.* This follows indeed from the Tannakian results from section 4 above, exactly as in the easy case, by using this time Proposition 7.26 as technical ingredient.

To be more precise, Proposition 7.26 shows that the linear spaces on the right form a Tannakian category, and so the results in section 4 apply, and give the result. □

We can unify the easy quantum groups, or at least the examples coming from categories $D\subset P_{even}$, with the quantum groups constructed above, as follows:

**Definition 7.28.** *A closed subgroup $G\subset U_N^+$ is called $q$-easy, or quizzy, with deformation parameter $q=\pm1$, when its tensor category appears as follows,*

$$Hom(u^{\otimes k},u^{\otimes l})=span\left(\dot{T}_\pi\Big|\pi\in D(k,l)\right)$$

*for a certain category of partitions $D\subset P_{even}$, where, for $q=-1,1$:*

$$\dot{T}=\bar{T},T$$

*The Schur-Weyl twist of $G$ is the quizzy quantum group $\bar{G}\subset U_N^+$ obtained via $q\to-q$.*

We will see later on that the easy quantum group associated to $P_{even}$ itself is the hyperochahedral group $H_N$, and so that our assumption $D\subset P_{even}$, replacing $D\subset P$, simply corresponds to $H_N\subset G$, replacing the usual condition $S_N\subset G$.

In relation now with the basic quantum groups, we first have the following result:

**Proposition 7.29.** *The linear map associated to the basic crossing is:*

$$\bar{T}_{/\!\!\!\backslash}(e_i \otimes e_j) = \begin{cases} -e_j \otimes e_i & \text{for } i \neq j \\ e_j \otimes e_i & \text{otherwise} \end{cases}$$

*The linear map associated to the half-liberating permutation is:*

$$\bar{T}_{/\!\!\!|\!\!\!\backslash}(e_i \otimes e_j \otimes e_k) = \begin{cases} -e_k \otimes e_j \otimes e_i & \text{for } i,j,k \text{ distinct} \\ e_k \otimes e_j \otimes e_i & \text{otherwise} \end{cases}$$

*Also, for any noncrossing pairing $\pi \in NC_2$, we have $\bar{T}_\pi = T_\pi$.*

*Proof.* We have to compute the signature of the various partitions involved, and we can use here (1,2,3) in Proposition 7.24, which give the results. □

The relation with the basic quantum groups comes from:

**Proposition 7.30.** *For an orthogonal quantum group $G$, the following hold:*

(1) *$\dot{T}_{/\!\!\!\backslash} \in End(u^{\otimes 2})$ precisely when $G \subset \dot{O}_N$.*
(2) *$\dot{T}_{/\!\!\!|\!\!\!\backslash} \in End(u^{\otimes 3})$ precisely when $G \subset \dot{O}_N^*$.*

*Proof.* We know this in the untwisted case. In the twisted case, the proof is as follows:

(1) By using the formula of $\bar{T}_{/\!\!\!\backslash}$ in Proposition 7.29, we obtain:

$$\begin{aligned}(\bar{T}_{/\!\!\!\backslash} \otimes 1)u^{\otimes 2}(e_i \otimes e_j \otimes 1) &= \sum_k e_k \otimes e_k \otimes u_{ki}u_{kj} \\ &- \sum_{k \neq l} e_l \otimes e_k \otimes u_{ki}u_{lj}\end{aligned}$$

We have as well the following formula:

$$u^{\otimes 2}(\bar{T}_{/\!\!\!\backslash} \otimes 1)(e_i \otimes e_j \otimes 1) = \begin{cases} \sum_{kl} e_l \otimes e_k \otimes u_{li}u_{ki} & \text{if } i = j \\ -\sum_{kl} e_l \otimes e_k \otimes u_{lj}u_{ki} & \text{if } i \neq j \end{cases}$$

For $i = j$ the conditions are $u_{ki}^2 = u_{ki}^2$ for any $k$, and $u_{ki}u_{li} = -u_{li}u_{ki}$ for any $k \neq l$. For $i \neq j$ the conditions are $u_{ki}u_{kj} = -u_{kj}u_{ki}$ for any $k$, and $u_{ki}u_{lj} = u_{lj}u_{ki}$ for any $k \neq l$. Thus we have exactly the relations between the coordinates of $\bar{O}_N$, and we are done.

(2) By using the formula of $\bar{T}_{/\!\!\!|\!\!\!\backslash}$ in Proposition 7.29, we obtain:

$$\begin{aligned}&(\bar{T}_{/\!\!\!|\!\!\!\backslash} \otimes 1)u^{\otimes 2}(e_i \otimes e_j \otimes e_k \otimes 1) \\ = &\sum_{abc \text{ not distinct}} e_c \otimes e_b \otimes e_a \otimes u_{ai}u_{bj}u_{ck} \\ - &\sum_{a,b,c \text{ distinct}} e_c \otimes e_b \otimes e_a \otimes u_{ai}u_{bj}u_{ck}\end{aligned}$$

On the other hand, we have as well the following formula:

$$\begin{aligned}
&u^{\otimes 2}(\bar{T}_{\times} \otimes 1)(e_i \otimes e_j \otimes e_k \otimes 1)\\
&= \begin{cases} \sum_{abc} e_c \otimes e_b \otimes e_a \otimes u_{ck}u_{bj}u_{ai} & \text{for } i,j,k \text{ not distinct}\\ -\sum_{abc} e_c \otimes e_b \otimes e_a \otimes u_{ck}u_{bj}u_{ai} & \text{for } i,j,k \text{ distinct}\end{cases}
\end{aligned}$$

For $i,j,k$ not distinct the conditions are $u_{ai}u_{bj}u_{ck} = u_{ck}u_{bj}u_{ai}$ for $a,b,c$ not distinct, and $u_{ai}u_{bj}u_{ck} = -u_{ck}u_{bj}u_{ai}$ for $a,b,c$ distinct. For $i,j,k$ distinct the conditions are $u_{ai}u_{bj}u_{ck} = -u_{ck}u_{bj}u_{ai}$ for $a,b,c$ not distinct, and $u_{ai}u_{bj}u_{ck} = u_{ck}u_{bj}u_{ai}$ for $a,b,c$ distinct. Thus we have exactly the relations between the coordinates of $\bar{O}_N^*$, and we are done. ☐

We can now formulate our first Schur-Weyl twisting result, as follows:

**Theorem 7.31.** *The twisted quantum groups introduced before,*

$$\begin{tikzcd}
\bar{U}_N \arrow[r] & \bar{U}_N^* \arrow[r] & U_N^+ \\
\bar{O}_N \arrow[r] \arrow[u] & \bar{O}_N^* \arrow[r] \arrow[u] & O_N^+ \arrow[u]
\end{tikzcd}$$

*appear as twists of the basic quantum groups, namely*

$$\begin{tikzcd}
U_N \arrow[r] & U_N^* \arrow[r] & U_N^+ \\
\bar{O}_N \arrow[r] \arrow[u] & \bar{O}_N^* \arrow[r] \arrow[u] & O_N^+ \arrow[u]
\end{tikzcd}$$

*via the Schur-Weyl twisting procedure described above.*

*Proof.* This follows indeed from Proposition 7.30 above. ☐

In order for our twisting theory to be complete, let us discuss as well the computation of the quantum isometry groups of the twisted spheres. We have here:

**Theorem 7.32.** *The quantum isometry groups of the twisted spheres,*

$$\begin{tikzcd}
\bar{S}^{N-1}_{\mathbb{C}} \arrow[r] & \bar{S}^{N-1}_{\mathbb{C},*} \arrow[r] & S^{N-1}_{\mathbb{C},+} \\
\bar{S}^{N-1}_{\mathbb{R}} \arrow[r] \arrow[u] & \bar{S}^{N-1}_{\mathbb{R},*} \arrow[r] \arrow[u] & S^{N-1}_{\mathbb{R},+} \arrow[u]
\end{tikzcd}$$

*are the above twisted orthogonal and unitary groups.*

*Proof.* The proof in the classical twisted cases is similar to the proof in the classical untwisted cases, by adding signs. Indeed, for the twisted real sphere $\bar{S}^{N-1}_{\mathbb{R}}$ we have:

$$\begin{aligned}\Phi(z_iz_j) &= \sum_k z_k^2\otimes u_{ki}u_{kj}\\ &+ \sum_{k<l} z_kz_l\otimes(u_{ki}u_{lj}-u_{li}u_{kj})\end{aligned}$$

We deduce that with $[[a,b]]=ab+ba$ we have the following formula:

$$\begin{aligned}\Phi([[z_i,z_j]]) &= \sum_k z_k^2\otimes [[u_{ki},u_{kj}]]\\ &+ \sum_{k<l} z_kz_l\otimes([u_{ki},u_{lj}]-[u_{li},u_{kj}])\end{aligned}$$

Now assuming $i\neq j$, we have $[[z_i,z_j]]=0$, and we therefore obtain:

$$[[u_{ki},u_{kj}]]=0\quad,\quad\forall k$$

$$[u_{ki},u_{lj}]=[u_{li},u_{kj}]\quad,\quad\forall k<l$$

By using now the standard trick, namely applying the antipode and then relabelling, the latter relation gives:

$$[u_{ki},u_{lj}]=0$$

Thus, we obtain the result. The proof for $\bar{S}^{N-1}_{\mathbb{C}}$ is similar, by using the above-mentioned categorical trick, in order to deduce from the relations $ab=\pm ba$ the remaining relations $ab^*=\pm b^*a$. Finally, the proof in the half-classical twisted cases is similar to the proof in the half-classical untwisted cases, by adding signs where needed. □

As a conclusion, we have a quite interesting notion of easy quantum group, basically coming from the Brauer philosophy for $O_N,U_N$, and notably covering $O_N^+,U_N^+$, along with some theory and examples, and with a twisting extension as well.

We will be back to this later on, in sections 11-12 below, with a negative result this time, stating that the easy quantum reflection groups are invariant under twisting.

## 8. Probabilistic aspects

We discuss here the computation of the various integrals over the compact quantum groups, with respect to the Haar measure. In order to formulate our results in a conceptual form, we use the modern measure theory language, namely probability theory. In the general noncommutative setting, the starting definition is as follows:

**Definition 8.1.** *Let $A$ be a $C^*$-algebra, given with a trace $tr$.*

(1) *The elements $a \in A$ are called random variables.*
(2) *The moments of such a variable are the numbers $M_k(a) = tr(a^k)$.*
(3) *The law of such a variable is the functional $\mu : P \to tr(P(a))$.*

Here $k = \circ \bullet \bullet \circ \ldots$ is as usual a colored integer, and the powers $a^k$ are defined by multiplicativity and the usual formulae, namely:

$$a^\emptyset = 1 \quad , \quad a^\circ = a \quad , \quad a^\bullet = a^*$$

As for the polynomial $P$, this is a noncommuting $*$-polynomial in one variable:

$$P \in \mathbb{C} < X, X^* >$$

Observe that the law is uniquely determined by the moments, because:

$$P(X) = \sum_k \lambda_k X^k \implies \mu(P) = \sum_k \lambda_k M_k(a)$$

Generally speaking, the above definition is something quite abstract, but there is no other way of doing things, at least at this level of generality. We have indeed:

**Theorem 8.2.** *Given a $C^*$-algebra with a faithful trace $(A, tr)$, any normal variable,*

$$aa^* = a^*a$$

*has a usual law, namely a complex probability measure $\mu \in \mathcal{P}(\mathbb{C})$ satisfying:*

$$tr(a^k) = \int_{\mathbb{C}} z^k d\mu(z)$$

*This law is unique, and is supported by the spectrum $\sigma(a) \subset \mathbb{C}$. In the non-normal case, $aa^* \neq a^*a$, such a usual law does not exist.*

*Proof.* We have two assertions here, the idea being as follows:

(1) In the normal case, $aa^* = a^*a$, the Gelfand theorem, or rather the subsequent continuous functional calculus theorem, tells us that we have $< a >= C(\sigma(a))$. Thus the functional $f(a) \to tr(f(a))$ can be regarded as an integration functional on the algebra $C(\sigma(a))$, and by the Riesz theorem, this latter functional must come from a probability measure $\mu$ on the spectrum $\sigma(a)$, in the sense that we must have:

$$tr(f(a)) = \int_{\sigma(a)} f(z) d\mu(z)$$

We are therefore led to the conclusions in the statement, with the uniqueness assertion coming from the fact that the elements $a^k$, taken as usual with respect to colored integer exponents, $k = \circ\bullet\bullet\circ\ldots$, generate the whole $C^*$-algebra $C(\sigma(a))$.

(2) In the non-normal case now, $aa^* \neq a^*a$, we must show that such a law does not exist. For this purpose, we can use a positivity trick, as follows:

$$\begin{aligned}
aa^* - a^*a \neq 0 &\implies (aa^* - a^*a)^2 > 0\\
&\implies aa^*aa^* - aa^*a^*a - a^*aaa^* + a^*aa^*a > 0\\
&\implies tr(aa^*aa^* - aa^*a^*a - a^*aaa^* + a^*aa^*a) > 0\\
&\implies tr(aa^*aa^* + a^*aa^*a) > tr(aa^*a^*a + a^*aaa^*)\\
&\implies tr(aa^*aa^*) > tr(aaa^*a^*)
\end{aligned}$$

Now assuming that $a$ has a law $\mu \in \mathcal{P}(\mathbb{C})$, in the sense that the moment formula in the statement holds, the above two different numbers would have to both appear by integrating $|z|^2$ with respect to this law $\mu$, which is contradictory, as desired. □

Summarizing, we have a beginning of a theory, generalizing that of the compact probability spaces $(X,\mu)$. A noncommutative probability space corresponds by definition to a pair $(A, tr)$, according to the formulae $A = C(X)$ and $tr(f) = \int_X f(x)d\mu(x)$. We can talk about moments and laws in this setting. And when $A$ is commutative, we recover in this way the usual probability theory. Let us discuss now the independence, and its noncommutative versions. As a starting point here, we have the following notion:

**Definition 8.3.** *Two subalgebras $B, C \subset A$ are called independent when the following condition is satisfied, for any $b \in B$ and $c \in C$:*

$$tr(bc) = tr(b)tr(c)$$

*Equivalently, the following condition must be satisfied, for any $b \in B$ and $c \in C$:*

$$tr(b) = tr(c) = 0 \implies tr(bc) = 0$$

*Also, two variables $b, c \in A$ are called independent when the algebras that they generate,*

$$B = < b > \quad , \quad C = < c >$$

*are independent inside $A$, in the above sense.*

Observe that the above two conditions are indeed equivalent. In one sense this is clear, and in the other sense, with $a' = a - tr(a)$, this follows from:

$$\begin{aligned}
tr(bc) &= tr[(b' + tr(b))(c' + tr(c))]\\
&= tr(b'c') + t(b')tr(c) + tr(b)tr(c') + tr(b)tr(c)\\
&= tr(b'c') + tr(b)tr(c)\\
&= tr(b)tr(c)
\end{aligned}$$

The other remark is that the above notion generalizes indeed the usual notion of independence, from the classical case, the result here being as follows:

**Theorem 8.4.** *Given two compact measured spaces $Y,Z$, the algebras*

$$C(Y)\subset C(Y\times Z)\quad,\quad C(Z)\subset C(Y\times Z)$$

*are independent in the above sense, and a converse of this fact holds too.*

*Proof.* We have two assertions here, the idea being as follows:

(1) First of all, given two arbitrary compact spaces $Y,Z$, we have embeddings of algebras as in the statement, defined by the following formulae:

$$f\to[(y,z)\to f(y)]\quad,\quad g\to[(y,z)\to g(z)]$$

In the measured space case now, the Fubini theorems tells us that:

$$\int_{Y\times Z}f(y)g(z)=\int_Y f(y)\int_Z g(z)$$

Thus, the algebras $C(Y),C(Z)$ are independent in the sense of Definition 8.3.

(2) Conversely now, assume that $B,C\subset A$ are independent, with $A$ being commutative. Let us write our algebras as follows, with $X,Y,Z$ being certain compact spaces:

$$A=C(X)\quad,\quad B=C(Y)\quad,\quad C=C(Z)$$

In this picture, the inclusions $B,C\subset A$ must come from quotient maps, as follows:

$$p:Z\to X\quad,\quad q:Z\to Y$$

Regarding now the independence condition from Definition 8.3, in the above picture, this tells us that the folowing equality must happen:

$$\int_X f(p(x))g(q(x))=\int_X f(p(x))\int_X g(q(x))$$

Thus we are in a Fubini type situation, and we obtain from this $Y\times Z\subset X$. Thus, the independence of $B,C\subset A$ appears as in (1) above. □

It is possible to develop some theory here, but this is ultimately not very interesting. As a much more interesting notion now, we have the freeness:

**Definition 8.5.** *Two subalgebras $B,C\subset A$ are called free when the following condition is satisfied, for any $b_i\in B$ and $c_i\in C$:*

$$tr(b_i)=tr(c_i)=0\implies tr(b_1c_1b_2c_2\ldots)=0$$

*Also, two variables $b,c\in A$ are called free when the algebras that they generate,*

$$B=<b>\quad,\quad C=<c>$$

*are free inside $A$, in the above sense.*

In short, freeness appears by definition as a kind of "free analogue" of independence, taking into account the fact that the variables do not necessarily commute. We will see in a moment examples, theory, applications, and other reasons for studying freeness. As a first observation, of theoretical nature, there is actually a certain lack of symmetry between Definition 8.3 and Definition 8.5, because in contrast to the former, the latter does not include an explicit formula for the quantities of the following type:

$$tr(b_1c_1b_2c_2\ldots)$$

However, this is not an issue, and is simply due to the fact that the formula in the free case is something more complicated, the result being as follows:

**Proposition 8.6.** *Assuming that $B,C\subset A$ are free, the restriction of $tr$ to $<B,C>$ can be computed in terms of the restrictions of $tr$ to $B,C$. To be more precise,*

$$tr(b_1c_1b_2c_2\ldots)=P\Big(\{tr(b_{i_1}b_{i_2}\ldots)\}_i,\{tr(c_{j_1}c_{j_2}\ldots)\}_j\Big)$$

*where $P$ is certain polynomial in several variables, depending on the length of the word $b_1c_1b_2c_2\ldots$, and having as variables the traces of products of type*

$$b_{i_1}b_{i_2}\ldots\quad,\quad c_{j_1}c_{j_2}\ldots$$

*with the indices being chosen increasing, $i_1<i_2<\ldots$ and $j_1<j_2<\ldots$*

*Proof.* This is something quite theoretical, so let us begin with an example. Our claim is that if $b,c$ are free then, exactly as in the case where we have independence:

$$tr(bc)=tr(b)tr(c)$$

Indeed, let us go back to the computation performed after Definition 8.3, which was as follows, with the convention $a'=a-tr(a)$:

$$\begin{aligned}
tr(bc)&=tr[(b'+tr(b))(c'+tr(c))]\\
&=tr(b'c')+t(b')tr(c)+tr(b)tr(c')+tr(b)tr(c)\\
&=tr(b'c')+tr(b)tr(c)\\
&=tr(b)tr(c)
\end{aligned}$$

Our claim is that this computation perfectly works under the sole freeness assumption. Indeed, the only non-trivial equality is the last one, which follows from:

$$tr(b')=tr(c')=0\implies tr(b'c')=0$$

In general now, the situation is of course more complicated, but the same trick applies. To be more precise, we can start our computation as follows:

$$\begin{aligned}
tr(b_1c_1b_2c_2\ldots)&=tr\big[(b_1'+tr(b_1))(c_1'+tr(c_1))(b_2'+tr(b_2))(c_2'+tr(c_2))\ldots\ldots\big]\\
&=tr(b_1'c_1'b_2'c_2'\ldots)+\text{other terms}\\
&=\text{other terms}
\end{aligned}$$

Observe that we have used here the freeness condition, in the following form:

$$tr(b_i') = tr(c_i') = 0 \implies tr(b_1'c_1'b_2'c_2'\ldots) = 0$$

Now regarding the "other terms", those which are left, each of them will consist of a product of traces of type $tr(b_i)$ and $tr(c_i)$, and then a trace of a product still remaining to be computed, which is of the following form, with $\beta_i \in B$ and $\gamma_i \in C$:

$$tr(\beta_1\gamma_1\beta_2\gamma_2\ldots)$$

To be more precise, the variables $\beta_i \in B$ appear as ordered products of those $b_i \in B$ not getting into individual traces $tr(b_i)$, and the variables $\gamma_i \in C$ appear as ordered products of those $c_i \in C$ not getting into individual traces $tr(c_i)$. Now since the length of each such alternating product $\beta_1\gamma_1\beta_2\gamma_2\ldots$ is smaller than the length of the original alternating product $b_1c_1b_2c_2\ldots$, we are led into of recurrence, and this gives the result. □

Let us discuss now some models for independence and freeness. We first have the following result, which clarifies the analogy between independence and freeness:

**Theorem 8.7.** *Given two algebras $(B, tr)$ and $(C, tr)$, the following hold:*

(1) *$B, C$ are independent inside their tensor product $B \otimes C$, endowed with its canonical tensor product trace, given on basic tensors by $tr(b \otimes c) = tr(b)tr(c)$.*
(2) *$B, C$ are free inside their free product $B * C$, endowed with its canonical free product trace, given by the formulae in Proposition 8.6.*

*Proof.* Both the assertions are clear from definitions, as follows:

(1) This is clear with either of the definitions of the independence, from Definition 8.3 above, because we have by construction of the trace:

$$tr(bc) = tr[(b \otimes 1)(1 \otimes c)] = tr(b \otimes c) = tr(b)tr(c)$$

Observe that there is a relation here with Theorem 8.4 as well, due to the following formula for compact spaces, with $\otimes$ being a topological tensor product:

$$C(Y \times Z) = C(Y) \otimes C(Z)$$

To be more precise, the present statement generalizes the first assertion in Theorem 8.4, and the second assertion tells us that this generalization is more or less the same thing as the original statement. All this comes of course from basic measure theory.

(2) This is clear from definitions, the only point being that of showing that the notion of freeness, or the recurrence formulae in Proposition 8.6, can be used in order to construct a canonical free product trace, on the free product of the two algebras involved:

$$tr : B * C \to \mathbb{C}$$

But this can be checked for instance by using a GNS construction. Indeed, consider the GNS constructions for the algebras $(B,tr)$ and $(C,tr)$:

$$B\to B(l^2(B))\quad,\quad C\to B(l^2(C))$$

By taking the free product of these representations, we obtain a representation as follows, with the $*$ symbol on the right being a free product of pointed Hilbert spaces:

$$B*C\to B(l^2(B)*l^2(C))$$

Now by composing with the linear form $T\to<T\xi,\xi>$, where $\xi=1_B=1_C$ is the common distinguished vector of $l^2(B)$ and $l^2(C)$, we obtain a linear form, as follows:

$$tr:B*C\to\mathbb C$$

It is routine then to check that $tr$ is indeed a trace, and this is the "canonical free product trace" from the statement. Then, an elementary computation shows that $B,C$ are indeed free inside $B*C$, with respect to this trace, and this finishes the proof. □

As an concrete application of the above results, we have:

**Theorem 8.8.** *We have a free convolution operation $\boxplus$ for the distributions*

$$\mu:\mathbb C<X,X^*>\to\mathbb C$$

*which is well-defined by the following formula, with $b,c$ taken to be free:*

$$\mu_b\boxplus\mu_c=\mu_{b+c}$$

*This restricts to an operation, still denoted $\boxplus$, on the real probability measures.*

*Proof.* We have several verifications to be performed here, as follows:

(1) We first have to check that given two variables $b,c$ which live respectively in certain $C^*$-algebras $B,C$, we can recover inside some $C^*$-algebra $A$, with exactly the same distributions $\mu_b,\mu_c$, as to be able to sum them and then talk about $\mu_{b+c}$. But this comes from Theorem 8.7, because we can set $A=B*C$, as explained there.

(2) The other verification which is needed is that of the fact that if $b,c$ are free, then the distribution $\mu_{b+c}$ depends only on the distributions $\mu_b,\mu_c$. But for this purpose, we can use the general formula from Proposition 8.6, namely:

$$tr(b_1c_1b_2c_2\ldots)=P\Big(\{tr(b_{i_1}b_{i_2}\ldots)\}_i,\{tr(c_{j_1}c_{j_2}\ldots)\}_j\Big)$$

Here $P$ is certain polynomial, depending on the length of $b_1c_1b_2c_2\ldots$, having as variables the traces of products $b_{i_1}b_{i_2}\ldots$ and $c_{j_1}c_{j_2}\ldots$, with $i_1<i_2<\ldots$ and $j_1<j_2<\ldots$

Now by plugging in arbitrary powers of $b,c$ as variables $b_i,c_j$, we obtain a family of formulae of the following type, with $Q$ being certain polyomials:

$$tr(b^{k_1}c^{l_1}b^{k_2}c^{l_2}\ldots)=P\Big(\{tr(b^k)\}_k,\{tr(c^l)\}_l\Big)$$

Thus the moments of $b+c$ depend only on the moments of $b,c$, with of course colored exponents in all this, according to our moment conventions, and this gives the result.

(3) Finally, in what regards the last assertion, regarding the real measures, this is clear from the fact that if $b,c$ are self-adjoint, then so is their sum $b+c$. □

We would like to have a linearization result for $\boxplus$, in the spirit of the known result for $*$. We will do this slowly, in several steps. As a first observation, both the independence and the freeness are nicely modelled inside group algebras, as follows:

**Theorem 8.9.** *We have the following results, valid for group algebras:*

(1) $C^*(\Gamma),C^*(\Lambda)$ *are independent inside* $C^*(\Gamma\times\Lambda)$*.*
(2) $C^*(\Gamma),C^*(\Lambda)$ *are free inside* $C^*(\Gamma*\Lambda)$*.*

*Proof.* In order to prove these results, we have two possible methods:

(1) We can use here the general results in Theorem 8.7 above, along with the following two isomorphisms, which are both standard:

$$C^*(\Gamma\times\Lambda)=C^*(\Lambda)\otimes C^*(\Gamma)$$

$$C^*(\Gamma*\Lambda)=C^*(\Lambda)*C^*(\Gamma)$$

(2) We can prove this directly as well, by using the fact that each group algebra is spanned by the corresponding group elements. Indeed, it is enough to check the independence and freeness formulae on group elements, which is in turn trivial. □

Regarding now the linearization problem for $\boxplus$, the situation here is quite tricky, and the above models do not provide good results. We must use instead:

**Theorem 8.10.** *Consider the shift operator on the space $H=l^2(\mathbb{N})$, given by:*

$$S(e_i)=e_{i+1}$$

*The variables of the following type, with $f\in\mathbb{C}[X]$ being a polynomial,*

$$S^*+f(S)$$

*model then in moments, up to finite order, all the distributions $\mu:\mathbb{C}[X]\to\mathbb{C}$.*

*Proof.* We have already met the shift $S$ in section 1 above, as the simplest example of an isometry which is not a unitary, $S^*S=1,SS^*=1$, with this coming from:

$$S^*(e_i)=\begin{cases}e_{i-1}&(i>0)\\0&(i=0)\end{cases}$$

Consider now a variable as in the statement, namely:

$$T=S^*+a_0+a_1S+a_2S^2+\ldots+a_nS^n$$

We have then $tr(T)=a_0$, then $tr(T^2)$ will involve $a_1$, then $tr(T^3)$ will involve $a_2$, and so on. Thus, we are led to a certain recurrence, that we will not attempt to solve now, with bare hands, but which definitely gives the conclusion in the statement. □

Before getting further, with taking free products of such models, let us work out a very basic example, which is something fundamental, that we will need in what follows:

**Proposition 8.11.** *In the context of the above correspondence, the variable*

$$T=S+S^*$$

*follows the Wigner semicircle law on* $[-2,2]$.

*Proof.* This is something that we already know from section 6, the idea being that the combinatorics of $(S+S^*)^k$ leads us into paths on $\mathbb N$, and to the Catalan numbers. □

Getting back now to our linearization program for $\boxplus$, the next step is that of taking a free product of the model found in Theorem 8.10 with itself. We have here:

**Proposition 8.12.** *We can define the algebra of creation operators*

$$S_x:v\to x\otimes v$$

*on the free Fock space associated to a real Hilbert space $H$, given by*

$$F(H)=\mathbb C\Omega\oplus H\oplus H^{\otimes 2}\oplus\ldots$$

*and at the level of examples, we have:*

(1) *With $H=\mathbb C$ we recover the shift algebra $A=<S>$ on $H=l^2(\mathbb N)$.*
(2) *With $H=\mathbb C^2$, we obtain the algebra $A=<S_1,S_2>$ on $H=l^2(\mathbb N*\mathbb N)$.*

*Proof.* We can talk indeed about the algebra $A(H)$ of creation operators on the free Fock space $F(H)$ associated to a real Hilbert space $H$, with the remark that, in terms of the abstract semigroup notions from section 6 above, we have:

$$A(\mathbb C^k)=C^*(\mathbb N^{*k})$$

$$F(\mathbb C^k)=l^2(\mathbb N^{*k})$$

As for the assertions (1,2) in the statement, these are both clear. □

With the above notions in hand, we have the following key freeness result:

**Proposition 8.13.** *Given a real Hilbert space $H$, and two orthogonal vectors $x,y\in H$,*

$$x\perp y$$

*the corresponding creation operators $S_x$ and $S_y$ are free with respect to*

$$tr(T)=<T\Omega,\Omega>$$

*called trace associated to the vacuum vector.*

*Proof.* In standard tensor notation for the elements of the free Fock space $F(H)$, the formula of a creation operator associated to a vector $x \in H$ is as follows:

$$S_x(y_1 \otimes \ldots \otimes y_n) = x \otimes y_1 \otimes \ldots \otimes y_n$$

As for the formula of the adjoint of this creation operator, this is as follows:

$$S_x^*(y_1 \otimes \ldots \otimes y_n) =< x, y_1 > \otimes y_2 \otimes \ldots \otimes y_n$$

We obtain from this the following formula, valid for any two vectors $x, y \in H$:

$$S_x^* S_y =< x, y > id$$

With these formulae in hand, the result follows by doing some elementary computations, in the spirit of those done before for the group algebras. □

With this technology in hand, let us go back to our linearization program for $\boxplus$. We have the following key result, further building on Proposition 8.13:

**Theorem 8.14.** *Given two polynomials $f, g \in \mathbb{C}[X]$, consider the variables*

$$R^* + f(R) \quad , \quad S^* + g(S)$$

*where $R, S$ are two creation operators, or shifts, associated to a pair of orthogonal norm 1 vectors. These variables are then free, and their sum has the same law as*

$$T^* + (f + g)(T)$$

*with $T$ being the usual shift on $l^2(\mathbb{N})$.*

*Proof.* We have two assertions here, the idea being as follows:

(1) The freeness assertion comes from the general freeness result from Proposition 8.13, via the various identifications coming from the previous results.

(2) Regarding now the second assertion, the idea is that this comes from a 45° rotation trick. Let us write indeed the two variables in the statement as follows:

$$X = R^* + a_0 + a_1 R + a_2 R^2 + \ldots$$

$$Y = S^* + b_0 + b_1 S + a_2 S^2 + \ldots$$

Now let us perform the following 45° base change, on the real span of the vectors $r, s \in H$ producing our two shifts $R, S$:

$$t = \frac{r + s}{\sqrt{2}} \quad , \quad u = \frac{r - s}{\sqrt{2}}$$

The new shifts, associated to these vectors $t, u \in H$, are then given by:

$$T = \frac{R + S}{\sqrt{2}} \quad , \quad U = \frac{R - S}{\sqrt{2}}$$

By using now these new shifts, which are free as well according to Proposition 8.13, we obtain the following equality of distributions:

$$\begin{aligned}
X+Y &= R^*+S^*+\sum_k a_kR^k+b_kS^k\\
&= \sqrt{2}T^*+\sum_k a_k\left(\frac{T+U}{\sqrt{2}}\right)^k+b_k\left(\frac{T-U}{\sqrt{2}}\right)^k\\
&\sim \sqrt{2}T^*+\sum_k a_k\left(\frac{T}{\sqrt{2}}\right)^k+b_k\left(\frac{T}{\sqrt{2}}\right)^k\\
&\sim T^*+\sum_k a_kT^k+b_kT^k
\end{aligned}$$

To be more precise, here in the last two lines we have used the freeness property of $T,U$ in order to cut $U$ from the computation, as it cannot bring anything, and then we did a basic rescaling at the end. Thus, we are led to the conclusion in the statement. □

We can now solve the linearization problem. Following [132], we have:

**Theorem 8.15.** *Given a real probability measure $\mu$, define its $R$-transform as follows:*

$$G_\mu(\xi)=\int_\mathbb{R}\frac{d\mu(t)}{\xi-t}\implies G_\mu\left(R_\mu(\xi)+\frac{1}{\xi}\right)=\xi$$

*The free convolution operation is then linearized by this $R$-transform.*

*Proof.* This can be done by using the above results, in several steps, as follows:

(1) According to Theorem 8.14, the operation $\mu\to f$ from Theorem 8.10 above linearizes the free convolution operation $\boxplus$. We are therefore left with a computation inside $C^*(\mathbb{N})$. To be more precise, consider a variable as in Theorem 8.14 above:

$$X=S^*+f(X)$$

In order to establish the result, we must prove that the $R$-transform of $X$, constructed according to the procedure in the statement, is the function $f$ itself.

(2) In order to do so, fix $|z|<1$ in the complex plane, and let us set:

$$w_z=\delta_0+\sum_{k=1}^\infty z_k\delta_k$$

The shift and its adjoint act then as follows, on this vector:

$$Sw_z=z^{-1}(w_z-\delta_0)\quad,\quad S^*w_z=zw_z$$

It follows that the adjoint of our operator $X$ acts as follows on this vector:

$$\begin{aligned} X^*w_z &= (S+f(S^*))w_z \\ &= z^{-1}(w_z-\delta_0)+f(z)w_z \\ &= (z^{-1}+f(z))w_z - z^{-1}\delta_0 \end{aligned}$$

Now observe that this formula can be written as follows:

$$z^{-1}\delta_0 = (z^{-1}+f(z)-X^*)w_z$$

The point now is that when $|z|$ is small, the operator appearing on the right is invertible. Thus, we can rewrite this formula as follows:

$$(z^{-1}+f(z)-X^*)^{-1}\delta_0 = zw_z$$

Now by applying the trace, we are led to the following formula:

$$\begin{aligned} tr\left[(z^{-1}+f(z)-X^*)^{-1}\right] &= \left\langle (z^{-1}+f(z)-X^*)^{-1}\delta_0,\delta_0\right\rangle \\ &= < zw_z,\delta_0 > \\ &= z \end{aligned}$$

(3) Let us apply now the complex function procedure in the statement to the real probability measure $\mu$ modelled by $X$. The Cauchy transform $G_\mu$ is given by:

$$\begin{aligned} G_\mu(\xi) &= tr((\xi-X)^{-1}) \\ &= \overline{tr\Big((\bar{\xi}-X^*)^{-1}\Big)} \\ &= tr((\xi-X^*)^{-1}) \end{aligned}$$

Now observe that, with the choice $\xi = z^{-1}+f(z)$ for our complex variable, the trace formula found in (2) above tells us precisely that we have:

$$G_\mu\big(z^{-1}+f(z)\big) = z$$

Thus, we have $R_\mu(z)=f(z)$, which finishes the proof, as explained in step (1). □

With the above linearization technology in hand, we can now establish the following free analogue of the CLT, also due to Voiculescu [132], [133]:

**Theorem 8.16** (Free CLT). *Given self-adjoint variables $x_1,x_2,x_3,\ldots$ which are f.i.d., centered, with variance $t>0$, we have, with $n\to\infty$, in moments,*

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n x_i \sim \gamma_t$$

*where $\gamma_t$ is the Wigner semicircle law of parameter $t$, having density:*

$$\gamma_t = \frac{1}{2\pi t}\sqrt{4t^2-x^2}dx$$

*Proof.* We follow the same idea as in the proof of the CLT:

(1) At $t = 1$, the $R$-transform of the variable in the statement on the left can be computed by using the linearization property from Theorem 8.15, and is given by:

$$R(\xi) = nR_x\left(\frac{\xi}{\sqrt{n}}\right) \simeq \xi$$

(2) Regarding now the right term, also at $t = 1$, our claim is that the $R$-transform of the Wigner semicircle law $\gamma_1$ is given by the following formula:

$$R_{\gamma_1}(\xi) = \xi$$

But this follows via some calculus, or directly from $S+S^* \sim \gamma_1$, coming from Proposition 8.11. Thus, the laws in the statement have the same $R$-transforms, as desired.

(4) Summarizing, we have proved the free CLT at $t = 1$. The passage to the general case, $t > 0$, is routine, by some standard dilation computations. □

Similarly, in the complex case, we have the following result:

**Theorem 8.17** (Free complex CLT). *Given variables $x_1, x_2, x_3, \ldots$, whose real and imaginary parts are f.i.d., centered, and with variance $t > 0$, we have, with $n \to \infty$,*

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n x_i \sim \Gamma_t$$

*where $\Gamma_t$ is the Voiculescu circular law of parameter $t$, appearing as the law of $\frac{1}{\sqrt{2}}(a+ib)$, where $a, b$ are self-adjoint and free, each following the law $\gamma_t$.*

*Proof.* This is clear from Theorem 8.16 above, by taking real and imaginary parts. □

We will be back later to theoretical free probability, with some further results on the subject. Now back to our quantum group questions, let us start with:

**Theorem 8.18.** *Given a Woronowicz algebra $(A, u)$, the law of the main character*

$$\chi = \sum_{i=1}^N u_{ii}$$

*with respect to the Haar integration has the following properties:*

1. *The moments of $\chi$ are the numbers $M_k = \dim(Fix(u^{\otimes k}))$.*
2. *$M_k$ counts as well the lenght $p$ loops at $1$, on the Cayley graph of $A$.*
3. *$law(\chi)$ is the Kesten measure of the associated discrete quantum group.*
4. *When $u \sim \bar{u}$ the law of $\chi$ is a usual measure, supported on $[-N, N]$.*
5. *The algebra $A$ is amenable precisely when $N \in supp(law(Re(\chi)))$.*
6. *Any morphism $f : (A, u) \to (B, v)$ must increase the numbers $M_k$.*
7. *Such a morphism $f$ is an isomorphism when $law(\chi_u) = law(\chi_v)$.*

*Proof.* These are things that we already know, the idea being as follows:

(1) This comes from the Peter-Weyl theory, which tells us the number of fixed points of $v = u^{\otimes k}$ can be recovered by integrating the character $\chi_v = \chi_u^k$.

(2) This is something true, and well-known, for $A = C^*(\Gamma)$, with $\Gamma =< g_1, \ldots, g_N >$ being a discrete group. In general, the proof is quite similar.

(3) This is actually the definition of the Kesten measure, in the case $A = C^*(\Gamma)$, with $\Gamma =< g_1, \ldots, g_N >$ being a discrete group. In general, this follows from (2).

(4) The equivalence $u \sim \bar{u}$ translates into $\chi_u = \chi_u^*$, and this gives the first assertion. As for the support claim, this follows from $uu^* = 1 \implies ||u_{ii}|| \leq 1$, for any $i$.

(5) This is the Kesten amenability criterion, which can be established as in the classical case, $A = C^*(\Gamma)$, with $\Gamma =< g_1, \ldots, g_N >$ being a discrete group.

(6) This is something elementary, which follows from (1) above, and from the fact that the morphisms of Woronowicz algebras increase the spaces of fixed points.

(7) This follows by using (6), and the Peter-Weyl theory, the idea being that if $f$ is not injective, then it must strictly increase one of the spaces $Fix(u^{\otimes k})$. □

As a conclusion to all this, given a compact quantum group $G$, computing $\mu = law(\chi)$ is the main question to be solved, and this regardless of our precise motivation for studying $G$. In what follows we will be interested in computing such laws, for the main examples of quantum groups that we have. In the easy quantum group case, we have:

**Theorem 8.19.** *For an easy quantum group $G = (G_N)$, coming from a category of partitions $D = (D(k,l))$, the asymptotic moments of the main character are given by*

$$\lim_{N\to\infty} \int_{G_N} \chi^k = |D(k)|$$

*where $D(k) = D(\emptyset, k)$, with the limiting sequence on the left consisting of certain integers, and being stationary at least starting from the $k$-th term.*

*Proof.* This follows indeed from the general formula from Theorem 8.18 (1), by using the linear independence result from section 5 above. □

Our next purpose will be that of understanding what happens for the basic classes of easy quantum groups. In the orthogonal case, we have:

**Theorem 8.20.** *In the $N \to \infty$ limit, the law of the main character $\chi_u$ is as follows:*

(1) *For $O_N$ we obtain a Gaussian law, $\frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$.*
(2) *For $O_N^+$ we obtain a Wigner semicircle law, $\frac{1}{2\pi}\sqrt{4-x^2}dx$.*

*Proof.* These are results that we both know, from section 5 above. □

In the unitary case now, we have:

**Theorem 8.21.** *In the $N \to \infty$ limit, the law of the main character $\chi_u$ is as follows:*

1. *For $U_N$ we obtain the complex Gaussian law $G_1$.*
2. *For $U_N^+$ we obtain the Voiculescu circular law $\Gamma_1$.*

*Proof.* These are once again results that we know, from section 6 above. □

Summarizing, we have seen so far that for $O_N, O_N^+, U_N, U_N^+$, the asymptotic laws of the main characters are the laws $g_1, \gamma_1, G_1, \Gamma_1$ coming from the various CLT. This is certainly nice, but there is still one conceptual problem, coming from:

**Proposition 8.22.** *The above convergences $law(\chi_u) \to g_1, \gamma_1, G_1, \Gamma_1$ are as follows:*

1. *They are non-stationary in the classical case.*
2. *They are stationary in the free case, starting from $N = 2$.*

*Proof.* This is something quite subtle, which can be proved as follows:

(1) Here we can use an amenability argument, based on the Kesten criterion. Indeed, $O_N, U_N$ being coamenable, the upper bound of the support of the law of $Re(\chi_u)$ is precisely $N$, and we obtain from this that the law of $\chi_u$ itself depends on $N \in \mathbb{N}$.

(2) Here the result follows from the computations in section 4 above, performed when working out the representation theory of $O_N^+, U_N^+$, which show that the linear maps $T_\pi$ associated to the noncrossing pairings are linearly independent, at any $N \geq 2$. □

In short, we are not over with our study, which seems to open more questions than it solves. Fortunately, the solution to this latest question is quite simple. The idea indeed will be that of improving our $g_1, \gamma_1, G_1, \Gamma_1$ results above with certain $g_t, \gamma_t, G_t, \Gamma_t$ results, which will require $N \to \infty$ in both the classical and free cases, in order to hold at any $t$. In practice, the definition that we will need is as follows:

**Definition 8.23.** *Given a Woronowicz algebra $(A, u)$, the variable*

$$\chi_t = \sum_{i=1}^{[tN]} u_{ii}$$

*is called truncation of the main character, with parameter $t \in (0, 1]$.*

Our purpose in what follows will be that of proving that for $O_N, O_N^+, U_N, U_N^+$, the asymptotic laws of the truncated characters $\chi_t$ with $t \in (0, 1]$ are the laws $g_t, \gamma_t, G_t, \Gamma_t$. This is something quite technical, motivated by the findings in Proposition 8.22 above, and also by a number of more advanced considerations, to become clear later on.

In order to start now, the basic result from Theorem 8.18 (1) is not useful in the general $t \in (0, 1]$ setting, and we must use instead general integration methods [58], [143]:

**Theorem 8.24.** *For an easy quantum group $G\subset U_N^+$, coming from a category of partitions $D=(D(k,l))$, we have the Weingarten integration formula*
$$\int_G u_{i_1j_1}^{e_1}\ldots u_{i_kj_k}^{e_k}=\sum_{\pi,\sigma\in D(k)}\delta_\pi(i)\delta_\sigma(j)W_{kN}(\pi,\sigma)$$
*for any colored integer $k=e_1\ldots e_k$ and any multi-indices $i,j$, where $D(k)=D(\emptyset,k)$, $\delta$ are usual Kronecker symbols, and*
$$W_{kN}=G_{kN}^{-1}$$
*with $G_{kN}(\pi,\sigma)=N^{|\pi\vee\sigma|}$, where $|.|$ is the number of blocks.*

*Proof.* We already know from section 3 above that any closed subgroup $G\subset U_N^+$ is subject to an abstract Weingarten formula, coming from Peter-Weyl theory, via some elementary linear algebra. With the notations there, the Kronecker symbols are given by:
$$\begin{aligned}\delta_{\xi_\pi}(i)&=\ <\xi_\pi,e_{i_1}\otimes\ldots\otimes e_{i_k}>\\&=\ \delta_\pi(i_1,\ldots,i_k)\end{aligned}$$
The Gram matrix being as well the correct one, we obtain the result. See [22]. □

We can apply the above formula to truncated characters, and we obtain:

**Proposition 8.25.** *The moments of truncated characters are given by the formula*
$$\int_G(u_{11}+\ldots+u_{ss})^k=Tr(W_{kN}G_{ks})$$
*and with $N\to\infty$ this quantity equals $(s/N)^k|D(k)|$.*

*Proof.* The first assertion follows from the following computation:
$$\begin{aligned}\int_G(u_{11}+\ldots+u_{ss})^k&=\sum_{i_1=1}^s\ldots\sum_{i_k=1}^s\int u_{i_1i_1}\ldots u_{i_ki_k}\\&=\sum_{\pi,\sigma\in D(k)}W_{kN}(\pi,\sigma)\sum_{i_1=1}^s\ldots\sum_{i_k=1}^s\delta_\pi(i)\delta_\sigma(i)\\&=\sum_{\pi,\sigma\in D(k)}W_{kN}(\pi,\sigma)G_{ks}(\sigma,\pi)\\&=Tr(W_{kN}G_{ks})\end{aligned}$$
The point now is that we have the following trivial estimates:
$$G_{kN}(\pi,\sigma):\begin{cases}=N^k&(\pi=\sigma)\\\leq N^{k-1}&(\pi\neq\sigma)\end{cases}$$

Thus with $N\to\infty$ we have the following estimate:

$$G_{kN}\sim N^k1$$

But this gives the folowing estimate, for our moment:

$$\begin{aligned}\int_G(u_{11}+\ldots+u_{ss})^k&=Tr(G_{kN}^{-1}G_{ks})\\&\sim Tr((N^k1)^{-1}G_{ks})\\&=N^{-k}Tr(G_{ks})\\&=N^{-k}s^k|D(k)|\end{aligned}$$

Thus, we have obtained the formula in the statement. See [22]. □

In order to process the above formula, we will need some more free probability theory. Following [123], [124], given a random variable $a$, we write:

$$\log F_a(\xi)=\sum_n k_n(a)\xi^n$$

$$R_a(\xi)=\sum_n\kappa_n(a)\xi^n$$

We call the coefficients $k_n(a),\kappa_n(a)$ cumulants, respectively free cumulants of $a$. With this notion in hand, we can define then more general quantities $k_\pi(a),\kappa_\pi(a)$, depending on partitions $\pi\in P(k)$, by multiplicativity over the blocks. We have then:

**Theorem 8.26.** *We have the classical and free moment-cumulant formulae*

$$M_k(a)=\sum_{\pi\in P(k)}k_\pi(a)$$

$$M_k(a)=\sum_{\pi\in NC(k)}\kappa_\pi(a)$$

*where $k_\pi(a),\kappa_\pi(a)$ are the generalized cumulants and free cumulants of $a$.*

*Proof.* This is standard, by using the formulae of $F_a,R_a$, or by doing some direct combinatorics, based on the Möbius inversion formula. See [115]. □

We can now improve our results about characters, as follows:

**Theorem 8.27.** *With $N\to\infty$, the laws of truncated characters are as follows:*

(1) *For $O_N$ we obtain the Gaussian law $g_t$.*
(2) *For $O_N^+$ we obtain the Wigner semicircle law $\gamma_t$.*
(3) *For $U_N$ we obtain the complex Gaussian law $G_t$.*
(4) *For $U_N^+$ we obtain the Voiculescu circular law $\Gamma_t$.*

*Proof.* With $s = [tN]$ and $N \to \infty$, the formula in Proposition 8.25 above gives:

$$\lim_{N\to\infty} \int_{G_N} \chi_t^k = \sum_{\pi \in D(k)} t^{|\pi|}$$

By using now the formulae in Theorem 8.26, this gives the results. Indeed:

(1) This is clear.

(2) This is clear as well.

(3) This follows by complexification.

(4) This follows by free complexification.

For details on all this, we refer to [22]. □

As an interesting consequence, related to [42], let us formulate as well:

**Theorem 8.28.** *The asymptotic laws of truncated characters for the liberation operations*

$$O_N \to O_N^+$$

$$U_N \to U_N^+$$

*are in Bercovici-Pata bijection, in the sense that the classical cumulants in the classical case equal the free cumulants in the free case.*

*Proof.* This follows indeed from the computations in the proof of Theorem 8.27. □

This result will be of great use for the liberation of more complicated compact Lie groups, because it provides us with a criterion for checking if our guesses are right. Let us discuss now the other easy quantum groups that we have. Regarding the half-liberations $O_N^*, U_N^*$ the situation is a bit complicated, but we have the following result, at $t = 1$:

**Proposition 8.29.** *The asymptotic laws of the main characters are as follows:*

(1) *For $O_N^*$ we obtain a symmetrized Rayleigh variable.*
(2) *For $U_N^*$ we obtain a complexification of this variable.*

*Proof.* The idea is to use a projective version trick. Indeed, assuming that $G = (G_N)$ is easy, coming from a category of pairings $D$, we have:

$$\lim_{N\to\infty} \int_{PG_N} (\chi\chi^*)^k = \#D((\circ\bullet)^k)$$

In our case, where $G_N = O_N^*, U_N^*$, we can therefore use Theorem 8.27 above at $t = 1$, and we are led to the conclusions in the statement. See [27], [28], [127]. □

The above result is of course something quite modest. We will be back to the quantum groups $O_N^*, U_N^*$ in section 16 below, with some better techniques for dealing with them. Next in our lineup, we have the bistochastic quantum groups. We have here:

**Proposition 8.30.** *For the bistochastic quantum groups*

$$B_N, B_N^+, C_N, C_N^+$$

*the asymptotic laws of truncated characters appear as modified versions of*

$$g_t, \gamma_t, G_t, \Gamma_t$$

*and the operations $O_N \to O_N^+$ and $U_N \to U_N^+$ are compatible with the Bercovici-Pata bijection.*

*Proof.* This follows indeed by using the same methods as for $O_N, O_N^+, U_N, U_N^+$, with the verification of the Bercovici-Pata bijection being elementary, and with the computation of the corresponding laws being routine as well. See [38], [28], [127]. □

Regarding now the twists, we have here the following general result:

**Proposition 8.31.** *The integration over $\bar{G}_N$ is given by the Weingarten type formula*

$$\int_{\bar{G}_N} u_{i_1j_1}\ldots u_{i_kj_k} = \sum_{\pi,\sigma\in D(k)} \bar{\delta}_\pi(i)\bar{\delta}_\sigma(j)W_{kN}(\pi,\sigma)$$

*where $W_{kN}$ is the Weingarten matrix of $G_N$.*

*Proof.* This follows from the general Weingarten formula from Theorem 8.24, with the corresponding Gram matrix being computed exactly as in the untwisted case. See [4]. □

As a consequence of the above result, we have another general result, as follows:

**Theorem 8.32.** *The Schur-Weyl twisting operation $G_N \leftrightarrow \bar{G}_N$ leaves invariant:*

1. *The law of the main character.*
2. *The coamenability property.*
3. *The asymptotic laws of truncated characters.*

*Proof.* This basically follows from Proposition 8.31, as follows:

(1) This is clear from the integration formula.

(2) This follows from (1), and from the Kesten criterion.

(3) This follows once again from the integration formula. □

To summarize, we have results for all the easy quantum groups introduced so far, and in each case we obtain Gaussian laws, and their versions.

There are many other probabilistic results that can be proved, by using the above technology, and we refer here to [26], [28], [29], [62], [63], [64], [65], [70], [71], [72], [73], [74], [80], [82], [83], [97], [98], [100], [104], [110], [111].

## 9. Quantum permutations

The quantum groups that we considered so far, namely $O_N, U_N$ and their liberations and twists, are of "continuous" nature. In order to have as well "discrete" examples, the idea will be that of looking at the corresponding quantum reflection groups. Let us start with a functional analytic description of the usual symmetric group:

**Proposition 9.1.** *Consider the symmetric group $S_N$.*

(1) *The standard coordinates $v_{ij} \in C(S_N)$, coming from the embedding $S_N \subset O_N$ given by the permutation matrices, are given by $v_{ij} = \chi(\sigma|\sigma(j) = i)$.*
(2) *The matrix $v = (v_{ij})$ is magic, in the sense that its entries are orthogonal projections, summing up to 1 on each row and each column.*
(3) *The algebra $C(S_N)$ is isomorphic to the universal commutative $C^*$-algebra generated by the entries of a $N \times N$ magic matrix.*

*Proof.* These results are all elementary, as follows:

(1) We recall that the canonical embedding $S_N \subset O_N$, coming from the standard permutation matrices, is given by $\sigma(e_j) = e_{\sigma(j)}$. Thus, we have $\sigma = \sum_j e_{\sigma(j)j}$, and it follows that the standard coordinates on $S_N \subset O_N$ are given by:

$$v_{ij}(\sigma) = \delta_{i,\sigma(j)}$$

(2) Any characteristic function $\chi \in \{0, 1\}$ being a projection in the operator algebra sense ($\chi^2 = \chi^* = \chi$), we have indeed a matrix of projections. As for the sum 1 condition on rows and columns, this is clear from the formula of the elements $v_{ij}$.

(3) Consider the universal algebra in the statement, namely:

$$A = C^*_{comm}\left((w_{ij})_{i,j=1,\ldots,N}\Big|w = \text{magic}\right)$$

We have a quotient map $A \to C(S_N)$, given by $w_{ij} \to v_{ij}$. On the other hand, by using the Gelfand theorem we can write $A = C(X)$, with $X$ being a compact space, and by using the coordinates $w_{ij}$ we have $X \subset O_N$, and then $X \subset S_N$. Thus we have as well a quotient map $C(S_N) \to A$ given by $v_{ij} \to w_{ij}$, and this gives (3). See Wang [140]. □

With the above result in hand, we can now formulate, following [140]:

**Theorem 9.2.** *The following is a Woronowicz algebra,*

$$C(S_N^+) = C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big|u = \text{magic}\right)$$

*and the underlying compact quantum group $S_N^+$ is called quantum permutation group.*

*Proof.* As a first remark, the algebra $C(S_N^+)$ is well-defined, because the magic condition forces $||u_{ij}|| \leq 1$, for any $C^*$-norm. Our claim now is that, by using the universal property of this algebra, we can define maps $\Delta, \varepsilon, S$. Consider indeed the following matrix:

$$U_{ij} = \sum_k u_{ik} \otimes u_{kj}$$

As a first observation, we have $U_{ij} = U_{ij}^*$. In fact the entries $U_{ij}$ are orthogonal projections, because we have as well:

$$\begin{aligned}U_{ij}^2 &= \sum_{kl} u_{ik}u_{il} \otimes u_{kj}u_{lj}\\ &= \sum_k u_{ik} \otimes u_{kj}\\ &= U_{ij}\end{aligned}$$

In order to prove now that the matrix $U = (U_{ij})$ is magic, it remains to verify that the sums on the rows and columns are 1. For the rows, this can be checked as follows:

$$\sum_j U_{ij} = \sum_{jk} u_{ik} \otimes u_{kj} = \sum_k u_{ik} \otimes 1 = 1 \otimes 1$$

For the columns the computation is similar, as follows:

$$\sum_i U_{ij} = \sum_{ik} u_{ik} \otimes u_{kj} = \sum_k 1 \otimes u_{kj} = 1 \otimes 1$$

Thus the matrix $U = (U_{ij})$ is magic indeed, and so we can define a comultiplication map by setting $\Delta(u_{ij}) = U_{ij}$. By using a similar reasoning, we can define as well a counit map by $\varepsilon(u_{ij}) = \delta_{ij}$, and an antipode map by $S(u_{ij}) = u_{ji}$. Thus the Woronowicz algebra axioms from section 2 are satisfied, and this finishes the proof. □

The terminology in the above result comes from the comparison with Proposition 9.1 (3), which tells us that we have an inclusion $S_N \subset S_N^+$, and that this inclusion is a liberation, in the sense that the classical version of $S_N^+$, obtained at the algebra level by dividing by the commutator ideal, is the usual symmetric group $S_N$. The terminology is further motivated by the following result, also from [140]:

**Proposition 9.3.** *The quantum permutation group $S_N^+$ acts on the set $X = \{1, \ldots, N\}$, the corresponding coaction map $\Phi : C(X) \to C(X) \otimes C(S_N^+)$ being given by:*

$$\Phi(\delta_i) = \sum_j \delta_j \otimes u_{ji}$$

*In fact, $S_N^+$ is the biggest compact quantum group acting on $X$, by leaving the counting measure invariant, in the sense that*

$$(tr \otimes id)\Phi = tr(.)1$$

*where $tr$ is the standard trace, given by $tr(\delta_i) = \frac{1}{N}, \forall i$.*

*Proof.* Our claim is that given a compact matrix quantum group $G$, the formula $\Phi(\delta_i)=\sum_j\delta_j\otimes u_{ji}$ defines a morphism of algebras, which is a coaction map, leaving the trace invariant, precisely when the matrix $u=(u_{ij})$ is a magic corepresentation of $C(G)$.

Indeed, let us first determine when $\Phi$ is multiplicative. We have:

$$\Phi(\delta_i)\Phi(\delta_k)=\sum_{jl}\delta_j\delta_l\otimes u_{ji}u_{lk}=\sum_j\delta_j\otimes u_{ji}u_{jk}$$

On the other hand, we have as well the following formula:

$$\Phi(\delta_i\delta_k)=\delta_{ik}\Phi(\delta_i)=\delta_{ik}\sum_j\delta_j\otimes u_{ji}$$

Thus, the multiplicativity of $\Phi$ is equivalent to the following conditions:

$$u_{ji}u_{jk}=\delta_{ik}u_{ji}\quad,\quad\forall i,j,k$$

Regarding now the unitality of $\Phi$, we have the following formula:

$$\begin{aligned}\Phi(1)&=\sum_i\Phi(\delta_i)\\&=\sum_{ij}\delta_j\otimes u_{ji}\\&=\sum_j\delta_j\otimes\left(\sum_i u_{ji}\right)\end{aligned}$$

Thus $\Phi$ is unital when the following conditions are satisfied:

$$\sum_i u_{ji}=1\quad,\quad\forall i$$

Finally, the fact that $\Phi$ is a $*$-morphism translates into:

$$u_{ij}=u_{ij}^*\quad,\quad\forall i,j$$

Summing up, in order for $\Phi(\delta_i)=\sum_j\delta_j\otimes u_{ji}$ to be a morphism of $C^*$-algebras, the elements $u_{ij}$ must be projections, summing up to 1 on each row of $u$. Regarding now the preservation of the trace condition, observe that we have:

$$(tr\otimes id)\Phi(\delta_i)=\frac{1}{N}\sum_j u_{ji}$$

Thus the trace is preserved precisely when the elements $u_{ij}$ sum up to 1 on each of the columns of $u$. We conclude from this that $\Phi(\delta_i)=\sum_j\delta_j\otimes u_{ji}$ is a morphism of $C^*$-algebras preserving the trace precisely when $u$ is magic, and since the coaction conditions on $\Phi$ are equivalent to the fact that $u$ must be a corepresentation, this finishes the proof of our claim. But this claim proves all the assertions in the statement. $\square$

As a perhaps quite surprising result now, also from [140], we have:

**Theorem 9.4.** *We have an embedding of compact quantum groups*

$$S_N \subset S_N^+$$

*given at the algebra level, $C(S_N^+) \to C(S_N)$, by the formula*

$$u_{ij} \to \chi\left(\sigma\middle|\sigma(j) = i\right)$$

*and this embedding is an isomorphism at $N \leq 3$, but not at $N \geq 4$, where $S_N^+$ is non-classical, infinite compact quantum group.*

*Proof.* The fact that we have indeed an embedding as above is clear from Proposition 9.1 and Theorem 9.2. Note that this follows as well from Proposition 9.3. Regarding now the second assertion, we can prove this in four steps, as follows:

Case $N = 2$. The result here is trivial, the $2 \times 2$ magic matrices being by definition as follows, with $p$ being a projection:

$$U = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$$

Indeed, this shows that the entries of a $2 \times 2$ magic matrix must pairwise commute, and so the algebra $C(S_2^+)$ follows to be commutative, which gives the result.

Case $N = 3$. This is more tricky, and we present here a simple, recent proof, from [105]. By using the same abstract argument as in the $N = 2$ case, and by permuting rows and columns, it is enough to check that $u_{11}, u_{22}$ commute. But this follows from:

$$\begin{aligned} u_{11}u_{22} &= u_{11}u_{22}(u_{11} + u_{12} + u_{13}) \\ &= u_{11}u_{22}u_{11} + u_{11}u_{22}u_{13} \\ &= u_{11}u_{22}u_{11} + u_{11}(1 - u_{21} - u_{23})u_{13} \\ &= u_{11}u_{22}u_{11} \end{aligned}$$

Indeed, by applying the involution to this formula, we obtain from this that we have as well $u_{22}u_{11} = u_{11}u_{22}u_{11}$. Thus we get $u_{11}u_{22} = u_{22}u_{11}$, as desired.

Case $N = 4$. In order to prove our various claims about $S_4^+$, consider the following matrix, with $p, q$ being projections, on some infinite dimensional Hilbert space:

$$U = \begin{pmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{pmatrix}$$

This matrix is magic, and if we choose $p, q$ as for the algebra $< p, q >$ to be not commutative, and infinite dimensional, we conclude that $C(S_4^+)$ is not commutative and infinite dimensional as well, and in particular is not isomorphic to $C(S_4)$.

Case $N \geq 5$. Here we can use the standard embedding $S_4^+ \subset S_N^+$, obtained at the level of the corresponding magic matrices in the following way:

$$u \to \begin{pmatrix}u&0\\0&1_{N-4}\end{pmatrix}$$

Indeed, with this embedding in hand, the fact that $S_4^+$ is a non-classical, infinite compact quantum group implies that $S_N^+$ with $N \geq 5$ has these two properties as well. $\square$

At the representation theory level now, we have the following result, from [23]:

**Theorem 9.5.** *For the quantum groups $S_N, S_N^+$, the intertwining spaces for the tensor powers of the fundamental corepresentation $u = (u_{ij})$ are given by*

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(T_\pi\Big|\pi \in D(k,l)\right)$$

*with $D = P, NC$. In other words, $S_N, S_N^+$ are easy, coming from the categories $P, NC$.*

*Proof.* We use the Tannakian duality results from section 4 above:

(1) $S_N^+$. According to Theorem 9.2, the algebra $C(S_N^+)$ appears as follows:

$$C(S_N^+) = C(O_N^+)\Big/\Big\langle u = \text{magic}\Big\rangle$$

Consider the one-block partition $\mu \in P(2,1)$. The linear map associated to it is:

$$T_\mu(e_i \otimes e_j) = \delta_{ij} e_i$$

We have $T_\mu = (\delta_{ijk})_{i,jk}$, and we obtain the following formula:

$$(T_\mu u^{\otimes 2})_{i,jk} = \sum_{lm} (T_\mu)_{i,lm} (u^{\otimes 2})_{lm,jk} = u_{ij} u_{ik}$$

On the hand, we have as well the following formula:

$$(uT_\mu)_{i,jk} = \sum_l u_{il} (T_\mu)_{l,jk} = \delta_{jk} u_{ij}$$

Thus, the relation defining $S_N^+ \subset O_N^+$ reformulates as follows:

$$T_\mu \in Hom(u^{\otimes 2}, u) \iff u_{ij} u_{ik} = \delta_{jk} u_{ij}, \forall i,j,k$$

The condition on the right being equivalent to the magic condition, we obtain:

$$C(S_N^+) = C(O_N^+)\Big/\Big\langle T_\mu \in Hom(u^{\otimes 2}, u)\Big\rangle$$

By using now the general theory from section 7, we conclude that the quantum group $S_N^+$ is indeed easy, with the corresponding category of partitions being $D =< \mu >$. But this latter category is $NC$, as one can see by "chopping" arbitrary noncrossing partitions into $\mu$-shaped components. Thus, we are led to the conclusion in the statement.

(2) $S_N$. Here the first part of the proof is similar, leading to the following formula:

$$C(S_N)=C(O_N)\Big/\Big\langle T_\mu\in Hom(u^{\otimes 2},u)\Big\rangle$$

But this shows that $S_N$ is easy, the corresponding category of partitions being:

$$D=<\mu,P_2>=<NC,P_2>=P$$

Alternatively, this latter formula follows directly for the result for $S_N^+$ proved above, via $S_N=S_N^+\cap O_N$, and the functoriality results explained in section 7. □

As a technical comment, there might seem to be a bit of a clash between the above results for $S_N,S_N^+$ at $N=2,3$, where we have $S_N=S_N^+$. However, there is no clash, because the implementation of the partitions is not faithful. In order to discuss now the representation theory of $S_N^+$, we will need precisely linear independence results for the vectors $\xi_\pi$ associated to the partitions $\pi\in NC$. Let us start with:

**Proposition 9.6.** *We have a bijection $NC(k)\simeq NC_2(2k)$, constructed as follows:*

1. *The application $NC(k)\to NC_2(2k)$ is the "fattening" one, obtained by doubling all the legs, and doubling all the strings as well.*
2. *Its inverse $NC_2(2k)\to NC(k)$ is the "shrinking" application, obtained by collapsing pairs of consecutive neighbors.*

*Proof.* The fact that the two operations in the statement are indeed inverse to each other is clear, by computing the corresponding two compositions, with the remark that the construction of the fattening operation requires the partitions to be noncrossing. □

Next in line, we have the following key result:

**Theorem 9.7.** *Consider the Temperley-Lieb algebra of index $N\geq 4$, defined as*

$$TL_N(k)=span(NC_2(k,k))$$

*with product given by the rule $\bigcirc=N$, when concatenating.*

1. *We have a representation $i:TL_N(k)\to B((\mathbb{C}^N)^{\otimes k})$, given by $\pi\to T_\pi$.*
2. *$Tr(T_\pi)=N^{loops(<\pi>)}$, where $\pi\to<\pi>$ is the closing operation.*
3. *The linear form $\tau=Tr\circ i:TL_N(k)\to\mathbb{C}$ is a faithful positive trace.*
4. *The representation $i:TL_N(k)\to B((\mathbb{C}^N)^{\otimes k})$ is faithful.*

*In particular, the vectors $\{\xi_\pi|\pi\in NC(k)\}\subset(\mathbb{C}^N)^{\otimes k}$ are linearly independent.*

*Proof.* All this is quite standard, but advanced, the idea being as follows:

(1) This is clear from the categorical properties of $\pi\to T_\pi$.

(2) This follows indeed from the following computation:

$$\begin{aligned}
Tr(T_\pi) &= \sum_{i_1\ldots i_k}\delta_\pi\begin{pmatrix}i_1\ldots i_k\\ i_1\ldots i_k\end{pmatrix}\\
&= \#\left\{i_1,\ldots,i_k\in\{1,\ldots,N\}\Big|\ker\begin{pmatrix}i_1\ldots i_k\\ i_1\ldots i_k\end{pmatrix}\geq\pi\right\}\\
&= N^{loops(<\pi>)}
\end{aligned}$$

(3) The traciality of $\tau$ is clear. Regarding now the faithfulness, this is something well-known, and we refer here to Jones' paper [90].

(4) This follows from (3) above, via a standard positivity argument. As for the last assertion, this follows from (4), by fattening the partitions. □

For our purposes, the final conclusion of Theorem 9.7 is exactly what we need. The problem, however, is that the proof of this fact remains quite heavy, based on [90]. We will be back to this a bit later, with the outline of a few alternative arguments. We can now work out the representation theory of $S_N^+$, as follows:

**Theorem 9.8.** *The quantum groups $S_N^+$ with $N\geq 4$ have the following properties:*

(1) *The moments of the main character are the Catalan numbers:*

$$\int_{S_N^+}\chi^k = C_k$$

(2) *The fusion rules for representations are as follows, exactly as for $SO_3$:*

$$r_k\otimes r_l = r_{|k-l|}+r_{|k-l|+1}+\ldots+r_{k+l}$$

(3) *The dimensions of the irreducible representations are given by*

$$\dim(r_k)=\frac{q^{k+1}-q^{-k}}{q-1}$$

*where $q,q^{-1}$ are the roots of $X^2-(N-2)X+1=0$.*

*Proof.* The proof, from [2], based on Theorem 9.7, goes as follows:

(1) We have indeed the following computation, coming from the $SU_2$ computations from section 5, and from Theorem 9.5, Proposition 9.6 and Theorem 9.7:

$$\begin{aligned}
\int_{S_N^+}\chi^k &= \dim(Fix(u^{\otimes k}))\\
&= |NC(k)|\\
&= |NC_2(2k)|\\
&= C_k
\end{aligned}$$

(2) This is standard, by using the formula in (1), and the known theory of $SO_3$. Let $A = span(\chi_k|k \in \mathbb{N})$ be the algebra of characters of $SO_3$. We can define a morphism as follows, where $f$ is the character of the fundamental representation of $S_N^+$:

$$\Psi : A \to C(S_N^+)$$

$$\chi_1 \to f - 1$$

The elements $f_k = \Psi(\chi_k)$ verify then the following formulae:

$$f_k f_l = f_{|k-l|} + f_{|k-l|+1} + \ldots + f_{k+l}$$

We prove now by recurrence that each $f_k$ is the character of an irreducible corepresentation $r_k$ of $C(S_N^+)$, non-equivalent to $r_0, \ldots, r_{k-1}$. At $k = 0, 1$ this is clear, so assume that the result holds at $k - 1$. By integrating characters we have, exactly as for $SO_3$:

$$r_{k-2}, r_{k-1} \subset r_{k-1} \otimes r_1$$

Thus there exists a certain corepresentation $r_k$ such that:

$$r_{k-1} \otimes r_1 = r_{k-2} + r_{k-1} + r_k$$

Once again by integrating characters, we conclude that $r_k$ is irreducible, and non-equivalent to $r_1, \ldots, r_{k-1}$, as for $SO_3$, which proves our claim. Finally, since any irreducible representation of $S_N^+$ must appear in some tensor power of $u$, and we have a formula for decomposing each $u^{\otimes k}$ into sums of representations $r_l$, we conclude that these representations $r_l$ are all the irreducible representations of $S_N^+$.

(3) From the Clebsch-Gordan rules we have, in particular:

$$r_k r_1 = r_{k-1} + r_k + r_{k+1}$$

We are therefore led to a recurrence, and the initial data being $\dim(r_0) = 1$ and $\dim(r_1) = N - 1 = q + 1 + q^{-1}$, we are led to the following formula:

$$\dim(r_k) = q^k + q^{k-1} + \ldots + q^{1-k} + q^{-k}$$

In more compact form, this gives the formula in the statement. □

The above result is quite surprising, and raises a massive number of questions. We would like to better understand the relation with $SO_3$, and more generally see what happens at values $N = n^2$ with $n \geq 2$, and also compute the law of $\chi$, and so on.

We will come up with answers to all these questions, but we will do this slowly. One way of understanding the relation with $SO_3$ comes from noncommutative geometry considerations. We recall that, according to the general theory from section 1, each finite dimensional $C^*$-algebra $A$ can be written as $A = C(F)$, with $F$ being a "finite quantum space". We make the convention that each such space $F$ is endowed with its counting measure, corresponding to the canonical trace $tr : A \subset \mathcal{L}(A) \to \mathbb{C}$.

Let us study the quantum group actions $G \curvearrowright F$. We denote by $\mu, \eta$ the multiplication and unit map of the algebra $C(F)$. Following [2], [140], we first have:

**Proposition 9.9.** *Consider a linear map $\Phi : C(F) \to C(F) \otimes C(G)$, written as*

$$\Phi(e_i) = \sum_j e_j \otimes u_{ji}$$

*with $\{e_i\}$ being a linear space basis of $C(F)$, orthonormal with respect to $tr$.*

1. *$\Phi$ is a linear space coaction $\iff$ $u$ is a corepresentation.*
2. *$\Phi$ is multiplicative $\iff$ $\mu \in Hom(u^{\otimes 2}, u)$.*
3. *$\Phi$ is unital $\iff$ $\eta \in Hom(1, u)$.*
4. *$\Phi$ leaves invariant $tr$ $\iff$ $\eta \in Hom(1, u^*)$.*
5. *If these conditions hold, $\Phi$ is involutive $\iff$ $u$ is unitary.*

*Proof.* This is a bit similar to the proof of Proposition 9.3 above, as follows:

(1) There are two axioms to be processed here. First, we have:

$$\begin{aligned}
(id \otimes \Delta)\Phi = (\Phi \otimes id)\Phi \quad &\iff \quad \sum_j e_j \otimes \Delta(u_{ji}) = \sum_k \Phi(e_k) \otimes u_{ki} \\
&\iff \quad \sum_j e_j \otimes \Delta(u_{ji}) = \sum_{jk} e_j \otimes u_{jk} \otimes u_{ki} \\
&\iff \quad \Delta(u_{ji}) = \sum_k u_{jk} \otimes u_{ki}
\end{aligned}$$

As for the axiom involving the counit, here we have as well, as desired:

$$\begin{aligned}
(id \otimes \varepsilon)\Phi = id \quad &\iff \quad \sum_j \varepsilon(u_{ji}) e_j = e_i \\
&\iff \quad \varepsilon(u_{ji}) = \delta_{ji}
\end{aligned}$$

(2) We have the following formula:

$$\begin{aligned}
\Phi(e_i) &= \sum_j e_j \otimes u_{ji} \\
&= \left( \sum_{ij} e_{ji} \otimes u_{ji} \right) (e_i \otimes 1) \\
&= u(e_i \otimes 1)
\end{aligned}$$

By using this formula, we obtain the following identity:

$$\begin{aligned}
\Phi(e_i e_k) &= u(e_i e_k \otimes 1) \\
&= u(\mu \otimes id)(e_i \otimes e_k \otimes 1)
\end{aligned}$$

On the other hand, we have as well the following identity, as desired:

$$\begin{aligned}\Phi(e_i)\Phi(e_k) &= \sum_{jl} e_je_l\otimes u_{ji}u_{lk}\\ &= (\mu\otimes id)\sum_{jl}e_j\otimes e_l\otimes u_{ji}u_{lk}\\ &= (\mu\otimes id)\left(\sum_{ijkl}e_{ji}\otimes e_{lk}\otimes u_{ji}u_{lk}\right)(e_i\otimes e_k\otimes 1)\\ &= (\mu\otimes id)u^{\otimes 2}(e_i\otimes e_k\otimes 1)\end{aligned}$$

(3) The formula $\Phi(e_i)=u(e_i\otimes 1)$ found above gives by linearity $\Phi(1)=u(1\otimes 1)$, which shows that $\Phi$ is unital precisely when $u(1\otimes 1)=1\otimes 1$, as desired.

(4) This follows from the following computation, by applying the involution:

$$\begin{aligned}(tr\otimes id)\Phi(e_i)=tr(e_i)1 &\iff \sum_j tr(e_j)u_{ji}=tr(e_i)1\\ &\iff \sum_j u_{ji}^*1_j=1_i\\ &\iff (u^*1)_i=1_i\\ &\iff u^*1=1\end{aligned}$$

(5) Assuming that (1-4) are satisfied, and that $\Phi$ is involutive, we have:

$$\begin{aligned}(u^*u)_{ik} &= \sum_l u_{li}^*u_{lk}\\ &= \sum_{jl}tr(e_j^*e_l)u_{ji}^*u_{lk}\\ &= (tr\otimes id)\sum_{jl}e_j^*e_l\otimes u_{ji}^*u_{lk}\\ &= (tr\otimes id)(\Phi(e_i)^*\Phi(e_k))\\ &= (tr\otimes id)\Phi(e_i^*e_k)\\ &= tr(e_i^*e_k)1\\ &= \delta_{ik}\end{aligned}$$

Thus $u^*u=1$, and since we know from (1) that $u$ is a corepresentation, it follows that $u$ is unitary. The proof of the converse is standard too, by using similar tricks. □

Following now [2], [140], we have the following result, extending the basic theory of $S_N^+$ to the present finite noncommutative space setting:

**Theorem 9.10.** *Given a finite quantum space $F$, there is a universal compact quantum group $S_F^+$ acting on $F$, leaving the counting measure invariant. We have*

$$C(S_F^+)=C(U_N^+)\Big/\Big/\Big\langle\mu\in Hom(u^{\otimes 2},u),\eta\in Fix(u)\Big\rangle$$

*where $N=|F|$ and where $\mu,\eta$ are the multiplication and unit maps of $C(F)$. For $F=\{1,\ldots,N\}$ we have $S_F^+=S_N^+$. Also, for the space $F=M_2$ we have $S_F^+=SO_3$.*

*Proof.* This result is from [2], the idea being as follows:

(1) This follows from Proposition 9.9 above, by using the standard fact that the complex conjugate of a corepresentation is a corepresentation too.

(2) Regarding now the main example, for $F=\{1,\ldots,N\}$ we obtain indeed the quantum permutation group $S_N^+$, due to the abstract result in Proposition 9.3 above.

(3) In order to do now the computation for $F=M_2$, we use some standard facts about $SU_2,SO_3$. We have an action by conjugation $SU_2\curvearrowright M_2(\mathbb{C})$, and this action produces, via the canonical quotient map $SU_2\to SO_3$, an action $SO_3\curvearrowright M_2(\mathbb{C})$.

On the other hand, it is routine to check, by using arguments like those in the proof of Theorem 9.4 at $N=2,3$, that any action $G\curvearrowright M_2(\mathbb{C})$ must come from a classical group. We conclude that the action $SO_3\curvearrowright M_2(\mathbb{C})$ is universal, as claimed. □

Regarding now the representation theory of these generalized quantum permutation groups $S_F^+$, the result here, from [2], is very similar to the one for $S_N^+$, as follows:

**Theorem 9.11.** *The quantum groups $S_F^+$ have the following properties:*

1. *The associated Tannakian categories are $TL_N$, with $N=|F|$.*
2. *The main character follows the Marchenko-Pastur law $\pi_1$, when $N\geq 4$.*
3. *The fusion rules for $S_F^+$ with $|F|\geq 4$ are the same as for $SO_3$.*

*Proof.* Once again this result is from [2], the idea being as follows:

(1) Our first claim is that the fundamental representation is equivalent to its adjoint, $u\sim\bar{u}$. Indeed, let us go back to the coaction formula from Proposition 9.9:

$$\Phi(e_i)=\sum_j e_j\otimes u_{ji}$$

We can pick our orthogonal basis $\{e_i\}$ to be the stadard multimatrix basis of $C(F)$, so that we have $e_i^*=e_{i^*}$, for a certain involution $i\to i^*$ on the index set. With this convention made, by conjugating the above formula of $\Phi(e_i)$, we obtain:

$$\Phi(e_{i^*})=\sum_j e_{j^*}\otimes u_{ji}^*$$

Now by interchanging $i \leftrightarrow i^*$ and $j \leftrightarrow j^*$, this latter formula reads:

$$\Phi(e_i) = \sum_j e_j \otimes u^*_{j^*i^*}$$

We therefore conclude, by comparing with the original formula, that we have:

$$u^*_{ji} = u_{j^*i^*}$$

But this shows that we have an equivalence $u \sim \bar{u}$, as claimed. Now with this result in hand, the proof goes as for the proof for $S_N^+$. To be more precise, the result follows from the fact that the multiplication and unit of any complex algebra, and in particular of $C(F)$, can be modelled by the following two diagrams:

$$m = |\cup| \quad , \quad u = \cap$$

Indeed, this is certainly true algebrically, and this is something well-known. As in what regards the $*$-structure, things here are fine too, because our choice for the trace leads to the following formula, which must be satisfied as well:

$$\mu\mu^* = N \cdot id$$

But the above diagrams $m, u$ generate the Temperley-Lieb algebra $TL_N$, as stated.

(2) The proof here is exactly as for $S_N^+$, by using moments. To be more precise, according to (1) these moments are the Catalan numbers, which are the moments of $\pi_1$.

(3) Once again same proof as for $S_N^+$, by using the fact that the moments of $\chi$ are the Catalan numbers, which naturally leads to the Clebsch-Gordan rules. □

It is quite clear now that our present formalism, and the above results, provide altogether a good and conceptual explanation for our $SO_3$ result regarding $S_N^+$. To be more precise, we can merge and reformulate the above results in the following way:

**Theorem 9.12.** *The quantun groups $S_F^+$ have the following properties:*

1. *For $F = \{1, \ldots, N\}$ we have $S_F^+ = S_N^+$.*
2. *For the space $F = M_N$ we have $S_F^+ = PO_N^+ = PU_N^+$.*
3. *In particular, for the space $F = M_2$ we have $S_F^+ = SO_3$.*
4. *The fusion rules for $S_F^+$ with $|F| \geq 4$ are independent of $F$.*
5. *Thus, the fusion rules for $S_F^+$ with $|F| \geq 4$ are the same as for $SO_3$.*

*Proof.* This is basically a compact form of what has been said above, with a new result added, and with some technicalities left aside:

(1) This is something that we know from Theorem 9.10.

(2) This is new, the idea being as follows. First of all, we know from section 4 above that the inclusion $PO_N^+ \subset PU_N^+$ is an isomorphism, with this coming from the free complexification formula $\widetilde{O}_N^+ = U_N^+$, but we will actually reprove this result. Consider indeed

the standard vector space action $U_N^+ \curvearrowright \mathbb{C}^N$, and then its adjoint action $PU_N^+ \curvearrowright M_N(\mathbb{C})$. By universality of $S_{M_N}^+$, we have inclusions as follows:

$$PO_N^+ \subset PU_N^+ \subset S_{M_N}^+$$

On the other hand, the main character of $O_N^+$ with $N \geq 2$ being semicircular, the main character of $PO_N^+$ must be Marchenko-Pastur. Thus the inclusion $PO_N^+ \subset S_{M_N}^+$ has the property that it keeps fixed the law of main character, and by Peter-Weyl theory we conclude that this inclusion must be an isomorphism, as desired.

(3) This is something that we know from Theorem 9.10, and that can be deduced as well from (2), by using the formula $PO_2^+ = SO_3$, which is something elementary.

(4) This is something that we know from Theorem 9.11.

(5) This follows from (3,4), as already pointed out in Theorem 9.11. □

All this is certainly quite conceptual, but perhaps a bit too abstract. At $N = 4$ we can formulate a more concrete result on the subject, by using the following construction:

**Definition 9.13.** *$C(SO_3^{-1})$ is the universal $C^*$-algebra generated by the entries of a $3\times 3$ orthogonal matrix $a = (a_{ij})$, with the following relations:*

1. *Skew-commutation: $a_{ij}a_{kl} = \pm a_{kl}a_{ij}$, with sign $+$ if $i \neq k, j \neq l$, and $-$ otherwise.*
2. *Twisted determinant condition: $\Sigma_{\sigma\in S_3} a_{1\sigma(1)}a_{2\sigma(2)}a_{3\sigma(3)} = 1$.*

Observe the similarity with the twisting constructions from section 7. However, $SO_3$ being not easy, we are not exactly in the Schur-Weyl twisting framework from there.

Our first task would be to prove that $C(SO_3^{-1})$ is a Woronowicz algebra. This is of course possible, by doing some computations, but we will not need to do these computations, because the result follows from the following theorem, from [13]:

**Theorem 9.14.** *We have an isomorphism of compact quantum groups*

$$S_4^+ = SO_3^{-1}$$

*given by the Fourier transform over the Klein group $K = \mathbb{Z}_2 \times \mathbb{Z}_2$.*

*Proof.* Consider indeed the matrix $a^+ = diag(1, a)$, corresponding to the action of $SO_3^{-1}$ on $\mathbb{C}^4$, and apply to it the Fourier transform over the Klein group $K = \mathbb{Z}_2 \times \mathbb{Z}_2$:

$$u = \frac{1}{4}\begin{pmatrix}1&1&1&1\\1&-1&-1&1\\1&-1&1&-1\\1&1&-1&-1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&a_{11}&a_{12}&a_{13}\\0&a_{21}&a_{22}&a_{23}\\0&a_{31}&a_{32}&a_{33}\end{pmatrix}\begin{pmatrix}1&1&1&1\\1&-1&-1&1\\1&-1&1&-1\\1&1&-1&-1\end{pmatrix}$$

It is routine to check that this matrix is magic, and vice versa, i.e. that the Fourier transform over $K$ converts the relations in Definition 9.13 into the magic relations in Definition 8.1. Thus, we obtain the identification from the statement. □

Yet another extension of Theorem 9.8, which is however quite technical, comes by looking at the general case $N = n^2$, with $n \geq 2$. It is possible indeed to complement Theorem 9.12 above with a general twisting result of the following type:

$$G^+(\widehat{F}_\sigma) = G^+(\widehat{F})^\sigma$$

To be more precise, this formula is valid indeed, for any finite group $F$ and any 2-cocycle $\sigma$ on it. In the case $F = \mathbb{Z}_n^2$ with Fourier cocycle on it, this leads to the conclusion that $PO_n^+$ appears as a cocycle twist of $S_{n^2}^+$. See [19]. In relation with this, we have:

**Proposition 9.15.** *The Gram matrices of $NC_2(2k), NC(k)$ are related by the formula*

$$G_{2k,n}(\pi,\sigma) = n^k(\Delta_{kn}^{-1} G_{k,n^2} \Delta_{kn}^{-1})(\pi',\sigma')$$

*where $\pi \to \pi'$ is the shrinking operation, and $\Delta_{kn}$ is the diagonal of $G_{kn}$.*

*Proof.* In the context of Proposition 9.6 above, it is elementary to see that we have:

$$|\pi \vee \sigma| = k + 2|\pi' \vee \sigma'| - |\pi'| - |\sigma'|$$

We therefore have the following formula, valid for any $n \in \mathbb{N}$:

$$n^{|\pi \vee \sigma|} = n^{k+2|\pi' \vee \sigma'| - |\pi'| - |\sigma'|}$$

Thus, we obtain the formula in the statement. □

We have the following interesting probabilistic fact, from [19] as well:

**Theorem 9.16.** *The following families of variables have the same joint law,*

(1) $\{u_{ij}^2\} \in C(O_n^+)$,
(2) $\{X_{ij} = \frac{1}{n}\sum_{ab} p_{ia,jb}\} \in C(S_{n^2}^+)$,

*where $u = (u_{ij})$ and $p = (p_{ia,jb})$ are the corresponding fundamental corepresentations.*

*Proof.* This result can be obtained via twisting methods. An alternative approach is by using the Weingarten formula for our two quantum groups, and the shrinking operation $\pi \to \pi'$. Indeed, we obtain the following moment formulae:

$$\begin{aligned}
\int_{O_n^+} u_{ij}^{2k} &= \sum_{\pi,\sigma \in NC_2(2k)} W_{2k,n}(\pi,\sigma) \\
\int_{S_{n^2}^+} X_{ij}^k &= \sum_{\pi,\sigma \in NC_2(2k)} n^{|\pi'|+|\sigma'|-k} W_{k,n^2}(\pi',\sigma')
\end{aligned}$$

According to Proposition 9.15 the summands coincide, and so the moments are equal, as desired. The proof in general, dealing with joint moments, is similar. □

The above result is quite interesting, because it makes a connection between free hyperspherical and free hypergeometric laws. We refer here to [19], [25].

Let us go back now to our main result so far, namely Theorem 9.8, and further build on that. Following [12], we have the following result:

**Theorem 9.17.** *The spectral measure of the main character of $S_N^+$ with $N\geq 4$ is the Marchenko-Pastur law of parameter* 1*, having the following density:*

$$\pi_1=\frac{1}{2\pi}\sqrt{4x^{-1}-1}dx$$

*Also, $S_4^+$ is coamenable, and $S_N^+$ with $N\geq 5$ is not coamenable.*

*Proof.* Here the first assertion follows from the following formula, which can be established by doing some calculus, and more specifically by setting $x=4\sin^2 t$:

$$\frac{1}{2\pi}\int_0^4\sqrt{1-4x^{-1}}x^kdx=C_k$$

As for the second assertion, this follows from this, and from the Kesten criterion. □

Our next purpose will be that of understanding, probabilistically speaking, the liberation operation $S_N\to S_N^+$. In what regards $S_N$, we have the following basic result:

**Theorem 9.18.** *Consider the symmetric group $S_N$, regarded as a compact group of matrices, $S_N\subset O_N$, via the standard permutation matrices.*

(1) *The main character $\chi\in C(S_N)$, defined as usual as $\chi=\sum_i u_{ii}$, counts the number of fixed points, $\chi(\sigma)=\#\{i|\sigma(i)=i\}$.*
(2) *The probability for a permutation $\sigma\in S_N$ to be a derangement, meaning to have no fixed points at all, becomes, with $N\to\infty$, equal to $1/e$.*
(3) *The law of the main character $\chi\in C(S_N)$ becomes, with $N\to\infty$, a Poisson law of parameter* 1*, with respect to the counting measure.*

*Proof.* This is something very classical, and beautiful, as follows:

(1) We have indeed the following computation:

$$\chi(\sigma)=\sum_i u_{ii}(\sigma)=\sum_i\delta_{\sigma(i)i}=\#\left\{i\Big|\sigma(i)=i\right\}$$

(2) This is best viewed by using the inclusion-exclusion principle. Let us set:

$$S_N^{i_1\ldots i_k}=\left\{\sigma\in S_N\Big|\sigma(i_1)=i_1,\ldots,\sigma(i_k)=i_k\right\}$$

By using the inclusion-exclusion principle, we have:

$$\begin{aligned}\mathbb{P}(\chi=0)&=\frac{1}{N!}|(S_1\cup\ldots\cup S_N)^c|\\&=\frac{1}{N!}\left(|S_N|-\sum_i|S_N^i|+\sum_{i<j}|S_N^{ij}|-\ldots+(-1)^N\sum_{i_1<\ldots<i_N}|S_N^{i_1\ldots i_N}|\right)\end{aligned}$$

For any $i_1 < \ldots < i_k$, we have $|S_N^{i_1\ldots i_k}| = (N-k)!$, and we obtain:

$$\begin{aligned}\mathbb{P}(\chi = 0) &= \frac{1}{N!}\sum_{k=0}^N (-1)^k \binom{N}{k}(N-k)! \\ &= 1 - \frac{1}{1!} + \frac{1}{2!} - \ldots + (-1)^{N-1}\frac{1}{(N-1)!} + (-1)^N \frac{1}{N!}\end{aligned}$$

Since on the right we have the expansion of $\frac{1}{e}$, we conclude that we have:

$$\lim_{N\to\infty} \mathbb{P}(\chi = 0) = \frac{1}{e}$$

(3) This follows by generalizing the computation in (2). To be more precise, a similar application of the inclusion-exclusion principle gives the following formula:

$$\lim_{N\to\infty} \mathbb{P}(\chi = k) = \frac{1}{k!e}$$

Thus, we obtain in the limit a Poisson law, as stated. □

In order to talk about free analogues of this, we will need some theory:

**Theorem 9.19.** *The following Poisson type limits converge, for any $t > 0$,*

$$p_t = \lim_{n\to\infty}\left(\left(1 - \frac{t}{n}\right)\delta_0 + \frac{t}{n}\delta_1\right)^{*n}$$

$$\pi_t = \lim_{n\to\infty}\left(\left(1 - \frac{t}{n}\right)\delta_0 + \frac{t}{n}\delta_1\right)^{\boxplus n}$$

*the limiting measures being the Poisson law $p_t$, and the Marchenko-Pastur law $\pi_t$,*

$$p_t = \frac{1}{e^t}\sum_{k=0}^\infty \frac{t^k \delta_k}{k!}$$

$$\pi_t = \max(1-t, 0)\delta_0 + \frac{\sqrt{4t - (x-1-t)^2}}{2\pi x}\,dx$$

*whose moments are given by the following formula*

$$M_k(p_t) = \sum_{\pi \in D(k)} t^{|\pi|}$$

*with $D = P, NC$. The Marchenko-Pastur measure $\pi_t$ is also called free Poisson law.*

*Proof.* This is something standard, which follows by using either $\log F, R$ and calculus, or classical and free cumulants, and combinatorics. The point indeed is that the limiting measures must be those having classical and free cumulants $t, t, t, \ldots$ But this gives all the assertions, the density computations being standard. See [109], [115], [136], [145]. □

We can now formulate a conceptual result about $S_N \to S_N^+$, as follows:

**Theorem 9.20.** *The law of the main character $\chi_u$ is as follows:*

(1) *For $S_N$ with $N \to \infty$ we obtain a Poisson law $p_1$.*
(2) *For $S_N^+$ with $N \geq 4$ we obtain a free Poisson law $\pi_1$.*

*In addition, these laws are related by the Bercovici-Pata correspondence.*

*Proof.* This follows indeed from the computations that we have, from Theorem 9.17 and Theorem 9.18, by using the various theoretical results from Theorem 9.19. □

As in the continuous case, our purpose now will be that of extending this result to the truncated characters. In order to discuss the classical case, we first have:

**Proposition 9.21.** *Consider the symmetric group $S_N$, together with its standard matrix coordinates $u_{ij} = \chi(\sigma \in S_N | \sigma(j) = i)$. We have the formula*

$$\int_{S_N} u_{i_1 j_1} \ldots u_{i_k j_k} = \begin{cases} \frac{(N - |\ker i|)!}{N!} & \text{if } \ker i = \ker j \\ 0 & \text{otherwise} \end{cases}$$

*where* $\ker i$ *denotes as usual the partition of* $\{1, \ldots, k\}$ *whose blocks collect the equal indices of* $i$*, and where* $|.|$ *denotes the number of blocks.*

*Proof.* According to the definition of $u_{ij}$, the integrals in the statement are given by:

$$\int_{S_N} u_{i_1 j_1} \ldots u_{i_k j_k} = \frac{1}{N!} \# \left\{ \sigma \in S_N \middle| \sigma(j_1) = i_1, \ldots, \sigma(j_k) = i_k \right\}$$

The existence of $\sigma \in S_N$ as above requires $i_m = i_n \iff j_m = j_n$. Thus, the integral vanishes when $\ker i \neq \ker j$. As for the case $\ker i = \ker j$, if we denote by $b \in \{1, \ldots, k\}$ the number of blocks of this partition, we have $N - b$ points to be sent bijectively to $N - b$ points, and so $(N - b)!$ solutions, and the integral is $\frac{(N-b)!}{N!}$, as claimed. □

We can now compute the laws of truncated characters, and we obtain:

**Proposition 9.22.** *For the symmetric group $S_N \subset O_N$, regarded as a compact group of matrices, $S_N \subset O_N$, via the standard permutation matrices, the truncated character*

$$\chi_t = \sum_{i=1}^{[tN]} u_{ii}$$

*counts the number of fixed points among $\{1, \ldots, [tN]\}$, and its law with respect to the counting measure becomes, with $N \to \infty$, a Poisson law of parameter $t$.*

*Proof.* With $S_{k,b}$ being the Stirling numbers, we have:

$$\begin{aligned}\int_{S_N}\chi_t^k &= \sum_{i_1\ldots i_k=1}^{[tN]}\int_{S_N}u_{i_1i_1}\ldots u_{i_ki_k}\\ &= \sum_{\pi\in P_k}\frac{[tN]!}{([tN]-|\pi|!)}\cdot\frac{(N-|\pi|!)}{N!}\\ &= \sum_{b=1}^{[tN]}\frac{[tN]!}{([tN]-b)!}\cdot\frac{(N-b)!}{N!}\cdot S_{k,b}\end{aligned}$$

In particular with $N\to\infty$ we obtain the following formula:

$$\lim_{N\to\infty}\int_{S_N}\chi_t^k=\sum_{b=1}^k S_{k,b}t^b$$

But this is a Poisson$(t)$ moment, and so we are done. □

We can now finish our computations, and generalize Theorem 9.20, as follows:

**Theorem 9.23.** *The laws of truncated characters $\chi_t=\sum_{i=1}^{[tN]}u_{ii}$ are as follows:*

(1) *For $S_N$ with $N\to\infty$ we obtain a Poisson law $p_t$.*
(2) *For $S_N^+$ with $N\to\infty$ we obtain a free Poisson law $\pi_t$.*

*In addition, these laws are related by the Bercovici-Pata correspondence.*

*Proof.* This follows from the above results:

(1) This is something that we already know, from Proposition 9.22.

(2) This is something that we know so far only at $t=1$, from Theorem 9.20. In order to deal with the general $t\in(0,1]$ case, we can use the same method as for the orthogonal and unitary quantum groups, from section 8, and we obtain the following moments:

$$M_k=\sum_{\pi\in NC(k)}t^{|\pi|}$$

But these numbers being the moments of the free Poisson law of parameter $t$, as explained in Theorem 9.19 above, we obtain the result. See [23]. □

Summarizing, the liberation operation $S_N\to S_N^+$ has many common features with the liberation operations $O_N\to O_N^+$ and $U_N\to U_N^+$, studied in section 8 above.

There are many other things that can be said about $S_N^+$ and its subgroups, with all this being related to the subfactor and planar algebra theory of Jones [90], [91], [92], [93], [94], [95], [96]. For an introduction to this, we refer to [2], [3] and related papers.

## 10. Quantum reflections

Many interesting examples of quantum permutation groups appear as particular cases of the following general construction from [3], involving finite graphs:

**Proposition 10.1.** *Given a finite graph $X$, with adjacency matrix $d \in M_N(0,1)$, the following construction produces a quantum permutation group,*

$$C(G^+(X)) = C(S_N^+)\Big/\Big\langle du = ud\Big\rangle$$

*whose classical version $G(X)$ is the usual automorphism group of $X$.*

*Proof.* The fact that we have a quantum group comes from the fact that $du = ud$ reformulates as $d \in End(u)$, which makes it clear that we are dividing by a Hopf ideal. Regarding the second assertion, we must establish here the following equality:

$$C(G(X)) = C(S_N)\Big/\Big\langle du = ud\Big\rangle$$

For this purpose, recall that we have $u_{ij}(\sigma) = \delta_{\sigma(j)i}$. By using this formula, we have:

$$(du)_{ij}(\sigma) = \sum_k d_{ik}u_{kj}(\sigma) = \sum_k d_{ik}\delta_{\sigma(j)k} = d_{i\sigma(j)}$$

On the other hand, we have as well:

$$(ud)_{ij}(\sigma) = \sum_k u_{ik}(\sigma)d_{kj} = \sum_k \delta_{\sigma(k)i}d_{kj} = d_{\sigma^{-1}(i)j}$$

Thus the condition $du = ud$ reformulates as $d_{ij} = d_{\sigma(i)\sigma(j)}$, and we are led to the usual notion of an action of a permutation group on $X$, as claimed. ∎

Let us work out some basic examples. We have the following result:

**Theorem 10.2.** *The construction $X \to G^+(X)$ has the following properties:*

1. *For the $N$-point graph, having no edges at all, we obtain $S_N^+$.*
2. *For the $N$-simplex, having edges everywhere, we obtain as well $S_N^+$.*
3. *We have $G^+(X) = G^+(X^c)$, where $X^c$ is the complementary graph.*
4. *For a disconnected union, we have $G^+(X) \,\hat{*}\, G^+(Y) \subset G^+(X \sqcup Y)$.*
5. *For the square, we obtain a non-classical, proper subgroup of $S_4^+$.*

*Proof.* All these results are elementary, the proofs being as follows:

(1) This follows from definitions, because here we have $d = 0$.

(2) Here $d = \mathbb{I}$ is the all-one matrix, and the magic condition gives $u\mathbb{I} = \mathbb{I}u = N\mathbb{I}$. We conclude that $du = ud$ is automatic in this case, and so $G^+(X) = S_N^+$.

(3) The adjacency matrices of $X, X^c$ being related by the formula $d_X + d_{X^c} = \mathbb{I}$. We can use here the above formula $u\mathbb{I} = \mathbb{I}u = N\mathbb{I}$, and we conclude that $d_X u = ud_X$ is equivalent to $d_{X^c}u = ud_{X^c}$. Thus, we obtain, as claimed, $G^+(X) = G^+(X^c)$.

(4) The adjacency matrix of a disconnected union is given by $d_{X\sqcup Y} = diag(d_X, d_Y)$. Now let $w = diag(u,v)$ be the fundamental corepresentation of $G^+(X) \hat{*} G^+(Y)$. Then $d_X u = u d_X$ and $d_Y v = v d_Y$ imply, as desired, $d_{X\sqcup Y} w = w d_{X\sqcup Y}$.

(5) We know from (3) that we have $G^+(\square) = G^+(|\ |)$. We know as well from (4) that we have $\mathbb{Z}_2 \hat{*} \mathbb{Z}_2 \subset G^+(|\ |)$. It follows that $G^+(\square)$ is non-classical. Finally, the inclusion $G^+(\square) \subset S_4^+$ is indeed proper, because $S_4 \subset S_4^+$ does not act on the square. $\square$

In order to further advance, and to explicitely compute various quantum automorphism groups, we can use the spectral decomposition of $d$, as follows:

**Proposition 10.3.** *A closed subgroup $G \subset S_N^+$ acts on a graph $X$ precisely when*

$$P_\lambda u = u P_\lambda \quad , \quad \forall \lambda \in \mathbb{R}$$

*where $d = \sum_\lambda \lambda \cdot P_\lambda$ is the spectral decomposition of the adjacency matrix of $X$.*

*Proof.* Since $d \in M_N(0,1)$ is a symmetric matrix, we can consider indeed its spectral decomposition, $d = \sum_\lambda \lambda \cdot P_\lambda$. We have then the following formula:

$$< d >= span\left\{ P_\lambda \Big| \lambda \in \mathbb{R} \right\}$$

But this shows that we have the following equivalence:

$$d \in End(u) \iff P_\lambda \in End(u), \forall \lambda \in \mathbb{R}$$

Thus, we are led to the conclusion in the statement. $\square$

In order to exploit this, we will often combine it with the following standard fact:

**Proposition 10.4.** *Consider a closed subgroup $G \subset S_N^+$, with associated coaction map*

$$\Phi : \mathbb{C}^N \to \mathbb{C}^N \otimes C(G)$$

*For a linear subspace $V \subset \mathbb{C}^N$, the following are equivalent:*

(1) *The magic matrix $u = (u_{ij})$ commutes with $P_V$.*
(2) *$V$ is invariant, in the sense that $\Phi(V) \subset V \otimes C(G)$.*

*Proof.* Let $P = P_V$. For any $i \in \{1, \ldots, N\}$ we have the following formula:

$$\begin{aligned}
\Phi(P(e_i)) &= \Phi\left(\sum_k P_{ki} e_k\right) \\
&= \sum_{jk} P_{ki} e_j \otimes u_{jk} \\
&= \sum_j e_j \otimes (uP)_{ji}
\end{aligned}$$

On the other hand the linear map $(P\otimes id)\Phi$ is given by a similar formula:

$$\begin{aligned}(P\otimes id)(\Phi(e_i)) &= \sum_k P(e_k)\otimes u_{ki}\\ &= \sum_{jk} P_{jk}e_j\otimes u_{ki}\\ &= \sum_j e_j\otimes (Pu)_{ji}\end{aligned}$$

Thus $uP=Pu$ is equivalent to $\Phi P=(P\otimes id)\Phi$, and the conclusion follows. □

We have as well the following useful complementary result, from [3]:

**Proposition 10.5.** *Let $p\in M_N(\mathbb{C})$ be a matrix, and consider its "color" decomposition, obtained by setting $(p_c)_{ij}=1$ if $p_{ij}=c$ and $(p_c)_{ij}=0$ otherwise:*

$$p=\sum_{c\in\mathbb{C}} c\cdot p_c$$

*Then $u=(u_{ij})$ commutes with $p$ if and only if it commutes with all matrices $p_c$.*

*Proof.* Consider the multiplication and counit maps of the algebra $\mathbb{C}^N$:

$$M:e_i\otimes e_j\to e_ie_j\quad,\quad C:e_i\to e_i\otimes e_i$$

Since $M,C$ intertwine $u,u^{\otimes 2}$, their iterations $M^{(k)},C^{(k)}$ intertwine $u,u^{\otimes k}$, and so:

$$p^{(k)}=M^{(k)}p^{\otimes k}C^{(k)}=\sum_{c\in\mathbb{C}} c^kp_c\in End(u)$$

Let $S=\{c\in\mathbb{C}|p_c\neq 0\}$, and $f(c)=c$. By Stone-Weierstrass we have $S=<f>$, and so for any $e\in S$ the Dirac mass at $e$ is a linear combination of powers of $f$:

$$\delta_e=\sum_k\lambda_kf^k=\sum_k\lambda_k\left(\sum_{c\in S}c^k\delta_c\right)=\sum_{c\in S}\left(\sum_k\lambda_kc^k\right)\delta_c$$

The corresponding linear combination of matrices $p^{(k)}$ is given by:

$$\sum_k\lambda_kp^{(k)}=\sum_k\lambda_k\left(\sum_{c\in S}c^kp_c\right)=\sum_{c\in S}\left(\sum_k\lambda_kc^k\right)p_c$$

The Dirac masses being linearly independent, in the first formula all coefficients in the right term are 0, except for the coefficient of $\delta_e$, which is 1. Thus the right term in the second formula is $p_e$, and it follows that we have $p_e\in End(u)$, as claimed. □

The above results can be combined, and we are led into a "color-spectral" decomposition method for $d$, which can lead to a number of nontrivial results. See [3].

As a basic application of this, we can further study $G^+(\square)$, as follows:

**Proposition 10.6.** *The quantum automorphism group of the $N$-cycle is as follows:*

(1) *At $N\neq 4$ we have $G^+(X)=D_N$.*
(2) *At $N=4$ we have $D_4\subset G^+(X)\subset S_4^+$, with proper inclusions.*

*Proof.* We already know that the results hold at $N\leq 4$, so let us assume $N\geq 5$. Given a $N$-th root of unity, $w^N=1$, consider the following vector:

$$\xi=(w^i)$$

This is an eigenvector of $d$, with eigenvalue $w+w^{N-1}$. With $w=e^{2\pi i/N}$, it follows that $1,f,f^2,\ldots,f^{N-1}$ are eigenvectors of $d$. More precisely, the invariant subspaces of $d$ are as follows, with the last subspace having dimension 1 or 2, depending on $N$:

$$\mathbb C1,\,\mathbb Cf\oplus\mathbb Cf^{N-1},\,\mathbb Cf^2\oplus\mathbb Cf^{N-2},\ldots$$

Consider now the associated coaction $\Phi:\mathbb C^N\to\mathbb C^N\otimes C(G)$, and write:

$$\Phi(f)=f\otimes a+f^{N-1}\otimes b$$

By taking the square of this equality we obtain:

$$\Phi(f^2)=f^2\otimes a^2+f^{N-2}\otimes b^2+1\otimes(ab+ba)$$

It follows that $ab=-ba$, and that $\Phi(f^2)$ is given by the following formula:

$$\Phi(f^2)=f^2\otimes a^2+f^{N-2}\otimes b^2$$

By multiplying this with $\Phi(f)$ we obtain:

$$\Phi(f^3)=f^3\otimes a^3+f^{N-3}\otimes b^3+f^{N-1}\otimes ab^2+f\otimes ba^2$$

Now since $N\geq 5$ implies that $1,N-1$ are different from $3,N-3$, we must have $ab^2=ba^2=0$. By using this and $ab=-ba$, we obtain by recurrence on $k$ that:

$$\Phi(f^k)=f^k\otimes a^k+f^{N-k}\otimes b^k$$

In particular at $k=N-1$ we obtain:

$$\Phi(f^{N-1})=f^{N-1}\otimes a^{N-1}+f\otimes b^{N-1}$$

On the other hand we have $f^*=f^{N-1}$, so by applying $*$ to $\Phi(f)$ we get:

$$\Phi(f^{N-1})=f^{N-1}\otimes a^*+f\otimes b^*$$

Thus $a^*=a^{N-1}$ and $b^*=b^{N-1}$. Together with $ab^2=0$ this gives:

$$\begin{aligned}(ab)(ab)^*&=abb^*a^*\\&=ab^Na^{N-1}\\&=(ab^2)b^{N-2}a^{N-1}\\&=0\end{aligned}$$

From positivity we get from this $ab=0$, and together with $ab=-ba$, this shows that $a,b$ commute. On the other hand $C(G)$ is generated by the coefficients of $\Phi$, which are powers of $a,b$, and so $C(G)$ must be commutative, and we obtain the result. $\square$

Summarizing, this was a bad attempt in understanding $G^+(\square)$, which appears to be "exceptional" among the quantum symmetry groups of the $N$-cycles. An alternative approach to $G^+(\square)$ comes by regarding the square as the $N=2$ particular case of the $N$-hypercube $\square_N$. Indeed, the usual symmetry group of $\square_N$ is the hyperoctahedral group $H_N$, so we should have a formula of the following type:

$$G(\square)=H_2^+$$

In order to clarify this, let us start with the following simple fact:

**Proposition 10.7.** *We have an embedding as follows, $g_i$ being the generators of $\mathbb{Z}_2^N$,*

$$\widehat{\mathbb{Z}_2^N}\subset S^{N-1}_{\mathbb{R},+}\quad,\quad x_i=\frac{g_i}{\sqrt{N}}$$

*whose image is the geometric hypercube:*

$$\square_N=\left\{x\in\mathbb{R}^N\Big|x_i=\pm\frac{1}{\sqrt{N}},\forall i\right\}$$

*Proof.* This is something that we already know, from section 1 above. Consider indeed the following standard group algebra generators:

$$g_i\in C^*(\mathbb{Z}_2^N)=C(\widehat{\mathbb{Z}_2^N})$$

These generators satisfy satisfy then $g_i=g_i^*$, $g_i^2=1$, and when rescaling by $1/\sqrt{N}$, we obtain the relations defining $\square_N$. $\square$

We can now study the quantum symmetry groups $G^+(\square_N)$, and we are led to the quite surprising conclusion, from [17], that these are the twisted orthogonal groups $\bar{O}_N$:

**Theorem 10.8.** *With $\mathbb{Z}_2^N=<g_1,\ldots,g_N>$ we have a coaction map*

$$\Phi:C^*(\mathbb{Z}_2^N)\to C^*(\mathbb{Z}_2^N)\otimes C(\bar{O}_N)\quad,\quad g_i\to\sum_j g_j\otimes u_{ji}$$

*which makes $\bar{O}_N$ the quantum isometry group of the hypercube $\square_N=\widehat{\mathbb{Z}_2^N}$, as follows:*

1. *With $\square_N$ viewed as an algebraic manifold, $\square_N\subset S^{N-1}_{\mathbb{R}}\subset S^{N-1}_{\mathbb{R},+}$.*
2. *With $\square_N$ viewed as a graph, with $2^N$ vertices and $2^{N-1}N$ edges.*
3. *With $\square_N$ viewed as a metric space, with metric coming from $\mathbb{R}^N$.*

*Proof.* Observe first that $\square_N$ is indeed an algebraic manifold, so (1) as formulated above makes sense, in the general framework of section 2. The cube $\square_N$ is also a graph, as indicated, and so (2) makes sense as well, in the framework of Proposition 10.1. Finally,

(3) makes sense as well, because we can define the quantum isometry group of a finite metric space exactly as for graphs, but with $d$ being this time the distance matrix.

(1) In order for $G\subset O_N^+$ to act affinely on $\square_N$, the variables $G_i=\sum_j g_j\otimes u_{ji}$ must satisfy the same relations as the generators $g_i\in\mathbb Z_2^N$. The self-adjointness being automatic, the relations to be checked are therefore:

$$G_i^2=1\quad,\quad G_iG_j=G_jG_i$$

We have the following computation:

$$\begin{aligned}G_i^2&=\sum_{kl}g_kg_l\otimes u_{ik}u_{il}\\&=1+\sum_{k<l}g_kg_l\otimes(u_{ik}u_{il}+u_{il}u_{ik})\end{aligned}$$

As for the commutators, these are given by:

$$[G_i,G_j]=\sum_{k<l}g_kg_l\otimes(u_{ik}u_{jl}-u_{jk}u_{il}+u_{il}u_{jk}-u_{jl}u_{ik})$$

From the first relation we obtain $ab=0$ for $a\neq b$ on the same row of $u$, and by using the antipode, the same happens for the columns. From the second relation we obtain:

$$[u_{ik},u_{jl}]=[u_{jk},u_{il}]\quad,\quad\forall k\neq l$$

Now by applying the antipode we obtain:

$$[u_{lj},u_{ki}]=[u_{li},u_{kj}]$$

By relabelling, this gives the following formula:

$$[u_{ik},u_{jl}]=[u_{il},u_{jk}]\quad,\quad j\neq i$$

Thus for $i\neq j,k\neq l$ we must have:

$$[u_{ik},u_{jl}]=[u_{jk},u_{il}]=0$$

We are therefore led to $G\subset\bar O_N$, as claimed.

(2) We can use here the fact that the cube $\square_N$, when regarded as a graph, is the Cayley graph of the group $\mathbb Z_2^N$. The eigenvectors and eigenvalues of $\square_N$ are as follows:

$$\begin{aligned}v_{i_1\ldots i_N}&=\sum_{j_1\ldots j_N}(-1)^{i_1j_1+\ldots+i_Nj_N}g_1^{j_1}\ldots g_N^{j_N}\\\lambda_{i_1\ldots i_N}&=(-1)^{i_1}+\ldots+(-1)^{i_N}\end{aligned}$$

With this picture in hand, and by using Proposition 10.3 and Proposition 10.4 above, the result follows from the same computations as in the proof of (1). See [17].

(3) Our claim here is that we obtain the same symmetry group as in (2). Indeed, observe that distance matrix of the cube has a color decomposition as follows:

$$d=d_1+\sqrt{2}d_2+\sqrt{3}d_3+\ldots+\sqrt{N}d_N$$

Since the powers of $d_1$ can be computed by counting loops on the cube, we have formulae as follows, with $x_{ij}\in\mathbb N$ being certain positive integers:

$$\begin{aligned}
d_1^2&=x_{21}1_N+x_{22}d_2\\
d_1^3&=x_{31}1_N+x_{32}d_2+x_{33}d_3\\
&\ldots\\
d_1^N&=x_{N1}1_N+x_{N2}d_2+x_{N3}d_3+\ldots+x_{NN}d_N
\end{aligned}$$

But this shows that we have $<d>=<d_1>$. Now since $d_1$ is the adjacency matrix of $\square_N$, viewed as graph, this proves our claim, and we obtain the result via (2). $\square$

Now back to our questions regarding the square, we have $G^+(\square)=\bar{O}_2$, and this formula appears as the $N=2$ particular case of a general formula, namely $G^+(\square_N)=\bar{O}_N$. This is quite conceptual, but still not ok. The problem is that we have $G(\square_N)=H_N$, and so for our theory to be complete, we would need a formula of type $H_N^+=\bar{O}_N$. And this latter formula is obviously wrong, because for $\bar{O}_N$ the character computations lead to Gaussian laws, who cannot appear as liberations of the character laws for $H_N$, that we have not computed yet, but which can only be something Poisson-related.

Summarizing, the problem of conceptually understanding $G(\square)$ remains open. In order to present now the correct, final solution, the idea will be that to look at the quantum group $G^+(|\ |)$ instead, which is equal to it, according to Proposition 10.2 (3). We first have the following result, extending Proposition 10.2 (4) above:

**Proposition 10.9.** *For a disconnected union of graphs we have*

$$G^+(X_1)\,\hat{*}\,\ldots\,\hat{*}\,G^+(X_k)\subset G^+(X_1\sqcup\ldots\sqcup X_k)$$

*and this inclusion is in general not an isomorphism.*

*Proof.* The proof of the first assertion is nearly identical to the proof of Proposition 10.2 (4) above. Indeed, the adjacency matrix of the disconnected union is given by:

$$d_{X_1\sqcup\ldots\sqcup X_k}=diag(d_{X_1},\ldots,d_{X_k})$$

$$w=diag(u_1,\ldots,u_k)$$

We have then $d_{X_i}u_i=u_id_{X_i}$, and this implies $dw=wd$, which gives the result. As for the last assertion, this is something that we already know, from Proposition 10.6 (2). $\square$

In the case where the graphs $X_1,\ldots,X_k$ are identical, which is the one that we are truly interested in, we can further build on this. We recall from [45] that we have:

**Proposition 10.10.** *Given closed subgroups $G\subset U_N^+$, $H\subset S_k^+$, with fundamental corepresentations $u,v$, the following construction produces a closed subgroup of $U_{Nk}^+$:*

$$C(G\wr_*H)=(C(G)^{*k}*C(H))/<[u_{ij}^{(a)},v_{ab}]=0>$$

*In the case where $G,H$ are classical, the classical version of $G\wr_*H$ is the usual wreath product $G\wr H$. Also, when $G$ is a quantum permutation group, so is $G\wr_*H$.*

*Proof.* Consider indeed the matrix $w_{ia,jb}=u_{ij}^{(a)}v_{ab}$, over the quotient algebra in the statement. It is routine to check that $w$ is unitary, and in the case $G\subset S_N^+$, our claim is that this matrix is magic. Indeed, the entries are projections, because they appear as products of commuting projections, and the row sums are as follows:

$$\sum_{jb}w_{ia,jb}=\sum_{jb}u_{ij}^{(a)}v_{ab}=\sum_bv_{ab}\sum_ju_{ij}^{(a)}=1$$

As for the column sums, these are as follows:

$$\sum_{ia}w_{ia,jb}=\sum_{ia}u_{ij}^{(a)}v_{ab}=\sum_av_{ab}\sum_iu_{ij}^{(a)}=1$$

With these observations in hand, it is routine to check that $G\wr_*H$ is indeed a quantum group, with fundamental corepresentation $w$, by constructing maps $\Delta,\varepsilon,S$ as in section 1, and in the case $G\subset S_N^+$, we obtain in this way a closed subgroup of $S_{Nk}^+$. Finally, the assertion regarding the classical version is standard as well. See [45]. $\square$

We refer to [12], [45], [126] for more details regarding the above construction. Now with this notion in hand, following [12], we have the following result:

**Theorem 10.11.** *Given a connected graph $X$, and $k\in\mathbb N$, we have the formulae*

$$G(kX)=G(X)\wr S_k$$

$$G^+(kX)=G^+(X)\wr_*S_k^+$$

*where $kX=X\sqcup\ldots\sqcup X$ is the $k$-fold disjoint union of $X$ with itself.*

*Proof.* The first formula is something well-known, which follows as well from the second formula, by taking the classical version. Regarding now the second formula, it is elementary to check that we have an inclusion as follows, for any finite graph $X$:

$$G^+(X)\wr_*S_k^+\subset G^+(kX)$$

Indeed, we want to construct an action $G^+(X)\wr_*S_k^+\curvearrowright kX$, and this amounts in proving that we have $[w,d]=0$. But, the matrices $w,d$ are given by:

$$w_{ia,jb}=u_{ij}^{(a)}v_{ab}\quad,\quad d_{ia,jb}=\delta_{ij}d_{ab}$$

With these formulae in hand, we have the following computation:

$$\begin{aligned}(dw)_{ia,jb} &= \sum_k d_{ik}w_{ka,jb}\\ &= \sum_k d_{ik}u_{kj}^{(a)}v_{ab}\\ &= (du^{(a)})_{ij}v_{ab}\end{aligned}$$

On the other hand, we have as well the following computation:

$$\begin{aligned}(wd)_{ia,jb} &= \sum_k w_{ia,kb}d_{kj}\\ &= \sum_k u_{ik}^{(a)}v_{ab}d_{kj}\\ &= (u^{(a)}d)_{ij}v_{ab}\end{aligned}$$

Thus we have $[w,d]=0$, and from this we obtain:

$$G^+(X)\wr_* S_k^+ \subset G^+(kX)$$

Regarding now the reverse inclusion, which requires $X$ to be connected, this follows by doing some matrix analysis, by using the commutation with $u$. To be more precise, let us denote by $w$ the fundamental corepresentation of $G^+(kX)$, and set:

$$u_{ij}^{(a)} = \sum_b w_{ia,jb} \quad , \quad v_{ab} = \sum_i v_{ab}$$

It is then routine to check, by using the fact that $X$ is indeed connected, that we have here magic unitaries, as in the definition of the free wreath products. Thus we obtain the reverse inclusion, that we were looking for, namely:

$$G^+(kX)\subset G^+(X)\wr_* S_k^+$$

To be more precise, the key ingredient is the fact that when $X$ is connected, the $*$-algebra generated by $d_X$ contains a matrix having nonzero entries. See [12]. $\square$

We are led in this way to the following result, from [17]:

**Theorem 10.12.** *Consider the graph consisting of $N$ segments.*

(1) *Its symmetry group is the hyperoctahedral group $H_N=\mathbb{Z}_2\wr S_N$.*
(2) *Its quantum symmetry group is the quantum group $H_N^+=\mathbb{Z}_2\wr_* S_N^+$.*

*Proof.* This comes from the above results, as follows:

(1) This is clear from definitions, with the remark that the relation with the formula $H_N=G(\square_N)$ comes by viewing the $N$ segments as being the $[-1,1]$ segments on each

of the $N$ coordinate axes of $\mathbb{R}^N$. Indeed, a symmetry of the $N$-cube is the same as a symmetry of the $N$ segments, and so, as desired:

$$G(\square_N)=\mathbb{Z}_2\wr S_N$$

(2) This follows from Theorem 10.11 above, applied to the segment graph. Observe also that (2) implies (1), by taking the classical version. $\square$

Now back to the square, we have $G^+(\square)=H_2^+$, and our claim is that this is the "good" and final formula. In order to prove this, we must work out the easiness theory for $H_N,H_N^+$, and find a compatibility there. We first have the following result:

**Proposition 10.13.** *The algebra $C(H_N^+)$ can be presented in two ways, as follows:*

1. *As the universal algebra generated by the entries of a $2N\times 2N$ magic unitary having the "sudoku" pattern $w=\begin{pmatrix}a&b\\ b&a\end{pmatrix}$, with $a,b$ being square matrices.*
2. *As the universal algebra generated by the entries of a $N\times N$ orthogonal matrix which is "cubic", in the sense that $u_{ij}u_{ik}=u_{ji}u_{ki}=0$, for any $j\neq k$.*

*As for $C(H_N)$, this has similar presentations, among the commutative algebras.*

*Proof.* Here the first assertion follows from Theorem 10.12, via Proposition 10.10, and the last assertion is clear as well, because $C(H_N)$ is the abelianization of $C(H_N^+)$. Thus, we are left with proving that the algebras $A_s,A_c$ coming from (1,2) coincide.

We construct first the arrow $A_c\to A_s$. The elements $a_{ij},b_{ij}$ being self-adjoint, their differences are self-adjoint as well. Thus $a-b$ is a matrix of self-adjoint elements. We have the following formula for the products on the columns of $a-b$:

$$\begin{aligned}(a-b)_{ik}(a-b)_{jk}&=a_{ik}a_{jk}-a_{ik}b_{jk}-b_{ik}a_{jk}+b_{ik}b_{jk}\\ &=\begin{cases}0&\text{for } i\neq j\\ a_{ik}+b_{ik}&\text{for } i=j\end{cases}\end{aligned}$$

In the $i=j$ case the elements $a_{ik}+b_{ik}$ sum up to 1, so the columns of $a-b$ are orthogonal. A similar computation works for rows, so $a-b$ is orthogonal.

Now by using the $i\neq j$ computation, along with its row analogue, we conclude that $a-b$ is cubic. Thus we can define a morphism $A_c\to A_s$ by the following formula:

$$\varphi(u_{ij})=a_{ij}-b_{ij}$$

We construct now the inverse morphism. Consider the following elements:

$$\alpha_{ij}=\frac{u_{ij}^2+u_{ij}}{2}\quad,\quad\beta_{ij}=\frac{u_{ij}^2-u_{ij}}{2}$$

These are projections, and the following matrix is a sudoku unitary:

$$M=\begin{pmatrix}(\alpha_{ij})&(\beta_{ij})\\ (\beta_{ij})&(\alpha_{ij})\end{pmatrix}$$

Thus we can define a morphism $A_s \to A_c$ by the following formula:

$$\psi(a_{ij}) = \frac{u_{ij}^2 + u_{ij}}{2} \quad , \quad \psi(b_{ij}) = \frac{u_{ij}^2 - u_{ij}}{2}$$

We check now the fact that $\psi, \varphi$ are indeed inverse morphisms:

$$\begin{aligned} \psi\varphi(u_{ij}) &= \psi(a_{ij} - b_{ij}) \\ &= \frac{u_{ij}^2 + u_{ij}}{2} - \frac{u_{ij}^2 - u_{ij}}{2} \\ &= u_{ij} \end{aligned}$$

As for the other composition, we have the following computation:

$$\begin{aligned} \varphi\psi(a_{ij}) &= \varphi\left(\frac{u_{ij}^2 + u_{ij}}{2}\right) \\ &= \frac{(a_{ij} - b_{ij})^2 + (a_{ij} - b_{ij})}{2} \\ &= a_{ij} \end{aligned}$$

A similar computation gives $\varphi\psi(b_{ij}) = b_{ij}$, which completes the proof. □

We can now work out the easiness property of $H_N, H_N^+$, with respect to the cubic representations, and we are led to the following result, which is fully satisfactory:

**Theorem 10.14.** *The quantum groups $H_N, H_N^+$ are both easy, as follows:*

1. *$H_N$ corresponds to the category $P_{even}$.*
2. *$H_N^+$ corresponds to the category $NC_{even}$.*

*Proof.* These assertions follow indeed from the fact that the cubic relations are implemented by the one-block partition in $P(2,2)$, which generates $NC_{even}$. See [17]. □

There is a similarity here with the easiness results for permutations and quantum permutations, obtained in sections 7 and 9 above. In fact, the basic algebraic results regarding $S_N, S_N^+$ and $H_N, H_N^+$ appear as the $s = 1, 2$ particular cases of:

**Theorem 10.15.** *The complex reflection groups $H_N^s = \mathbb{Z}_s \wr S_N$ and their free analogues $H_N^{s+} = \mathbb{Z}_s \wr_* S_N^+$, defined for any $s \in \mathbb{N}$, have the following properties:*

1. *They have $N$-dimensional coordinates $u = (u_{ij})$, which are subject to the relations $u_{ij}u_{ij}^* = u_{ij}^*u_{ij}$, $p_{ij} = u_{ij}u_{ij}^* = $ magic, and $u_{ij}^s = p_{ij}$.*
2. *They are easy, the corresponding categories $P^s \subset P, NC^s \subset NC$ being given by the fact that we have $\# \circ - \# \bullet = 0(s)$, as a weighted sum, in each block.*

*Proof.* We already know that the results hold at $s = 1, 2$, and the proof in general is similar. With respect to the above proof at $s = 2$, the situation is as follows:

(1) Observe first that the result holds at $s = 1$, where we obtain the magic condition, and at $s = 2$ as well, where we obtain something equivalent to the cubic condition. In general, this follows from a $\mathbb{Z}_s$-analogue of Proposition 10.13. See [41].

(2) Once again, the result holds at $s = 1$, trivially, and at $s = 2$ as well, where our condition is equivalent to $\# \circ + \# \bullet = 0(2)$, in each block. In general, this follows as in the proof of Theorem 10.14, by using the one-block partition in $P(s, s)$. See [10]. $\square$

The above proof is of course quite brief, but we will not be really interested here in the case $s \geq 3$, which is quite technical. In fact, the above result, dealing with the general case $s \in \mathbb{N}$, is here for providing an introduction to the case $s = \infty$, where we have:

**Theorem 10.16.** *The pure complex reflection groups $K_N = \mathbb{T} \wr S_N$ and their free analogues $K_N^+ = \mathbb{T} \wr_* S_N^+$ have the following properties:*

1. *They have $N$-dimensional coordinates $u = (u_{ij})$, which are subject to the relations $u_{ij} u_{ij}^* = u_{ij}^* u_{ij}$ and $p_{ij} = u_{ij} u_{ij}^* =$ magic.*
2. *They are easy, the corresponding categories $\mathcal{P}_{even} \subset P, \mathcal{NC}_{even} \subset NC$ being given by the fact that we have $\# \circ = \# \bullet$, as a weighted equality, in each block.*

*Proof.* The assertions here appear as an $s = \infty$ extension of (1,2) in Theorem 10.15 above, and their proof can be obtained along the same lines, as follows:

(1) This follows indeed by working out a $\mathbb{T}$-analogue of the computations in the proof of Proposition 10.13 above. We refer here to [41].

(2) Once again, this appears as a $s = \infty$ extension of the results that we already have, and for details here, we refer once again to [10]. $\square$

The above results at $s = 2, \infty$ are quite interesting for us, because we can now focus on the quantum reflection groups $H_N, H_N^+, K_N, K_N^+$, with the idea in mind of completing the orthogonal and unitary quantum group picture from section 7 above.

Before doing this, we have two more quantum groups to be introduced and studied, namely the half-liberations $H_N^*, K_N^*$. We have here the following result:

**Theorem 10.17.** *We have quantum groups $H_N^*, K_N^*$, which are both easy, as follows,*

1. *$H_N^* = H_N^+ \cap O_N^*$, corresponding to the category $P_{even}^*$,*
2. *$K_N^* = K_N^+ \cap U_N^*$, corresponding to the category $\mathcal{P}_{even}^*$,*

*with the symbol $*$ standing for the fact that the corresponding partitions, when relabelled clockwise $\circ \bullet \circ \bullet \ldots$, must contain the same number of $\circ, \bullet$, in each block.*

*Proof.* This is standard, from the results that we already have, regarding the various quantum groups involved, because the interesection operations at the quantum group level correspond to generation operations, at the category of partitions level. $\square$

We can now complete the "continuous" picture from section 7 above, as follows:

**Theorem 10.18.** *The basic orthogonal and unitary quantum groups are related to the basic real and complex quantum reflection groups as follows,*

$$\begin{tikzcd} U_N \arrow[r] & U_N^* \arrow[r] & U_N^+ \\ O_N \arrow[u] \arrow[r] & O_N^* \arrow[u] \arrow[r] & O_N^+ \arrow[u] \end{tikzcd} \quad \leftrightarrow \quad \begin{tikzcd} K_N \arrow[r] & K_N^* \arrow[r] & K_N^+ \\ H_N \arrow[u] \arrow[r] & H_N^* \arrow[u] \arrow[r] & H_N^+ \arrow[u] \end{tikzcd}$$

*the connecting operations $U \leftrightarrow K$ being given by $K = U \cap K_N^+$ and $U = \{K, O_N\}$.*

*Proof.* According to the general results in section 7 above, in terms of categories of partitions, the operations introduced in the statement reformulate as follows:

$$D_K = < D_U, \mathcal{NC}_{even} > \quad , \quad D_U = D_K \cap P_2$$

On the other hand, by putting together the various easiness results that we have, the categories of partitions for the quantum groups in the statement are as follows:

$$\begin{tikzcd} \mathcal{P}_2 \arrow[d] & \mathcal{P}_2^* \arrow[l] \arrow[d] & \mathcal{NC}_2 \arrow[l] \arrow[d] \\ P_2 & P_2^* \arrow[l] & NC_2 \arrow[l] \end{tikzcd} \quad : \quad \begin{tikzcd} \mathcal{P}_{even} \arrow[d] & \mathcal{P}_{even}^* \arrow[l] \arrow[d] & \mathcal{NC}_{even} \arrow[l] \arrow[d] \\ P_{even} & P_{even}^* \arrow[l] & NC_{even} \arrow[l] \end{tikzcd}$$

It is elementary to check that these categories are related by the above intersection and generation operations, and we conclude that the correspondence holds indeed. □

Our purpose now will be that of showing that a twisted analogue of the above result holds. It is convenient to include in our discussion two more quantum groups, coming from [27], [117] and denoted $H_N^{[\infty]}, K_N^{[\infty]}$, which are constructed as follows:

**Theorem 10.19.** *We have intermediate liberations $H_N^{[\infty]}, K_N^{[\infty]}$ as follows, constructed by using the relations $\alpha\beta\gamma = 0$, for any $a \neq c$ on the same row or column of $u$,*

$$\begin{tikzcd} K_N \arrow[r] & K_N^* \arrow[r] & K_N^{[\infty]} \arrow[r] & K_N^+ \\ H_N \arrow[u] \arrow[r] & H_N^* \arrow[u] \arrow[r] & H_N^{[\infty]} \arrow[u] \arrow[r] & H_N^+ \arrow[u] \end{tikzcd}$$

*with the convention $\alpha = a, a^*$, and so on. These quantum groups are easy, the corresponding categories $P_{even}^{[\infty]} \subset P_{even}$ and $\mathcal{P}_{even}^{[\infty]} \subset \mathcal{P}_{even}$ being generated by $\eta = \ker\binom{iij}{jii}$.*

*Proof.* This is routine, by using the fact that the relations $\alpha\beta\gamma = 0$ in the statement are equivalent to the condition $\eta \in End(u^{\otimes k})$, with $|k| = 3$. We refer here to [27], [117]. ☐

In order to discuss the twisting, we will need the following technical result:

**Proposition 10.20.** *We have the following equalities,*

$$\begin{aligned} P_{even}^* &= \left\{\pi \in P_{even}\Big|\varepsilon(\tau) = 1, \forall\tau \leq \pi, |\tau| = 2\right\} \\ P_{even}^{[\infty]} &= \left\{\pi \in P_{even}\Big|\sigma \in P_{even}^*, \forall\sigma \subset \pi\right\} \\ P_{even}^{[\infty]} &= \left\{\pi \in P_{even}\Big|\varepsilon(\tau) = 1, \forall\tau \leq \pi\right\} \end{aligned}$$

*where $\varepsilon : P_{even} \to \{\pm 1\}$ is the signature of even permutations.*

*Proof.* This is routine combinatorics, from [5], [117], the idea being as follows:

(1) Given $\pi \in P_{even}$, we have $\tau \leq \pi, |\tau| = 2$ precisely when $\tau = \pi^\beta$ is the partition obtained from $\pi$ by merging all the legs of a certain subpartition $\beta \subset \pi$, and by merging as well all the other blocks. Now observe that $\pi^\beta$ does not depend on $\pi$, but only on $\beta$, and that the number of switches required for making $\pi^\beta$ noncrossing is $c = N_\bullet - N_\circ$ modulo 2, where $N_\bullet/N_\circ$ is the number of black/white legs of $\beta$, when labelling the legs of $\pi$ counterclockwise $\circ\bullet\circ\bullet\ldots$ Thus $\varepsilon(\pi^\beta) = 1$ holds precisely when $\beta \in \pi$ has the same number of black and white legs, and this gives the result.

(2) This simply follows from the equality $P_{even}^{[\infty]} =< \eta >$ coming from Theorem 10.19, by computing $< \eta >$, and for the complete proof here we refer to [117].

(3) We use here the fact, also from [117], that the relations $g_i g_i g_j = g_j g_i g_i$ are trivially satisfied for real reflections. This leads to the following conclusion:

$$P_{even}^{[\infty]}(k,l) = \left\{\ker\begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{pmatrix}\Big|g_{i_1}\ldots g_{i_k} = g_{j_1}\ldots g_{j_l} \text{ inside } \mathbb{Z}_2^{*N}\right\}$$

In other words, the partitions in $P_{even}^{[\infty]}$ are those describing the relations between free variables, subject to the conditions $g_i^2 = 1$. We conclude that $P_{even}^{[\infty]}$ appears from $NC_{even}$ by "inflating blocks", in the sense that each $\pi \in P_{even}^{[\infty]}$ can be transformed into a partition $\pi' \in NC_{even}$ by deleting pairs of consecutive legs, belonging to the same block.

Now since this inflation operation leaves invariant modulo 2 the number $c \in \mathbb{N}$ of switches in the definition of the signature, it leaves invariant the signature $\varepsilon = (-1)^c$ itself, and we obtain in this way the inclusion "$\subset$" in the statement.

Conversely, given $\pi \in P_{even}$ satisfying $\varepsilon(\tau) = 1$, $\forall\tau \leq \pi$, our claim is that:

$$\rho \leq \sigma \subset \pi, |\rho| = 2 \implies \varepsilon(\rho) = 1$$

Indeed, let us denote by $\alpha, \beta$ the two blocks of $\rho$, and by $\gamma$ the remaining blocks of $\pi$, merged altogether. We know that the partitions $\tau_1 = (\alpha \wedge \gamma, \beta)$, $\tau_2 = (\beta \wedge \gamma, \alpha)$,

$\tau_3 = (\alpha, \beta, \gamma)$ are all even. On the other hand, putting these partitions in noncrossing form requires respectively $s+t, s'+t, s+s'+t$ switches, where $t$ is the number of switches needed for putting $\rho = (\alpha, \beta)$ in noncrossing form. Thus $t$ is even, and we are done.

With the above claim in hand, we conclude, by using the second equality in the statement, that we have $\sigma \in P^*_{even}$. Thus we have $\pi \in P^{[\infty]}_{even}$, which ends the proof of "$\supset$". $\square$

With the above result in hand, we can now prove:

**Theorem 10.21.** *We have the following results:*

1. *The quantum groups from Theorem 10.19 are equal to their own twists.*
2. *With input coming from this, a twisted version of Theorem 10.18 holds.*

*Proof.* This result, established in [5], basically comes from the results that we have.

(1) In the real case, the verifications are as follows:

– $H_N^+$. We know from section 7 above that for $\pi \in NC_{even}$ we have $\bar{T}_\pi = T_\pi$, and since we are in the situation $D \subset NC_{even}$, the definitions of $G, \bar{G}$ coincide.

– $H_N^{[\infty]}$. Here we can use the same argument as in (1), based this time on the description of $P^{[\infty]}_{even}$ involving the signature found in Proposition 10.20.

– $H_N^*$. We have $H_N^* = H_N^{[\infty]} \cap O_N^*$, so $\bar{H}_N^* \subset H_N^{[\infty]}$ is the subgroup obtained via the defining relations for $\bar{O}_N^*$. But all the $abc = -cba$ relations defining $\bar{H}_N^*$ are automatic, of type $0 = 0$, and it follows that $\bar{H}_N^* \subset H_N^{[\infty]}$ is the subgroup obtained via the relations $abc = cba$, for any $a, b, c \in \{u_{ij}\}$. Thus we have $\bar{H}_N^* = H_N^{[\infty]} \cap O_N^* = H_N^*$, as claimed.

– $H_N$. We have $H_N = H_N^* \cap O_N$, and by functoriality, $\bar{H}_N = \bar{H}_N^* \cap \bar{O}_N = H_N^* \cap \bar{O}_N$. But this latter intersection is easily seen to be equal to $H_N$, as claimed.

In the complex case the proof is similar, and we refer here to [5].

(2) This can be proved by proceeding as in the proof of Theorem 10.18 above, with of course some care when formulating the result. Once again, we refer here to [5]. $\square$

Regarding the probabilistic aspects, we will need some general theory. We have the following definition, extending the Poisson limit theory from section 9 above:

**Definition 10.22.** *Associated to any compactly supported positive measure $\rho$ on $\mathbb{R}$ are the probability measures*

$$p_\rho = \lim_{n\to\infty} \left( \left(1 - \frac{c}{n}\right)\delta_0 + \frac{1}{n}\rho \right)^{*n}$$

$$\pi_\rho = \lim_{n\to\infty} \left( \left(1 - \frac{c}{n}\right)\delta_0 + \frac{1}{n}\rho \right)^{\boxplus n}$$

*where $c = mass(\rho)$, called compound Poisson and compound free Poisson laws.*

In what follows we will be interested in the case where $\rho$ is discrete, as is for instance the case for $\rho=\delta_t$ with $t>0$, which produces the Poisson and free Poisson laws. The following result allows one to detect compound Poisson/free Poisson laws:

**Proposition 10.23.** *For $\rho=\sum_{i=1}^s c_i\delta_{z_i}$ with $c_i>0$ and $z_i\in\mathbb R$, we have*

$$F_{p_\rho}(y)=\exp\left(\sum_{i=1}^s c_i(e^{iyz_i}-1)\right)$$

$$R_{\pi_\rho}(y)=\sum_{i=1}^s\frac{c_iz_i}{1-yz_i}$$

*where $F,R$ denote respectively the Fourier transform, and Voiculescu's $R$-transform.*

*Proof.* Let $\mu_n$ be the measure appearing in Definition 10.22, under the convolution signs. In the classical case, we have the following computation:

$$\begin{aligned}
&& F_{\mu_n}(y)&=\left(1-\frac{c}{n}\right)+\frac{1}{n}\sum_{i=1}^s c_ie^{iyz_i}\\
&\implies& F_{\mu_n^{*n}}(y)&=\left(\left(1-\frac{c}{n}\right)+\frac{1}{n}\sum_{i=1}^s c_ie^{iyz_i}\right)^n\\
&\implies& F_{p_\rho}(y)&=\exp\left(\sum_{i=1}^s c_i(e^{iyz_i}-1)\right)
\end{aligned}$$

In the free case now, we use a similar method. The Cauchy transform of $\mu_n$ is:

$$G_{\mu_n}(\xi)=\left(1-\frac{c}{n}\right)\frac{1}{\xi}+\frac{1}{n}\sum_{i=1}^s\frac{c_i}{\xi-z_i}$$

Consider now the $R$-transform of the measure $\mu_n^{\boxplus n}$, which is given by:

$$R_{\mu_n^{\boxplus n}}(y)=nR_{\mu_n}(y)$$

The above formula of $G_{\mu_n}$ shows that the equation for $R=R_{\mu_n^{\boxplus n}}$ is as follows:

$$\begin{aligned}
&& \left(1-\frac{c}{n}\right)\frac{1}{y^{-1}+R/n}+\frac{1}{n}\sum_{i=1}^s\frac{c_i}{y^{-1}+R/n-z_i}&=y\\
&\implies& \left(1-\frac{c}{n}\right)\frac{1}{1+yR/n}+\frac{1}{n}\sum_{i=1}^s\frac{c_i}{1+yR/n-yz_i}&=1
\end{aligned}$$

Now multiplying by $n$, rearranging the terms, and letting $n\to\infty$, we get:

$$\begin{aligned}\frac{c+yR}{1+yR/n}&=\sum_{i=1}^s\frac{c_i}{1+yR/n-yz_i}\\ \implies\quad c+yR_{\pi_\rho}(y)&=\sum_{i=1}^s\frac{c_i}{1-yz_i}\\ \implies\quad R_{\pi_\rho}(y)&=\sum_{i=1}^s\frac{c_iz_i}{1-yz_i}\end{aligned}$$

This finishes the proof in the free case, and we are done. $\square$

We have as well the following result, providing an alternative to Definition 10.22:

**Theorem 10.24.** *For $\rho=\sum_{i=1}^sc_i\delta_{z_i}$ with $c_i>0$ and $z_i\in\mathbb R$, we have*

$$p_\rho/\pi_\rho=\mathrm{law}\left(\sum_{i=1}^sz_i\alpha_i\right)$$

*where the variables $\alpha_i$ are Poisson/free Poisson$(c_i)$, independent/free.*

*Proof.* Let $\alpha$ be the sum of Poisson/free Poisson variables in the statement. We will show that the Fourier/$R$-transform of $\alpha$ is given by the formulae in Proposition 10.23.

Indeed, by using some well-known Fourier transform formulae, we have:

$$\begin{aligned}F_{\alpha_i}(y)=\exp(c_i(e^{iy}-1))\quad&\implies\quad F_{z_i\alpha_i}(y)=\exp(c_i(e^{iyz_i}-1))\\ &\implies\quad F_\alpha(y)=\exp\left(\sum_{i=1}^sc_i(e^{iyz_i}-1)\right)\end{aligned}$$

Also, by using some well-known $R$-transform formulae, we have:

$$\begin{aligned}R_{\alpha_i}(y)=\frac{c_i}{1-y}\quad&\implies\quad R_{z_i\alpha_i}(y)=\frac{c_iz_i}{1-yz_i}\\ &\implies\quad R_\alpha(y)=\sum_{i=1}^s\frac{c_iz_i}{1-yz_i}\end{aligned}$$

Thus we have indeed the same formulae as those in Proposition 10.23. $\square$

We can go back now to quantum reflection groups, and we have:

**Theorem 10.25.** *The asymptotic laws of truncated characters are as follows, where $\varepsilon_s$ with $s\in\{1,2,\ldots,\infty\}$ is the uniform measure on the $s$-th roots of unity:*

1. *For $H_N^s$ we obtain the compound Poisson law $b_t^s=p_{t\varepsilon_s}$.*
2. *For $H_N^{s+}$ we obtain the compound free Poisson law $\beta_t^s=\pi_{t\varepsilon_s}$.*

*These measures are in Bercovici-Pata bijection.*

*Proof.* This follows from easiness, and from the Weingarten formula. To be more precise, at $t = 1$ this follows by counting the partitions, and at $t \in (0,1]$ general, this follows in the usual way, for instance by using cumulants. For details here, we refer to [10]. $\square$

The above measures are called Bessel and free Bessel laws. This is because at $s = 2$ we have $b_t^2 = e^{-t}\sum_{k=-\infty}^{\infty} f_k(t/2)\delta_k$, with $f_k$ being the Bessel function of the first kind:

$$f_k(t) = \sum_{p=0}^{\infty} \frac{t^{|k|+2p}}{(|k|+p)!p!}$$

The Bessel and free Bessel laws have particularly interesting properties at the parameter values $s = 2, \infty$. So, let us record the precise statement here:

**Theorem 10.26.** *The asymptotic laws of truncated characters are as follows:*

(1) *For $H_N$ we obtain the real Bessel law $b_t = p_{t\varepsilon_2}$.*
(2) *For $K_N$ we obtain the complex Bessel law $B_t = p_{t\varepsilon_\infty}$.*
(3) *For $H_N^+$ we obtain the free real Bessel law $\beta_t = \pi_{t\varepsilon_2}$.*
(4) *For $K_N^+$ we obtain the free complex Bessel law $\mathfrak{B}_t = \pi_{t\varepsilon_\infty}$.*

*Proof.* This follows indeed from Theorem 10.25 above, at $s = 2, \infty$. $\square$

In addition to what has been said above, there are as well some interesting results about the Bessel and free Bessel laws involving the multiplicative convolution $\times$, and the multiplicative free convolution $\boxtimes$ from [134]. For details, we refer here to [10].

Also, the study of the quantum automorphism groups of the finite graphs is something that was systematically developed, and goes well beyond the preliminary material explained here. We refer here to [44], [105], [112], [118], [119], [120], [126].

## 11. Classification results

We discuss in this section and in the next one the classification questions for the closed subgroups $G_N \subset U_N^+$, in the easy case, and beyond. There has been a lot of work on the subject, and our objective will be that of presenting a few basic results, with some discussion. We have already met a number of easy quantum groups, as follows:

**Proposition 11.1.** *We have the following basic examples of easy quantum groups:*

(1) *Unitary quantum groups:* $O_N, O_N^*, O_N^+, U_N, U_N^*, U_N^+$
(2) *Bistochastic versions:* $B_N, B_N^+, C_N, C_N^+$.
(3) *Quantum permutation groups:* $S_N, S_N^+$.
(4) *Quantum reflection groups:* $H_N, H_N^*, H_N^+, K_N, K_N^*, K_N^+$.

*Proof.* This is something that we already know, the partitions being as follows:

(1) $P_2, P_2^*, NC_2, \mathcal{P}_2, \mathcal{P}_2^*, \mathcal{NC}_2$.

(2) $P_{12}, NC_{12}, \mathcal{P}_{12}, \mathcal{NC}_{12}$.

(3) $P, NC$.

(4) $P_{even}, P_{even}^*, NC_{even}, \mathcal{P}_{even}, \mathcal{P}_{even}^*, \mathcal{NC}_{even}$. □

In addition to the above list, we have the quantum groups $H_N^s, H_N^{s+}$ with $3 \leq s < \infty$, as well as the related series $H_N^{s*} = H_N^{s+} \cap U_N^*$. Further examples can be constructed via free complexification, or via operations of type $G_N \to \mathbb{Z}_r \times G_N$, or $G_N \to \mathbb{Z}_r G_N$, with $r \in \{2, 3, \ldots, \infty\}$. There are as well many "exotic" intermediate liberation procedures, involving relations far more complicated than the half-commutation ones $abc = cba$.

All this makes the classification question particularly difficult. So, our first task in what follows will be that of cutting a bit from complexity, by adding some extra axioms, chosen as "natural" as possible. A first such axiom, very natural, is as follows:

**Proposition 11.2.** *For an easy quantum group $G = (G_N)$, coming from a category of partitions $D \subset P$, the following conditions are equivalent:*

(1) $G_{N-1} = G_N \cap U_{N-1}^+$, *via the embedding* $U_{N-1}^+ \subset U_N^+$ *given by* $u \to diag(u, 1)$.
(2) $G_{N-1} = G_N \cap U_{N-1}^+$, *via the $N$ possible diagonal embeddings* $U_{N-1}^+ \subset U_N^+$.
(3) *$D$ is stable under the operation which consists in removing blocks.*

*If these conditions are satisfied, we say that $G = (G_N)$ is "uniform".*

*Proof.* We use here the general easiness theory from section 7 above.

(1) $\iff$ (2) This is something standard, coming from the inclusion $S_N \subset G_N$, which makes everything $S_N$-invariant. The result follows as well from the proof of (1) $\iff$ (3) below, which can be converted into a proof of (2) $\iff$ (3), in the obvious way.

(1) $\iff$ (3) Given a subgroup $K\subset U_{N-1}^+$, with fundamental corepresentation $u$, consider the $N\times N$ matrix $v=diag(u,1)$. Our claim is that for any $\pi\in P(k)$ we have:

$$\xi_\pi\in Fix(v^{\otimes k})\iff\xi_{\pi'}\in Fix(v^{\otimes k'}),\forall\pi'\in P(k'),\pi'\subset\pi$$

In order to prove this, we must study the condition on the left. We have:

$$\begin{aligned}
&\xi_\pi\in Fix(v^{\otimes k})\\
\iff&(v^{\otimes k}\xi_\pi)_{i_1\dots i_k}=(\xi_\pi)_{i_1\dots i_k},\forall i\\
\iff&\sum_j(v^{\otimes k})_{i_1\dots i_k,j_1\dots j_k}(\xi_\pi)_{j_1\dots j_k}=(\xi_\pi)_{i_1\dots i_k},\forall i\\
\iff&\sum_j\delta_\pi(j_1,\dots,j_k)v_{i_1j_1}\dots v_{i_kj_k}=\delta_\pi(i_1,\dots,i_k),\forall i
\end{aligned}$$

Now let us recall that our corepresentation has the special form $v=diag(u,1)$. We conclude from this that for any index $a\in\{1,\dots,k\}$, we must have:

$$i_a=N\implies j_a=N$$

With this observation in hand, if we denote by $i',j'$ the multi-indices obtained from $i,j$ obtained by erasing all the above $i_a=j_a=N$ values, and by $k'\leq k$ the common length of these new multi-indices, our condition becomes:

$$\sum_{j'}\delta_\pi(j_1,\dots,j_k)(v^{\otimes k'})_{i'j'}=\delta_\pi(i_1,\dots,i_k),\forall i$$

Here the index $j$ is by definition obtained from $j'$ by filling with $N$ values. In order to finish now, we have two cases, depending on $i$, as follows:

Case 1. Assume that the index set $\{a|i_a=N\}$ corresponds to a certain subpartition $\pi'\subset\pi$. In this case, the $N$ values will not matter, and our formula becomes:

$$\sum_{j'}\delta_\pi(j'_1,\dots,j'_{k'})(v^{\otimes k'})_{i'j'}=\delta_\pi(i'_1,\dots,i'_{k'})$$

Case 2. Assume now the opposite, namely that the set $\{a|i_a=N\}$ does not correspond to a subpartition $\pi'\subset\pi$. In this case the indices mix, and our formula reads:

$$0=0$$

Thus, we are led to $\xi_{\pi'}\in Fix(v^{\otimes k'})$, for any subpartition $\pi'\subset\pi$, as claimed.

Now with this claim in hand, the result follows from Tannakian duality. $\square$

At the level of the basic examples, from Proposition 11.1 above, the classical and free quantum groups are uniform, while the half-liberations are not. Indeed, this can be seen either with categories of partitions, or with intersections, the point in the half-classical case being that the relations $abc=cba$, when applied to the coefficients of a matrix of type $v=diag(u,1)$, collapse with $c=1$ to the usual commutation relations $ab=ba$.

For classification purposes the uniformity axiom is something very natural and useful, substantially cutting from complexity, and we have the following result, from [38]:

**Theorem 11.3.** *The classical and free uniform orthogonal easy quantum groups, with inclusions between them, are as follows:*

$$\begin{tikzcd}
& H_N^+ \arrow[rr] & & O_N^+ \\
S_N^+ \arrow[rr] \arrow[ur] & & B_N^+ \arrow[ur] & \\
& H_N \arrow[rr] \arrow[uu, dotted] & & O_N \arrow[uu, dotted] \\
S_N \arrow[rr] \arrow[ur] \arrow[uu, dotted] & & B_N \arrow[ur] \arrow[uu, dotted] &
\end{tikzcd}$$

*Moreover, this is an intersection/easy generation diagram, in the sense that for any of its square subdiagrams $P \subset Q, R \subset S$ we have $P = Q \cap R$ and $\{Q, R\} = S$.*

*Proof.* We know that the quantum groups in the statement are indeed easy and uniform, the corresponding categories of partitions being as follows:

$$\begin{tikzcd}
& NC_{even} \arrow[dl] \arrow[dd, dotted] & & NC_2 \arrow[ll] \arrow[dl] \arrow[dd, dotted] \\
NC \arrow[dd, dotted] & & NC_{12} \arrow[ll] \arrow[dd, dotted] & \\
& P_{even} \arrow[dl] & & P_2 \arrow[ll] \arrow[dl] \\
P & & P_{12} \arrow[ll] &
\end{tikzcd}$$

Since this latter diagram is an intersection and generation diagram, we conclude that we have an intersection and easy generation diagram of quantum groups, as stated.

Regarding now the classification, consider an easy quantum group $S_N \subset G_N \subset O_N$. This most come from a category $P_2 \subset D \subset P$, and if we assume $G = (G_N)$ to be uniform, then $D$ is uniquely determined by the subset $L \subset \mathbb{N}$ consisting of the sizes of the blocks of the partitions in $D$. Our claim is that the admissible sets are as follows:

(1) $L = \{2\}$, producing $O_N$.
(2) $L = \{1, 2\}$, producing $B_N$.
(3) $L = \{2, 4, 6, \ldots\}$, producing $H_N$.
(4) $L = \{1, 2, 3, \ldots\}$, producing $S_N$.

In one sense, this follows from our easiness results for $O_N,B_N,H_N,S_N$. In the other sense now, assume that $L\subset\mathbb N$ is such that the set $P_L$ consisting of partitions whose sizes of the blocks belong to $L$ is a category of partitions. We know from the axioms of the categories of partitions that the semicircle $\cap$ must be in the category, so we have $2\in L$. We claim that the following conditions must be satisfied as well:

$$k,l\in L,\ k>l\implies k-l\in L$$

$$k\in L,\ k\geq 2\implies 2k-2\in L$$

Indeed, we will prove that both conditions follow from the axioms of the categories of partitions. Let us denote by $b_k\in P(0,k)$ the one-block partition:

$$b_k=\begin{Bmatrix}\sqcap\!\!\sqcap&\ldots&\sqcap\\ 1\ \ 2&\ldots&k\end{Bmatrix}$$

For $k>l$, we can write $b_{k-l}$ in the following way:

$$b_{k-l}=\begin{Bmatrix}\sqcap\!\!\sqcap&\ldots&\ldots&\ldots&\ldots&\sqcap\\ 1\ \ 2&\ldots&l&l+1&\ldots&k\\ \sqcup\!\!\sqcup&\ldots&\sqcup&|&\ldots&|\\ &&&1&\ldots&k-l\end{Bmatrix}$$

In other words, we have the following formula:

$$b_{k-l}=(b_l^*\otimes|^{\otimes k-l})b_k$$

Since all the terms of this composition are in $P_L$, we have $b_{k-l}\in P_L$, and this proves our first claim. As for the second claim, this can be proved in a similar way, by capping two adjacent $k$-blocks with a 2-block, in the middle.

With these conditions in hand, we can conclude in the following way:

Case 1. Assume $1\in L$. By using the first condition with $l=1$ we get:

$$k\in L\implies k-1\in L$$

This condition shows that we must have $L=\{1,2,\ldots,m\}$, for a certain number $m\in\{1,2,\ldots,\infty\}$. On the other hand, by using the second condition we get:

$$\begin{aligned}m\in L&\implies 2m-2\in L\\ &\implies 2m-2\leq m\\ &\implies m\in\{1,2,\infty\}\end{aligned}$$

The case $m=1$ being excluded by the condition $2\in L$, we reach to one of the two sets producing the groups $S_N,B_N$.

Case 2. Assume $1\notin L$. By using the first condition with $l=2$ we get:

$$k\in L\implies k-2\in L$$

This condition shows that we must have $L = \{2, 4, \ldots, 2p\}$, for a certain number $p \in \{1, 2, \ldots, \infty\}$. On the other hand, by using the second condition we get:

$$\begin{aligned}
2p \in L &\implies 4p - 2 \in L \\
&\implies 4p - 2 \leq 2p \\
&\implies p \in \{1, \infty\}
\end{aligned}$$

Thus $L$ must be one of the two sets producing $O_N, H_N$, and we are done.

In the free case, $S_N^+ \subset G_N \subset O_N^+$, the situation is quite similar, the admissible sets being once again the above ones, producing this time $O_N^+, B_N^+, H_N^+, S_N^+$. See [38]. $\square$

As already mentioned, when removing the uniformity axiom things become more complicated, and the classification result here, from [38], [117], is as follows:

**Theorem 11.4.** *The classical and free orthogonal easy quantum groups are*

$$\begin{tikzcd}
 & & H_N^+ \arrow[rrr] & & & O_N^+ \\
 & S_N'^+ \arrow[ur] & & & \mathcal{B}_N'^+ \arrow[ur] & \\
S_N^+ \arrow[ur] \arrow[rrr] & & & B_N^+ \arrow[ur] & & \\
 & & H_N \arrow[rrr] \arrow[uuu, dotted] & & & O_N \arrow[uuu, dotted] \\
 & S_N' \arrow[ur] & & & B_N' \arrow[ur] & \\
S_N \arrow[ur] \arrow[rrr] \arrow[uuu, dotted] & & & B_N \arrow[ur] \arrow[uuu, dotted] & &
\end{tikzcd}$$

*with $S_N' = S_N \times \mathbb{Z}_2$, $B_N' = B_N \times \mathbb{Z}_2$, and with $S_N'^+, \mathcal{B}_N'^+$ being their liberations, where $\mathcal{B}_N'^+$ stands for the two possible such liberations, $B_N'^+ \subset B_N''^+$.*

*Proof.* The idea here is that of jointly classifying the "classical" categories of partitions $P_2 \subset D \subset P$, and the "free" ones $NC_2 \subset D \subset NC$. At the classical level this leads to 2 more groups, namely $S_N', B_N'$. See [38]. At the free level we obtain 3 more quantum groups, $S_N'^+, B_N'^+, B_N''^+$, with the inclusion $B_N'^+ \subset B_N''^+$ being best thought of as coming from an inclusion $B_N' \subset B_N''$, which happens to be an isomorphism. See [38]. $\square$

Now back to the easy uniform case, the classification here remains a quite technical topic. The problem comes from the following negative result:

**Proposition 11.5.** *The cubic diagram from Theorem 11.3, and its unitary analogue,*

$$\begin{tikzcd}
 & K_N^+ \arrow[rr] & & U_N^+ \\
S_N^+ \arrow[ur] \arrow[rr] & & C_N^+ \arrow[ur] & \\
 & K_N \arrow[uu, dotted] \arrow[rr] & & U_N \arrow[uu, dotted] \\
S_N \arrow[uu, dotted] \arrow[ur] \arrow[rr] & & C_N \arrow[uu, dotted] \arrow[ur] &
\end{tikzcd}$$

*cannot be merged, without degeneration, into a 4-dimensional cubic diagram.*

*Proof.* All this is a bit philosophical, with the problem coming from the "taking the bistochastic version" operation, and more specifically, from the following equalities:

$$H_N \cap C_N = K_N \cap C_N = S_N$$

Indeed, these equalities do hold, and so the 3D cube obtained by merging the classical faces of the orthogonal and unitary cubes is something degenerate, as follows:

$$\begin{tikzcd}
 & K_N \arrow[rr] & & U_N \\
S_N \arrow[ur] \arrow[rr] & & C_N \arrow[ur] & \\
 & H_N \arrow[uu] \arrow[rr] & & O_N \arrow[uu] \\
S_N \arrow[uu] \arrow[ur] \arrow[rr] & & B_N \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

Thus, the 4D cube, having this 3D cube as one of its faces, is degenerate too. □

Summarizing, when positioning ourselves at $U_N^+$, we have 4 natural directions to be followed, namely taking the classical, discrete, real and bistochastic versions. And the problem is that, while the first three operations are "good", the fourth one is "bad".

In order to fix this problem, in a useful and efficient way, the natural choice is that of slashing the bistochastic quantum groups $B_N, B_N^+, C_N, C_N^+$, which are rather secondary objects anyway, as well the quantum permutation groups $S_N, S_N^+$.

In order to formulate now our second general axiom, doing the job, consider the cube $T_N = \mathbb{Z}_2^N$, regarded as diagonal torus of $O_N$. We have then:

**Proposition 11.6.** *For an easy quantum group $G=(G_N)$, coming from a category of partitions $D\subset P$, the following conditions are equivalent:*

1. *$T_N\subset G_N$.*
2. *$H_N\subset G_N$.*
3. *$D\subset P_{even}$.*

*If these conditions are satisfied, we say that $G_N$ is "twistable".*

*Proof.* We use the general easiness theory from section 7 above.

(1) $\iff$ (2) Here it is enough to check that the easy envelope $T_N'$ of the cube equals the hyperoctahedral group $H_N$. But this follows from:

$$T_N'=<T_N,S_N>'=H_N'=H_N$$

(2) $\iff$ (3) This follows by functoriality, from the fact that $H_N$ comes from the category of partitions $P_{even}$, that we know from section 10 above. $\square$

The teminology in the above result comes from the fact that, assuming $D\subset P_{even}$, we can indeed twist $G_N$, into a certain quizzy quantum group $\bar{G}_N$. We refer to section 7 above to full details regarding the construction $G_N\to\bar{G}_N$. In what follows we will not need this twisting procedure, and we will just use Proposition 11.6 as it is, as a statement providing us with a simple and natural condition to be imposed on $G_N$. In practice now, imposing this second axiom leads to something nice, namely:

**Theorem 11.7.** *The basic quantum unitary and quantum reflection groups, from Proposition 11.1 above, which are uniform and twistable, are as follows,*

$$\begin{tikzcd}
 & K_N^+ \arrow[rr] & & U_N^+ \\
H_N^+ \arrow[rr] \arrow[ur] & & O_N^+ \arrow[ur] & \\
 & K_N \arrow[rr] \arrow[uu] & & U_N \arrow[uu] \\
H_N \arrow[rr] \arrow[ur] \arrow[uu] & & O_N \arrow[ur] \arrow[uu] &
\end{tikzcd}$$

*and this is an intersection and easy generation diagram.*

*Proof.* The first assertion comes from discussion after Proposition 11.2, telling us that the uniformity condition eliminates $O_N^*,U_N^*,H_N^*,K_N^*$. Also, the twistability condition eliminates $B_N,B_N^+,C_N,C_N^+$ and $S_N,S_N^+$. Thus, we are left with the 8 quantum groups in the statement, which are indeed easy, coming from the following categories:

$$\begin{tikzcd}[row sep=small, column sep=small]
 & \mathcal{NC}_{even} \arrow[dl] \arrow[dd] & & \mathcal{NC}_2 \arrow[ll] \arrow[dl] \arrow[dd] \\
NC_{even} \arrow[dd] & & NC_2 \arrow[ll] \arrow[dd] & \\
 & \mathcal{P}_{even} \arrow[dl] & & \mathcal{P}_2 \arrow[ll] \arrow[dl] \\
P_{even} & & P_2 \arrow[ll] &
\end{tikzcd}$$

Since this latter diagram is an intersection and generation diagram, we conclude that we have an intersection and easy generation diagram of quantum groups, as stated. □

In the general case now, where we have an arbitrary uniform and twistable easy quantum group, this quantum group appears by definition as follows:

$$H_N \subset G_N \subset U_N^+$$

Thus, our quantum group can be imagined as sitting inside the above cube. The point now is that, by using the operations $\cap$ and $\{\, ,\}$, we can in principle "project" it on the faces and edges of the cube, and then use some kind of 3D orientation coming from this, in order to deduce some structure and classification results. Let us start with:

**Definition 11.8.** *Associated to any twistable easy quantum group*

$$H_N \subset G_N \subset U_N^+$$

*are its classical, discrete and real versions, given by the following formulae,*

$$G_N^c = G_N \cap U_N$$

$$G_N^d = G_N \cap K_N^+$$

$$G_N^r = G_N \cap O_N^+$$

*as well as its free, smooth and unitary versions, given by the following formulae,*

$$G_N^f = \{G_N, H_N^+\}$$

$$G_N^s = \{G_N, O_N\}$$

$$G_N^u = \{G_N, K_N\}$$

*where $\cap$ and $\{\, ,\}$ are respectively the intersection and easy generation operations.*

In this definition the classical, real and unitary versions are something quite standard. Regarding the discrete and smooth versions, here we have no abstract justification for our terminology, due to the fact that easy quantum groups do not have known differential geometry. However, in the classical case, where $G_N \subset U_N$, our constructions produce indeed discrete and smooth versions, and this is where our terminology comes from. Finally, regarding the free version, this comes once again from the known examples.

To be more precise, regarding the free version, the various results that we have show that the liberation operation $G_N \to G_N^+$ usually appears via the formula:

$$G_N^+ = \{G_N, S_N^+\}$$

This formula expresses the fact that the category of partitions of $G_N^+$ is obtained from the one of $G_N$ by removing the crossings. But in the twistable setting, where we have by definition $H_N \subset G_N$, this is the same as setting:

$$G_N^+ = \{G_N, H_N^+\}$$

All this is of course a bit theoretical, and this is why we use the symbol $f$ for free versions in the above sense, and keep $+$ for well-known, studied liberations.

In relation now with our questions, and our 3D plan, we can now formulate:

**Proposition 11.9.** *Given an intermediate quantum group $H_N \subset G_N \subset U_N^+$, we have a diagram of closed subgroups of $U_N^+$, obtained by inserting*

$$\begin{tikzcd}
 & & G_N^f & & \\
 & & & G_N^u & \\
G_N^d \arrow[rr] & & G_N \arrow[uu] \arrow[ur] \arrow[rr] & & G_N^s \\
 & G_N^r \arrow[ur] & & & \\
 & & G_N^c \arrow[uu] & &
\end{tikzcd}
\quad \dashrightarrow \quad
\begin{tikzcd}
 & K_N^+ \arrow[rr] & & U_N^+ \\
H_N^+ \arrow[ur] \arrow[rr] & & O_N^+ \arrow[ur] & \\
 & K_N \arrow[uu] \arrow[rr] & & U_N \arrow[uu] \\
H_N \arrow[uu] \arrow[ur] \arrow[rr] & & O_N \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

*in the obvious way, with each $G_N^x$ belonging to the main diagonal of each face.*

*Proof.* The fact that we have indeed the diagram of inclusions on the left is clear from Definition 11.8. Regarding now the insertion procedure, consider any of the faces of the cube, denoted $P \subset Q, R \subset S$. Our claim is that the corresponding quantum group $G = G_N^x$ can be inserted on the corresponding main diagonal $P \subset S$, as follows:

$$\begin{tikzcd}
Q \arrow[rr] & & S \\
 & G \arrow[ur] & \\
P \arrow[uu] \arrow[ur] \arrow[rr] & & R \arrow[uu]
\end{tikzcd}$$

We have to check here a total of $6 \times 2 = 12$ inclusions. But, according to Definition 11.8, these inclusions that must be checked are as follows:

(1) $H_N \subset G_N^c \subset U_N$, where $G_N^c = G_N \cap U_N$.

(2) $H_N \subset G_N^d \subset K_N^+$, where $G_N^d = G_N \cap K_N^+$.

(3) $H_N \subset G_N^r \subset O_N^+$, where $G_N^r = G_N \cap O_N^+$.

(4) $H_N^+ \subset G_N^f \subset U_N^+$, where $G_N^f = \{G_N, H_N^+\}$.

(5) $O_N \subset G_N^s \subset U_N^+$, where $G_N^s = \{G_N, O_N\}$.

(6) $K_N \subset G_N^u \subset U_N^+$, where $G_N^u = \{G_N, K_N\}$.

All these statements being trivial from the definition of $\cap$ and $\{\,,\}$, and from our assumption $H_N \subset G_N \subset U_N^+$, our insertion procedure works indeed, and we are done. $\square$

In order now to complete the diagram, we have to project as well $G_N$ on the edges of the cube. For this purpose we can basically assume, by replacing $G_N$ with each of its 6 projections on the faces, that $G_N$ actually lies on one of the six faces. The technical result that we will need here is as follows:

**Proposition 11.10.** *Given an intersection and easy generation diagram $P \subset Q, R \subset S$ and an intermediate easy quantum group $P \subset G \subset S$, as follows,*

$$\begin{tikzcd}
Q \arrow[rr] & & S \\
& G \arrow[ur] & \\
P \arrow[uu] \arrow[ur] \arrow[rr] & & R \arrow[uu]
\end{tikzcd}$$

*we can extend this diagram into a diagram as follows:*

$$\begin{tikzcd}
Q \arrow[r] & \{G,Q\} \arrow[r] & S \\
G \cap Q \arrow[u] \arrow[r] & G \arrow[u] \arrow[r] & \{G,R\} \arrow[u] \\
P \arrow[u] \arrow[r] & G \cap R \arrow[u] \arrow[r] & R \arrow[u]
\end{tikzcd}$$

*In addition, $G$ "slices the square", in the sense that this is an intersection and easy generation diagram, precisely when $G = \{G \cap Q, G \cap R\}$ and $G = \{G,Q\} \cap \{G,R\}$.*

*Proof.* This is indeed clear from definitions, because the intersection and easy generation conditions are automatic for the upper left and lower right squares, and so are half of the intersection and easy generation conditions for the lower left and upper right squares. Thus, we are left with two conditions only, which are those in the statement. □

Now back to 3 dimensions, and to the cube, we have the following result:

**Proposition 11.11.** *Assuming that $H_N \subset G_N \subset U_N^+$ satisfies the conditions*

$$G_N^{cs} = G_N^{sc} \quad , \quad G_N^{cu} = G_N^{uc} \quad , \quad G_N^{df} = G_N^{fd}$$
$$G_N^{du} = G_N^{ud} \quad , \quad G_N^{rf} = G_N^{fr} \quad , \quad G_N^{rs} = G_N^{sr}$$

*the diagram in Proposition 11.9 can be completed, via the construction in Proposition 11.10, into a diagram dividing the cube along the* 3 *coordinates axes, into* 8 *small cubes.*

*Proof.* We have to prove that the 12 projections on the edges are well-defined, with the problem coming from the fact that each of these projections can be defined in 2 possible ways, depending on the face that we choose first.

The verification goes as follows:

(1) Regarding the 3 edges emanating from $H_N$, the result here follows from:

$$G_N^{cd} = G_N^{dc} = G_N \cap K_N$$
$$G_N^{cr} = G_N^{rc} = G_N \cap O_N$$
$$G_N^{dr} = G_N^{rd} = G_N \cap H_N^+$$

These formulae are indeed all trivial, of type:

$$(G \cap Q) \cap R = (G \cap R) \cap Q = G \cap P$$

(2) Regarding the 3 edges landing into $U_N^+$, the result here follows from:

$$G_N^{fs} = G_N^{sf} = \{G_N, O_N^+\}$$
$$G_N^{fu} = G_N^{uf} = \{G_N, K_N^+\}$$
$$G_N^{su} = G_N^{us} = \{G_N, U_N\}$$

These formulae are once again trivial, of type:

$$\{\{G, Q\}, R\} = \{\{G, R\}, Q\} = \{G, S\}$$

(3) Finally, regarding the remaining 6 edges, not emanating from $H_N$ or landing into $U_N^+$, here the result follows from our assumptions in the statement. □

We are not done yet, because nothing guarantees that we obtain in this way an intersection and easy generation diagram.

Thus, we must add more axioms, as follows:

**Theorem 11.12.** *Assume that $H_N\subset G_N\subset U_N^+$ satisfies the following conditions, where by "intermediate" we mean in each case "parallel to its neighbors":*

(1) *The 6 compatibility conditions in Proposition 11.11 above,*
(2) *$G_N^c,G_N,G_N^f$ slice the classical/intermediate/free faces,*
(3) *$G_N^d,G_N,G_N^s$ slice the discrete/intermediate/smooth faces,*
(4) *$G_N^r,G_N,G_N^u$ slice the real/intermediate/unitary faces,*

*Then $G_N$ "slices the cube", in the sense that the diagram obtained in Proposition 11.11 above is an intersection and easy generation diagram.*

*Proof.* This follows indeed from Proposition 11.10 and Proposition 11.11 above. $\square$

It is quite clear that $G_N$ can be reconstructed from its edge projections, so in order to do the classification, we first need a "coordinate system". Common sense would suggest to use the one emanating from $H_N$, or perhaps the one landing into $U_N^+$. However, technically speaking, best is to use the coordinate system based at $O_N$, highlighted below:

$$\begin{tikzcd}
 & K_N^+ \arrow[rr] & & U_N^+ \\
H_N^+ \arrow[ur] \arrow[rr] & & O_N^+ \arrow[ur] & \\
 & K_N \arrow[uu] \arrow[rr] & & U_N \arrow[uu] \\
H_N \arrow[uu] \arrow[ur] \arrow[rr, Rightarrow] & & O_N \arrow[uu, Rightarrow] \arrow[ur, Rightarrow] &
\end{tikzcd}$$

This choice comes from the fact that the classification result for $O_N\subset O_N^+$, explained below, is something very simple. And this is not the case with the results for $H_N\subset H_N^+$ and for $U_N\subset U_N^+$, from [108], [117] which are quite complicated, with uncountably many solutions, in the general non-uniform case. As for the result for $K_N\subset K_N^+$, this is not available yet, but it is known that there are uncountably many solutions here as well.

So, here is now the key result, from [41], dealing with the vertical direction:

**Theorem 11.13.** *There is only one proper intermediate easy quantum group*

$$O_N\subset G_N\subset O_N^+$$

*namely the quantum group $O_N^*$, which is not uniform.*

*Proof.* We must compute here the categories of pairings $NC_2\subset D\subset P_2$, and this can be done via some standard combinatorics, in three steps, as follows:

(1) Let $\pi\in P_2-NC_2$, having $s\geq 4$ strings. Our claim is that:

– If $\pi\in P_2-P_2^*$, there exists a semicircle capping $\pi'\in P_2-P_2^*$.

– If $\pi \in P_2^* - NC_2$, there exists a semicircle capping $\pi' \in P_2^* - NC_2$.

Indeed, both these assertions can be easily proved, by drawing pictures.

(2) Consider now a partition $\pi \in P_2(k,l) - NC_2(k,l)$. Our claim is that:

– If $\pi \in P_2(k,l) - P_2^*(k,l)$ then $<\pi>= P_2$.
– If $\pi \in P_2^*(k,l) - NC_2(k,l)$ then $<\pi>= P_2^*$.

This can be indeed proved by recurrence on the number of strings, $s = (k+l)/2$, by using (1), which provides us with a descent procedure $s \to s-1$, at any $s \geq 4$.

(3) Finally, assume that we are given an easy quantum group $O_N \subset G \subset O_N^+$, coming from certain sets of pairings $D(k,l) \subset P_2(k,l)$. We have three cases:

– If $D \not\subset P_2^*$, we obtain $G = O_N$.
– If $D \subset P_2, D \not\subset NC_2$, we obtain $G = O_N^*$.
– If $D \subset NC_2$, we obtain $G = O_N^+$.

Thus, we have proved the uniquess result. As for the non-uniformity of the unique solution, $O_N^*$, this is something that we already know, from Theorem 11.7 above. □

The above result is something quite remarkable, and it is actually believed that the result could still hold, without the easiness assumption. We refer here to [18]. As already mentioned, the related inclusions $H_N \subset H_N^+$ and $U_N \subset U_N^+$, studied in [108] and [117], are far from being maximal, having uncountably many intermediate objects, and the same is known to hold for $K_N \subset K_N^+$. There are many interesting open questions here. It is conjectured for instance that there should be a contravariant duality $H_N^\times \leftrightarrow U_N^\times$, mapping the family and series from [117] to the series and family from [127].

Here is now another basic result that we will need, in order to perform our classification work here, dealing this time with the "discrete vs. continuous" direction:

**Theorem 11.14.** *There are no proper intermediate easy groups*

$$H_N \subset G_N \subset O_N$$

*except for $H_N, O_N$ themselves.*

*Proof.* We must prove that there are no proper intermediate categories $P_2 \subset D \subset P_{even}$. But this can done via some combinatorics, in the spirit of the proof of Theorem 11.3, and with the result itself coming from Theorem 11.4. For full details here, see [38]. □

As a comment here, the inclusion $H_N^+ \subset O_N^+$ is maximal as well, as explained once again in [38]. As for the complex versions of these results, regarding the inclusions $K_N \subset U_N$ and $K_N^+ \subset U_N^+$, here the classification, in the non-uniform case, is available from [127]. Summarizing, we have here once again something very basic and fundamental, providing some evidence for a kind of general "discrete vs. continuous" dichotomy.

Finally, here is a third and last result that we will need, for our classification work here, regarding the missing direction, namely the "real vs. complex" one:

**Theorem 11.15.** *The proper intermediate easy groups*

$$O_N \subset G_N \subset U_N$$

*are the groups $\mathbb{Z}_r O_N$ with $r \in \{2, 3, \ldots, \infty\}$, which are not uniform.*

*Proof.* This is standard and well-known, from [127], the proof being as follows:

(1) Our first claim is that the group $\mathbb{T}O_N \subset U_N$ is easy, the corresponding category of partitions being the subcategory $\bar{P}_2 \subset P_2$ consisting of the pairings having the property that when flatenning, we have the global formula $\#\circ = \#\bullet$.

(2) Indeed, if we denote the standard corepresentation by $u = zv$, with $z \in \mathbb{T}$ and with $v = \bar{v}$, then in order to have $Hom(u^{\otimes k}, u^{\otimes l}) \neq \emptyset$, the $z$ variabes must cancel, and in the case where they cancel, we obtain the same Hom-space as for $O_N$.

Now since the cancelling property for the $z$ variables corresponds precisely to the fact that $k, l$ must have the same numbers of $\circ$ symbols minus $\bullet$ symbols, the associated Tannakian category must come from the category of pairings $\bar{P}_2 \subset P_2$, as claimed.

(3) Our second claim is that, more generally, the group $\mathbb{Z}_r O_N \subset U_N$ is easy, with the corresponding category $P_2^r \subset P_2$ consisting of the pairings having the property that when flatenning, we have the global formula $\#\circ = \#\bullet\,(r)$.

(4) Indeed, this is something that we already know at $r = 1, \infty$, where the group in question is $O_N, \mathbb{T}O_N$. The proof in general is similar, by writing $u = zv$ as above.

(5) Let us prove now the converse, stating that the above groups $O_N \subset \mathbb{Z}_r O_N \subset U_N$ are the only intermediate easy groups $O_N \subset G \subset U_N$. According to our conventions for the easy quantum groups, which apply of course to the classical case, we must compute the following intermediate categories of pairings:

$$\mathcal{P}_2 \subset D \subset P_2$$

(6) So, assume that we have such a category, $D \neq \mathcal{P}_2$, and pick an element $\pi \in D - \mathcal{P}_2$, assumed to be flat. We can modify $\pi$, by performing the following operations:

– First, we can compose with the basic crossing, in order to assume that $\pi$ is a partition of type $\cap \ldots\ldots \cap$, consisting of consecutive semicircles. Our assumption $\pi \notin \mathcal{P}_2$ means that at least one semicircle is colored black, or white.

– Second, we can use the basic mixed-colored semicircles, and cap with them all the mixed-colored semicircles. Thus, we can assume that $\pi$ is a nonzero partition of type $\cap \ldots\ldots \cap$, consisting of consecutive black or white semicircles.

– Third, we can rotate, as to assume that $\pi$ is a partition consisting of an upper row of white semicircles, $\cup \ldots\ldots \cup$, and a lower row of white semicircles, $\cap \ldots\ldots \cap$. Our assumption $\pi \notin \mathcal{P}_2$ means that this latter partition is nonzero.

(7) For $a,b\in\mathbb N$ consider the partition consisting of an upper row of $a$ white semicircles, and a lower row of $b$ white semicircles, and set:

$$\mathcal C=\left\{\pi_{ab}\Big|a,b\in\mathbb N\right\}\cap D$$

According to the above we have $\pi\in<\mathcal C>$. The point now is that we have:

– There exists $r\in\mathbb N\cup\{\infty\}$ such that $\mathcal C$ equals the following set:

$$\mathcal C_r=\left\{\pi_{ab}\Big|a=b(r)\right\}$$

This is indeed standard, by using the categorical axioms.

– We have the following formula, with $P_2^r$ being as above:

$$<\mathcal C_r>=P_2^r$$

This is standard as well, by doing some diagrammatic work.

(8) With these results in hand, the conclusion now follows. Indeed, with $r\in\mathbb N\cup\{\infty\}$ being as above, we know from the beginning of the proof that any $\pi\in D$ satisfies:

$$\pi\in<\mathcal C>=<\mathcal C_r>=P_2^r$$

Thus we have an inclusion $D\subset P_2^r$. Conversely, we have as well:

$$P_2^r=<\mathcal C_r>=<\mathcal C>\subset<D>=D$$

Thus we have $D=P_2^r$, and this finishes the proof. See [127]. □

Once again, there are many comments that can be made here, with the whole subject in the easy case being generally covered by the classification results in [127]. As for the non-easy case, there are many interesting things here as well, as for instance the results in [18], stating that $PO_N\subset PU_N$, and $\mathbb TO_N\subset U_N$ as well, are maximal.

We can now formulate a classification result, as follows:

**Theorem 11.16** (Ground zero)**.** *There are exactly eight closed subgroups $G_N\subset U_N^+$ having the following properties,*

1. *Easiness,*
2. *Uniformity,*
3. *Twistability,*
4. *Slicing property,*

*namely the quantum groups $O_N,U_N,H_N,K_N$ and $O_N^+,U_N^+,H_N^+,K_N^+$.*

*Proof.* We already know, from Theorem 11.7 above, that the 8 quantum groups in the statement have indeed the properties (1-4), and form a cube, as follows:

$$\begin{tikzcd}
 & K_N^+ \arrow[rr] & & U_N^+ \\
H_N^+ \arrow[rr] \arrow[ur] & & O_N^+ \arrow[ur] & \\
 & K_N \arrow[rr] \arrow[uu] & & U_N \arrow[uu] \\
H_N \arrow[rr] \arrow[uu] \arrow[ur] & & O_N \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

Conversely now, assuming that an easy quantum group $G = (G_N)$ has the above properties (2-4), the twistability property, (3), tells us that we have:

$$H_N \subset G_N \subset U_N^+$$

Thus $G_N$ sits inside the cube, and the above discussion applies. To be more precise, by using Theorem 11.13, Theorem 11.14 and Theorem 11.15, along with the uniformity condition, (2), we conclude that the edge projections of $G_N$ must be among the vertices of the cube. Now by using the slicing axiom, (4), we deduce from this that $G_N$ itself must be a vertex of the cube. Thus, we have exactly 8 solutions to our problem, as claimed. $\square$

All this is quite philosophical. Bluntly put, by piling up a number of very natural axioms, namely those of Woronowicz from [148], then our assumption $S^2 = id$, and then the easiness, uniformity, twistability, and slicing properties, we have managed to destroy everything, or almost. The casualities include lots of interesting finite and compact Lie groups, the duals of all finitely generated discrete groups, plus of course lots of interesting quantum groups, which appear not to be strong enough to survive our axioms.

We should mention that the above result is in tune with free probability, and with noncommutative geometry, where the most important quantum groups which appear are precisely the above 8 ones. In what regards free probability, this comes from the various character computations performed in sections 8 and 10 above, which give:

**Theorem 11.17.** *The asymptotic character laws for the* 8 *main quantum groups are*

$$\begin{tikzcd}[row sep=small, column sep=small]
& \mathfrak{B}_t \arrow[rr, dash] & & \Gamma_t \\
\beta_t \arrow[ur, dash] \arrow[rr, dash] & & \gamma_t \arrow[ur, dash] & \\
& B_t \arrow[uu, dash] \arrow[rr, dash] & & G_t \arrow[uu, dash] \\
b_t \arrow[uu, dash] \arrow[ur, dash] \arrow[rr, dash] & & g_t \arrow[uu, dash] \arrow[ur, dash] &
\end{tikzcd}$$

*which are exactly the* 8 *main limiting laws in classical and free probability.*

*Proof.* This is something that we already know, explained in sections 8 and 10 above. To be more precise, the assertion regarding the characters is something which was proved there, and the last assertion, which is a bit informal, comes from the general classical and free probability theory explained as well in sections 8 and 10 above. $\square$

In what regards now noncommutative geometry, the idea is that our 8 main quantum groups correspond to the 4 possible "abstract noncommutative geometries", in the strongest possible sense, which are the real/complex, classical/free ones. In order to explain this, consider the following diagram, consisting of spheres and tori:

$$\begin{tikzcd}[row sep=small, column sep=small]
& \mathbb{T}_N^+ \arrow[rr] & & S^{N-1}_{\mathbb{C},+} \\
T_N^+ \arrow[ur] \arrow[rr] & & S^{N-1}_{\mathbb{R},+} \arrow[ur] & \\
& \mathbb{T}_N \arrow[uu] \arrow[rr] & & S^{N-1}_{\mathbb{C}} \arrow[uu] \\
T_N \arrow[uu] \arrow[ur] \arrow[rr] & & S^{N-1}_{\mathbb{R}} \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

These 4+4 spheres and tori add to the 4+4 unitary and reflection groups that we have, so we have a total of 16 basic geometric objects. But these objects can be arranged, in an obvious way, into 4 quadruplets of type $(S,T,U,K)$, consisting a sphere $S$, a torus $T$, a unitary group $U$, and a reflection group $K$, with relations between them, as follows:

$$\begin{array}{ccc}S&\longleftrightarrow&T\\ \updownarrow&\times&\updownarrow\\ U&\longleftrightarrow&K\end{array}$$

To be more precise, we obtain in this way the quadruplets $(S,T,U,K)$ corresponding to the real/complex, classical/free geometries. As mentioned above, it is possible to do some axiomatization and classification work here, with the conclusion that, under strong combinatorial axioms, including easiness, these 4 geometries are the only ones.

## 12. The standard cube

We discuss here a number of more specialized classification results, for the twistable easy quantum groups, $H_N\subset G\subset U_N^+$, and for more general such intermediate quantum groups. The general idea will be as before, namely that of viewing our quantum group as sitting inside the standard cube, discussed in section 11:

$$\begin{tikzcd}
& K_N^+ \arrow[rr] & & U_N^+ \\
H_N^+ \arrow[ur] \arrow[rr] & & O_N^+ \arrow[ur] & \\
& K_N \arrow[uu] \arrow[rr] & & U_N \arrow[uu] \\
H_N \arrow[uu] \arrow[ur] \arrow[rr] & & O_N \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

Let us first discuss the classification in the easy case, for the lower and upper faces of the cube. Following [127], in the uniform case, the result here is as follows:

**Theorem 12.1.** *The classical and free uniform twistable easy quantum groups are*

$$\begin{tikzcd}
& & K_N^+ \arrow[r] & K_N^{++} \arrow[r] & U_N^+ \\
& H_N^{s+} \arrow[ur] & & & \\
H_N^+ \arrow[ur] \arrow[rrr] & & & O_N^+ \arrow[uur] & \\
& & K_N \arrow[uuu, dotted] \arrow[rr] & & U_N \arrow[uuu, dotted] \\
& H_N^s \arrow[ur] & & & \\
H_N \arrow[uuu, dotted] \arrow[ur] \arrow[rrr] & & & O_N \arrow[uuu, dotted] \arrow[uur] &
\end{tikzcd}$$

*where* $H_s=\mathbb{Z}_s\wr S_N$, $H_N^{s+}=\mathbb{Z}_s\wr_* S_N^+$ *with* $s=4,6,8\ldots$, *and where* $K_N^+=\widetilde{K_N^+}$.

*Proof.* The idea here is that of jointly classifying the "classical" categories of partitions $\mathcal{P}_2\subset D\subset P_{even}$, and the "free" ones $\mathcal{NC}_2\subset D\subset NC_{even}$, under the assumption that the category is stable under the operation which consists in removing blocks. In the classical case, the new solutions appear on the edge $H_N\subset K_N$, and are the complex reflection groups $H_s=\mathbb{Z}_s\wr S_N$ with $s=4,6,8\ldots$, the cases $s=2,\infty$ corresponding respectively to $H_N,K_N$. In the free case we obtain as new solutions the standard liberattions of these groups, namely the quantum groups $H_N^{s+}=\mathbb{Z}_s\wr_* S_N^+$ with $s=4,6,8\ldots$, and we have as well an extra solution, appearing on the edge $K_N^+ \subset U_N^+$, which is the free complexification $\widetilde{K_N^+}$ of the quantum group $K_N^+$, which is easy, and bigger than $K_N^+$. See [127]. □

The above result can be generalized, by lifting both the uniformity and twistablility assumptions, and the result here, which is more technical, is explained in [127].

Another key result is the one from [117], dealing with the front face of the standard cube, the orthogonal one. We first have the folowing result:

**Proposition 12.2.** *The easy quantum groups $H_N \subset G \subset O_N^+$ are as follows,*

$$\begin{tikzcd}
H_N^+ \arrow[rr] & & O_N^+ \\
H_N^{[\infty]} \arrow[u, dotted] & & O_N^* \arrow[u] \\
H_N \arrow[u, dotted] \arrow[rr] & & O_N \arrow[u]
\end{tikzcd}$$

*with the dotted arrows indicating that we have intermediate quantum groups there.*

*Proof.* This is a key result in the classification of easy quantum groups, as follows:

(1) We have a first dichotomy concerning the quantum groups in the statement, namely $H_N \subset G \subset O_N^+$, which must fall into one of the following two classes:

$$O_N \subset G \subset O_N^+$$

$$H_N \subset G \subset H_N^+$$

This comes indeed from the early classification results for easy quantum groups, from [22], [38], [39]. In addition, these early classification results solve as well the first problem, $O_N \subset G \subset O_N^+$, with $G = O_N^*$ being the unique non-trivial solution.

(2) We have then a second dichotomy, concerning the quantum groups which are left, namely $H_N \subset G \subset H_N^+$, which must fall into one of the following two classes:

$$H_N \subset G \subset H_N^{[\infty]}$$

$$H_N^{[\infty]} \subset G \subset H_N^+$$

This comes indeed from various papers, and more specifically from the final classification paper of Raum and Weber [117], where the quantum groups $S_N \subset G \subset H_N^+$ with $G \not\subset H_N^{[\infty]}$ were classified, and shown to contain $H_N^{[\infty]}$. For full details, we refer to [117]. □

Regarding now the case $H_N^{[\infty]} \subset G \subset H_N^+$, the precise result here, from [117], is:

**Proposition 12.3.** *Let $H_N^{[r]}\subset H_N^+$ be the easy quantum group coming from:*

$$\pi_r=\ker\begin{pmatrix}1&\ldots&r&r&\ldots&1\\1&\ldots&r&r&\ldots&1\end{pmatrix}$$

*We have then inclusions of quantum groups as follows,*

$$H_N^+=H_N^{[1]}\supset H_N^{[2]}\supset H_N^{[3]}\supset\ldots\ldots\supset H_N^{[\infty]}$$

*and we obtain in this way all the intermediate easy quantum groups*

$$H_N^{[\infty]}\subset G\subset H_N^+$$

*satisfying the assumption $G\neq H_N^{[\infty]}$.*

*Proof.* Once again, this is something technical, and we refer here to [117]. □

It remains to discuss the easy quantum groups $H_N\subset G\subset H_N^{[\infty]}$, with the endpoints $G=H_N,H_N^{[\infty]}$ included. Once again, we follow here [117]. First, we have:

**Definition 12.4.** *A discrete group generated by real reflections, $g_i^2=1$,*

$$\Gamma=<g_1,\ldots,g_N>$$

*is called uniform if each $\sigma\in S_N$ produces a group automorphism, $g_i\to g_{\sigma(i)}$.*

Consider a uniform reflection group, $\mathbb Z_2^{*N}\to\Gamma\to\mathbb Z_2^N$. We can associate to it a family of subsets $D(k,l)\subset P(k,l)$, which form a category of partitions, as follows:

$$D(k,l)=\left\{\pi\in P(k,l)\Big|\ker\begin{pmatrix}i\\j\end{pmatrix}\leq\pi\implies g_{i_1}\ldots g_{i_k}=g_{j_1}\ldots g_{j_l}\right\}$$

Observe that we have $P_{even}^{[\infty]}\subset D\subset P_{even}$, with the inclusions coming respectively from $\eta\in D$, and from $\Gamma\to\mathbb Z_2^N$. Conversely, given $P_{even}^{[\infty]}\subset D\subset P_{even}$, we can associate to it a uniform reflection group $\mathbb Z_2^{*N}\to\Gamma\to\mathbb Z_2^N$, as follows:

$$\Gamma=\left\langle g_1,\ldots g_N\Big|g_{i_1}\ldots g_{i_k}=g_{j_1}\ldots g_{j_l},\forall i,j,k,l,\ker\begin{pmatrix}i\\j\end{pmatrix}\in D(k,l)\right\rangle$$

As explained in [117], the correspondences $\Gamma\to D$ and $D\to\Gamma$ are bijective, and inverse to each other, at $N=\infty$. We have in fact the following result, from [117]:

**Proposition 12.5.** *We have correspondences between:*

1. *Uniform reflection groups $\mathbb Z_2^{*\infty}\to\Gamma\to\mathbb Z_2^\infty$.*
2. *Categories of partitions $P_{even}^{[\infty]}\subset D\subset P_{even}$.*
3. *Easy quantum groups $G=(G_N)$, with $H_N^{[\infty]}\supset G_N\supset H_N$.*

*Proof.* This is something quite technical, which follows along the lines of the above discussion. As an illustration, if we denote by $\mathbb{Z}_2^{\circ N}$ the quotient of $\mathbb{Z}_2^{*N}$ by the relations of type $abc = cba$ between the generators, we have the following correspondences:

$$\begin{tikzcd}
\mathbb{Z}_2^N \arrow[d, leftrightsquigarrow] & \mathbb{Z}_2^{\circ N} \arrow[l] \arrow[d, leftrightsquigarrow] & \mathbb{Z}_2^{*N} \arrow[l] \arrow[d, leftrightsquigarrow] \\
H_N \arrow[r] & H_N^* \arrow[r] & H_N^{[\infty]}
\end{tikzcd}$$

More generally, for any $s \in \{2, 4, \ldots, \infty\}$, the quantum groups $H_N^{(s)} \subset H_N^{[s]}$ constructed in [22] come from the quotients of $\mathbb{Z}_2^{\circ N} \leftarrow \mathbb{Z}_2^{*N}$ by the relations $(ab)^s = 1$. See [117]. □

We can now formulate a final classification result, as follows:

**Theorem 12.6.** *The easy quantum groups $H_N \subset G \subset O_N^+$ are as follows,*

$$\begin{tikzcd}
H_N^+ \arrow[rr] & & O_N^+ \\
H_N^{[r]} \arrow[u] & & \\
H_N^{[\infty]} \arrow[u] & & O_N^* \arrow[uu] \\
H_N^{\Gamma} \arrow[u] & & \\
H_N \arrow[u] \arrow[rr] & & O_N \arrow[uu]
\end{tikzcd}$$

*with the family $H_N^\Gamma$ covering $H_N, H_N^{[\infty]}$, and with the series $H_N^{[r]}$ covering $H_N^+$.*

*Proof.* This follows indeed from Proposition 12.2, Proposition 12.3 and Proposition 12.5 above. For further details, we refer to the paper of Raum and Weber [117]. □

All the above is quite technical, and can be extended as well, as to cover all the orthogonal easy quantum groups, $S_N \subset G \subset O_N^+$. For full details here, we refer to [117].

Another interesting result, dealing this time with the unitary edge of the standard cube, is the one from [107], [108]. To be more precise, the problem here is that of classifying the intermediate easy quantum groups as follows:

$$U_N \subset G \subset U_N^+$$

A first construction of such quantum groups is as follows:

**Proposition 12.7.** *Associated to any $r \in \mathbb{N}$ is the quantum group $U_N \subset U_N^{(r)} \subset U_N^+$ coming from the category $\mathcal{P}_2^{(r)}$ of matching pairings having the property that $\#\circ = \#\bullet(r)$ holds between the legs of each string. These quantum groups have the following properties:*

1. *At $r = 1$ we obtain the usual unitary group, $U_N^{(1)} = U_N$.*
2. *At $r = 2$ we obtain the half-classical unitary group, $U_N^{(2)} = U_N^*$.*
3. *For any $r|s$ we have an embedding $U_N^{(r)} \subset U_N^{(s)}$.*
4. *In general, we have an embedding $U_N^{(r)} \subset U_N^r \rtimes \mathbb{Z}_r$.*
5. *We have as well a cyclic matrix model $C(U_N^{(r)}) \subset M_r(C(U_N^r))$.*
6. *In this latter model, $\int_{U_N^{(r)}}$ appears as the restriction of $tr_r \otimes \int_{U_N^r}$.*

*Proof.* This is something quite compact, summarizing the various findings from [16], [107]. Here are a few brief explanations on all this:

(1) This is clear from $\mathcal{P}_2^{(1)} = \mathcal{P}_2$, and from a well-known result of Brauer [54].

(2) This is because $\mathcal{P}_2^{(2)}$ is generated by the partitions with implement the relations $abc = cba$ between the variables $\{u_{ij}, u_{ij}^*\}$, used in [49] for constructing $U_N^*$.

(3) This simply follows from $\mathcal{P}_2^{(s)} \subset \mathcal{P}_2^{(r)}$, by functoriality.

(4) This is the original definition of $U_N^{(r)}$, from [16]. We refer to [16] for the exact formula of the embedding, and to [107] for the compatibility with the Tannakian definition.

(5) This is also from [16], more specifically it is an alternative definition for $U_N^{(r)}$.

(6) Once again, this is something from [16], and we will be back to it. □

Let us discuss now the second known construction of unitary quantum groups, from [108]. This construction uses an additive semigroup $D \subset \mathbb{N}$, but as pointed out there, using instead the complementary set $C = \mathbb{N} - D$ leads to several simplifications.

So, let us call "cosemigroup" any subset $C \subset \mathbb{N}$ which is complementary to an additive semigroup, $x, y \notin C \implies x + y \notin C$. The construction from [108] is then:

**Proposition 12.8.** *Associated to any cosemigroup $C \subset \mathbb{N}$ is the easy quantum group $U_N \subset U_N^C \subset U_N^+$ coming from the category $\mathcal{P}_2^C \subset P_2^{(\infty)}$ of pairings having the property $\#\circ - \#\bullet \in C$, between each two legs colored $\circ, \bullet$ of two strings which cross. We have:*

1. *For $C = \emptyset$ we obtain the quantum group $U_N^+$.*
2. *For $C = \{0\}$ we obtain the quantum group $U_N^\times$.*
3. *For $C = \{0, 1\}$ we obtain the quantum group $U_N^{**}$.*
4. *For $C = \mathbb{N}$ we obtain the quantum group $U_N^{(\infty)}$.*
5. *For $C \subset C'$ we have an inclusion $U_N^{C'} \subset U_N^C$.*
6. *Each quantum group $U_N^C$ contains each quantum group $U_N^{(r)}$.*

*Proof.* Once again this is something very compact, coming from recent work in [108], with our convention that the semigroup $D\subset\mathbb N$ which is used there is replaced here by its complement $C=\mathbb N-D$. Here are a few explanations on all this:

(1) The assumption $C=\emptyset$ means that the condition $\#\circ-\#\bullet\in C$ can never be applied. Thus, the strings cannot cross, we have $\mathcal P_2^\emptyset=\mathcal{NC}_2$, and so $U_N^\emptyset=U_N^+$.

(2) As explained in [108], here we obtain indeed the quantum group $U_N^\times$, constructed by using the relations $ab^*c=cb^*a$, with $a,b,c\in\{u_{ij}\}$.

(3) This is also explained in [108], with $U_N^{**}$ being the quantum group from [16], which is the biggest whose full projective version, in the sense there, is classical.

(4) Here the assumption $C=\mathbb N$ simply tells us that the condition $\#\circ-\#\bullet\in C$ in the statement is irrelevant. Thus, we have $\mathcal P_2^{\mathbb N}=\mathcal P_2^{(\infty)}$, and so $U_N^{\mathbb N}=U_N^{(\infty)}$.

(5) This is clear by functoriality, because $C\subset C'$ implies $\mathcal P_2^C\subset\mathcal P_2^{C'}$.

(6) This is clear from definitions, and from Proposition 12.7 above. □

We have the following key result, from [108]:

**Theorem 12.9.** *The easy quantum groups $U_N\subset G\subset U_N^+$ are as follows,*

$$U_N\subset\{U_N^{(r)}\}\subset\{U_N^C\}\subset U_N^+$$

*with the series covering $U_N$, and the family covering $U_N^+$.*

*Proof.* This is something non-trivial, and we refer here to [108]. The general idea is that $U_N^{(\infty)}$ produces a dichotomy for the quantum groups in the statement, and this leads, via combinatorial computations, to the series and the family. See [107], [108]. □

Observe that there is an obvious similarity here with the dichotomy for the liberations of $H_N$, coming from [117]. To be more precise, the above-mentioned classification results for the liberations of $H_N,U_N$ have some obvious similarity between them. We have indeed a family followed by a series, and a series followed by a family, and this suggests the existence of a "contravariant duality" between these quantum groups, as follows:

$$\begin{tikzcd}
U_N \arrow[r] \arrow[d, dash, dotted] & U_N^{(r)} \arrow[r] \arrow[d, dash, dotted] & U_N^C \arrow[r] \arrow[d, dash, dotted] & U_N^+ \arrow[d, dash, dotted] \\
H_N^+ & H_N^{[r]} \arrow[l] & H_N^\Gamma \arrow[l] & H_N \arrow[l]
\end{tikzcd}$$

At the first glance, this might sound a bit strange. Indeed, we have some natural and well-established correspondences $H_N\leftrightarrow U_N$ and $H_N^+\leftrightarrow U_N^+$, obtained in one sense by taking the real reflection subgroup, $H = U \cap H_N^+$, and in the other sense by setting $U =< H, U_N >$. Thus, our proposal of duality seems to go the wrong way.

On the other hand, obvious as well is the fact that these correspondences $H_N \leftrightarrow U_N$ and $H_N^+ \leftrightarrow U_N^+$ cannot be extended as to map the series to the series, and the family to the family, because the series/families would have to be "inverted", in order to do so.

Following [9], let us discuss now what happens inside the standard cube, first in the easy case, and then in general. Let us start with the following definition:

**Definition 12.10.** *An easy quantum group $H_N \subset G_N \subset U_N^+$ is called "bi-oriented" if*

$$\begin{tikzcd}
 & G_N^d \arrow[rr] & & G_N \\
G_N^{dr} \arrow[rr] \arrow[ur] & & G_N^r \arrow[ur] & \\
 & G_N^{cd} \arrow[rr] \arrow[uu] & & G_N^c \arrow[uu] \\
H_N \arrow[rr] \arrow[ur] \arrow[uu] & & G_N^{cr} \arrow[ur] \arrow[uu] &
\end{tikzcd}
\quad : \quad
\begin{tikzcd}
 & G_N^{fu} \arrow[rr] & & U_N^+ \\
G_N^{f} \arrow[rr] \arrow[ur] & & G_N^{sf} \arrow[ur] & \\
 & G_N^{u} \arrow[rr] \arrow[uu] & & G_N^{su} \arrow[uu] \\
G_N \arrow[rr] \arrow[ur] \arrow[uu] & & G_N^{s} \arrow[ur] \arrow[uu] &
\end{tikzcd}$$

*are both intersection and easy generation diagrams.*

Observe that the diagram on the left is automatically an intersection diagram, and that the diagram on the right is automatically an easy generation diagram.

The question of replacing the slicing axiom with the bi-orientability condition makes sense. In fact, we can even talk about weaker axioms, as follows:

**Definition 12.11.** *An easy quantum group $H_N \subset G_N \subset U_N^+$ is called "oriented" if*

$$G_N = \{G_N^{cd}, G_N^{cr}, G_N^{dr}\}$$

$$G_N = G_N^{fs} \cap G_N^{fu} \cap G_N^{su}$$

*and "weakly oriented" if the following weaker conditions hold,*

$$G_N = \{G_N^c, G_N^d, G_N^r\}$$

$$G_N = G_N^f \cap G_N^s \cap G_N^u$$

*where the various versions are those in section 11 above.*

In order to prove the uniqueness, in the bi-orientable case, we can still proceed as in the proof of the Ground Zero theorem, but we are no longer allowed to use the coordinate system there, based at $O_N$. To be more precise, we must use the 2 coordinate systems highlighted below, both taken in some weak sense, weaker than the slicing:

$$\begin{tikzcd}
 & K_N^+ \arrow[rr, Rightarrow] & & U_N^+ \\
H_N^+ \arrow[ur] \arrow[rr] & & O_N^+ \arrow[ur, Rightarrow] & \\
 & K_N \arrow[uu] \arrow[rr] & & U_N \arrow[uu, Rightarrow] \\
H_N \arrow[uu, Rightarrow] \arrow[ur, Rightarrow] \arrow[rr, Rightarrow] & & O_N \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

Skipping some details here, all this is viable, by using the known "edge results" surveyed above, and with the key fact being that the quantum group $H_N^{[\infty]}$ from [117] has no orthogonal counterpart. Thus, we obtain in principle some improvements of the Ground Zero theorem, under the bi-orientability assumption, and more generally under the orientability assumption. As for the weak orientability assumption, the situation here is more tricky, because we would need full "face results", which are not available yet.

Let us discuss now the general, non-easy case. We must find extensions of the notions of uniformity, twistability and orientability. Regarding the uniformity, we have:

**Definition 12.12.** *A series $G=(G_N)$ of closed subgroups $G_N\subset U_N^+$ is called:*

1. *Weakly uniform, if for any $N\in\mathbb{N}$ we have $G_{N-1}=G_N\cap U_{N-1}^+$, with respect to the embedding $U_{N-1}^+\subset U_N^+$ given by $u\to diag(u,1)$.*
2. *Uniform, if for any $N\in\mathbb{N}$ we have $G_{N-1}=G_N\cap U_{N-1}^+$, with respect to the $N$ possible embeddings $U_{N-1}^+\subset U_N^+$, of type $u\to diag(u,1)$.*

Regarding the examples, in the classical case we have substantially more examples than in the easy case, obtained by using the determinant, and its powers:

**Proposition 12.13.** *The following compact groups are uniform,*

1. *The complex reflection groups*
$$H_N^{s,d}=\left\{g\in\mathbb{Z}_s\wr S_N\Big|(\det g)^d=1\right\}$$
*for any values of the parameters $s\in\{1,2,\ldots,\infty\}$ and $d\in\mathbb{N}$, $d|s$,*
2. *The orthogonal group $O_N$, the special orthogonal group $SO_N$, and the series*
$$U_N^d=\left\{g\in U_N\Big|(\det g)^d=1\right\}$$
*of modified unitary groups, with $s\in\{1,2,\ldots,\infty\}$,*

*and so are the bistochastic versions of these groups.*

*Proof.* Both these assertions are clear from definitions. Observe that the groups in (1), which are well-known objects in finite group theory, and more precisely form the series of complex reflection groups, generalize the groups $H_N^s$ from section 10 above, which appear at $d = s$. See [121]. The groups in (2) are well-known as well, in compact Lie group theory, with $U_N^1$ being equal to $SU_N$, and with $U_N^\infty$ being by definition $U_N$ itself. $\square$

In the free case now, corresponding to the condition $S_N^+ \subset G_N \subset U_N^+$, it is widely believed that the only examples are the easy ones. A precise conjecture in this sense, which is a bit more general, valid for any $G_N \subset U_N^+$, states that we should have:

$$< G_N, S_N^+ >= \{G'_N, S_N^+\}$$

Here $G'_N$ denotes as usual the easy envelope of $G_N$, and $\{\,,\}$ is an easy generation operation. This conjecture is probably something quite difficult.

Now back to our questions, we have definitely no new examples in the free case. So, the basic examples will be those that we previously met, namely:

**Proposition 12.14.** *The following free quantum groups are uniform,*

1. *Liberations $H_N^{s+} = \mathbb{Z}_s \wr_* S_N^+$ of the complex reflection groups $H_N^s = \mathbb{Z}_s \wr S_N$,*
2. *Liberations $O_N^+, U_N^+$ of the continuous groups $O_N, U_N$,*

*and so are the bistochastic versions of these quantum groups.*

*Proof.* This is something that we basically know, with the uniformity check for $H_N^{s+}$ being the same as for $S_N^+, H_N^+, K_N^+$, which appear at $s = 1, 2, \infty$. $\square$

We would need a second axiom, such as the twistability condition $T_N \subset G_N$. However, if we look at Proposition 12.14, a condition of type $A_N \subset G_N$ would be more appropriate. In order to comment on this dillema, let us recall from section 11 that "taking the bistochastic version" is a bad direction, geometrically speaking. But the operations "taking the diagonal torus" and "taking the special version", that we are currently discussing, are bad too. Thus, we have 3 bad directions, and so a cube:

**Proposition 12.15.** *We have the following diagram of finite groups,*

$$\begin{tikzcd}
 & S_N \arrow[rr] & & H_N \\
A_N \arrow[ur] \arrow[rr] & & SH_N \arrow[ur] & \\
 & \{1\} \arrow[uu] \arrow[rr] & & T_N \arrow[uu] \\
\{1\} \arrow[uu] \arrow[ur] \arrow[rr] & & ST_N \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

*obtained from $H_N$ by taking bistochastic, special and diagonal versions.*

*Proof.* This is clear, with the operations of taking bistochastic versions, special versions and diagonal subgroups corresponding to going left, backwards, and downwards. □

Observe that the above cube is degenerate on the bottom left, but this is certainly not surprising, because what we are doing here is to combine 3 bad directions.

Now back to our classification questions, the vertices of the above cube are all interesting groups, and assuming that our quantum groups $G_N\subset U_N^+$ contain any of them is something quite natural. Let us just select here three such conditions, as follows:

**Definition 12.16.** *A closed subgroup $G_N\subset U_N^+$ is called:*

1. *Twistable, if $T_N\subset G_N$.*
2. *Homogeneous, if $S_N\subset G_N$.*
3. *Half-homogeneous, if $A_N\subset G_N$.*

Let us go ahead now, and formulate our third and last definition, regarding the orientability axiom. Things are quite tricky here, and we must start as follows:

**Definition 12.17.** *Associated to any closed subgroup $G_N\subset U_N^+$ are its classical, discrete and real versions, given by*

$$G_N^c=G_N\cap U_N$$

$$G_N^d=G_N\cap K_N^+$$

$$G_N^r=G_N\cap O_N^+$$

*as well as its free, smooth and unitary versions, given by*

$$G_N^f=<G_N,H_N^+>$$

$$G_N^s=<G_N,O_N>$$

$$G_N^u=<G_N,K_N>$$

*where $<\,,>$ is the usual, non-easy topological generation operation.*

Observe the difference, and notational clash, with some of the notions from section 11 above. As explained in section 7 above, it is believed that we should have $\{\,,\}=<\,,>$, but this is not clear at all, and the problem comes from this.

A second issue comes when composing the above operations, and more specifically those involving the generation operation, once again due to the conjectural status of the formula $\{\,,\}=<\,,>$. Due to this fact, instead of formulating a result here, we have to formulate a second definition, complementary to Definition 12.7, as follows:

**Definition 12.18.** *Associated to any closed subgroup $G_N\subset U_N^+$ are the mixes of its classical, discrete and real versions, given by*

$$G_N^{cd}=G_N\cap K_N$$
$$G_N^{cr}=G_N\cap O_N^+$$
$$G_N^{dr}=G_N\cap H_N^+$$

*as well as the mixes of its free, smooth and unitary versions, given by*

$$G_N^{fs}=<G_N,O_N^+>$$
$$G_N^{fu}=<G_N,K_N^+>$$
$$G_N^{us}=<G_N,U_N>$$

*where $<,>$ is the usual, non-easy topological generation operation.*

Now back to our orientation questions, the slicing and bi-orientability conditions lead us again into $\{,\}$ vs. $<,>$ troubles, and are therefore rather to be ignored. The orientability conditions from Definition 12.11, however, have the following analogue:

**Definition 12.19.** *A closed subgroup $G_N\subset U_N^+$ is called "oriented" if*

$$G_N=<G_N^{cd},G_N^{cr},G_N^{dr}>$$
$$G_N=G_N^{fs}\cap G_N^{fu}\cap G_N^{su}$$

*and "weakly oriented" if the following conditions hold,*

$$G_N=<G_N^c,G_N^d,G_N^r>$$
$$G_N=G_N^f\cap G_N^s\cap G_N^u$$

*where the various versions are those in Definition 12.17 and Definition 12.18.*

With these notions, our claim is that some classification results are possible:

(1) In the classical case, we believe that the uniform, half-homogeneous, oriented groups are those in Proposition 12.13, with some bistochastic versions excluded. This is of course something quite heavy, well beyond easiness, with the potential tools available for proving such things coming from advanced finite group theory and Lie algebra theory. Our uniformity axiom could play a key role here, when combined with [121], in order to exclude all the exceptional objects which might appear on the way.

(2) In the free case, under similar assumptions, we believe that the solutions should be those in Proposition 12.14, once again with some bistochastic versions excluded. This is something heavy, too, related to the above-mentioned conjecture $<G_N,S_N^+>=\{G_N',S_N^+\}$. Indeed, assuming that we would have such a formula, and perhaps some more formulae of the same type as well, we can in principle work out our way inside the cube, from the edge and face projections to $G_N$ itself, and in this process $G_N$ would become easy.

(3) In the group dual case, the orientability axiom simplifies, because the group duals are discrete in our sense. We believe that the uniform, twistable, oriented group duals should appear as combinations of certain abelian groups, which appear in the classical case, with duals of varieties of real reflection groups, which appear in the real case.

Let us go back now to the cube, and to edge problems, but without the easiness assumption, this time. We have the following result from [18], to start with:

**Theorem 12.20.** *The following inclusions are maximal:*

(1) $\mathbb{T}O_N \subset U_N$.
(2) $PO_N \subset PU_N$.

*Proof.* In order to prove these results, consider as well the group $\mathbb{T}SO_N$.

Observe that we have $\mathbb{T}SO_N = \mathbb{T}O_N$ if $N$ is odd. If $N$ is even the group $\mathbb{T}O_N$ has two connected components, with $\mathbb{T}SO_N$ being the component containing the identity.

Let us denote by $\mathfrak{so}_N, \mathfrak{u}_N$ the Lie algebras of $SO_N, U_N$. It is well-known that $\mathfrak{u}_N$ consists of the matrices $M \in M_N(\mathbb{C})$ satisfying $M^* = -M$, and that:

$$\mathfrak{so}_N = \mathfrak{u}_N \cap M_N(\mathbb{R})$$

Also, it is easy to see that the Lie algebra of $\mathbb{T}SO_N$ is $\mathfrak{so}_N \oplus i\mathbb{R}$.

Step 1. Our first claim is that if $N \geq 2$, the adjoint representation of $SO_N$ on the space of real symmetric matrices of trace zero is irreducible.

Let indeed $X \in M_N(\mathbb{R})$ be symmetric with trace zero. We must prove that the following space consists of all the real symmetric matrices of trace zero:

$$V = span\left\{UXU^t\middle|U \in SO_N\right\}$$

We first prove that $V$ contains all the diagonal matrices of trace zero. Since we may diagonalize $X$ by conjugating with an element of $SO_N$, our space $V$ contains a nonzero diagonal matrix of trace zero. Consider such a matrix:

$$D = diag(d_1, d_2, \ldots, d_N)$$

We can conjugate this matrix by the following matrix:

$$\begin{pmatrix}0&-1&0\\1&0&0\\0&0&I_{N-2}\end{pmatrix} \in SO_N$$

We conclude that our space $V$ contains as well the following matrix:

$$D' = diag(d_2, d_1, d_3, \ldots, d_N)$$

More generally, we see that for any $1 \leq i, j \leq N$ the diagonal matrix obtained from $D$ by interchanging $d_i$ and $d_j$ lies in $V$. Now since $S_N$ is generated by transpositions, it follows that $V$ contains any diagonal matrix obtained by permuting the entries of $D$. But it is well-known that this representation of $S_N$ on the diagonal matrices of trace zero is irreducible, and hence $V$ contains all such diagonal matrices, as claimed.

In order to conclude now, assume that $Y$ is an arbitrary real symmetric matrix of trace zero. We can find then an element $U \in SO_N$ such that $UYU^t$ is a diagonal matrix of trace zero. But we then have $UYU^t \in V$, and hence also $Y \in V$, as desired.

Step 2. Our claim is that the inclusion $\mathbb{T}SO_N \subset U_N$ is maximal in the category of connected compact groups.

Let indeed $G$ be a connected compact group satisfying $\mathbb{T}SO_N \subset G \subset U_N$. Then $G$ is a Lie group. Let $\mathfrak{g}$ denote its Lie algebra, which satisfies:

$$\mathfrak{so}_N \oplus i\mathbb{R} \subset \mathfrak{g} \subset \mathfrak{u}_N$$

Let $ad_G$ be the action of $G$ on $\mathfrak{g}$ obtained by differentiating the adjoint action of $G$ on itself. This action turns $\mathfrak{g}$ into a $G$-module. Since $SO_N \subset G$, $\mathfrak{g}$ is also a $SO_N$-module.

Now if $G \neq \mathbb{T}SO_N$, then since $G$ is connected we must have $\mathfrak{so}_N \oplus i\mathbb{R} \neq \mathfrak{g}$. It follows from the real vector space structure of the Lie algebras $\mathfrak{u}_N$ and $\mathfrak{so}_N$ that there exists a nonzero symmetric real matrix of trace zero $X$ such that:

$$iX \in \mathfrak{g}$$

We know that the space of symmetric real matrices of trace zero is an irreducible representation of $SO_N$ under the adjoint action. Thus $\mathfrak{g}$ must contain all such $X$, and hence $\mathfrak{g} = \mathfrak{u}_N$. But since $U_N$ is connected, it follows that $G = U_N$.

Step 3. Our claim is that the commutant of $SO_N$ in $M_N(\mathbb{C})$ is as follows:

(1) $SO_2' = \left\{ \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \Big| \alpha, \beta \in \mathbb{C} \right\}$.
(2) If $N \geq 3$, $SO_N' = \{\alpha I_N | \alpha \in \mathbb{C}\}$.

Indeed, at $N = 2$ this is a direct computation.

At $N \geq 3$, an element in $X \in SO_N'$ commutes with any diagonal matrix having exactly $N-2$ entries equal to 1 and two entries equal to $-1$. Hence $X$ is a diagonal matrix.

Now since $X$ commutes with any even permutation matrix and $N \geq 3$, it commutes in particular with the permutation matrix associated with the cycle $(i,j,k)$ for any $1 < i < j < k$, and hence all the entries of $X$ are the same.

We conclude that $X$ is a scalar matrix, as claimed.

Step 4. Our claim is that the set of matrices with nonzero trace is dense in $SO_N$.

At $N = 2$ this is clear, since the set of elements in $SO_2$ having a given trace is finite. So assume $N > 2$, and let:

$$T \in SO_N \simeq SO(\mathbb{R}^N)$$

$$Tr(T) = 0$$

Let $E\subset\mathbb R^N$ be a 2-dimensional subspace preserved by $T$, such that:

$$T_{|E}\in SO(E)$$

Let $\varepsilon>0$ and let $S_\varepsilon\in SO(E)$ with $||T_{|E}-S_\varepsilon||<\varepsilon$, and with $Tr(T_{|E})\neq Tr(S_\varepsilon)$, in the $N=2$ case. Now define $T_\varepsilon\in SO(\mathbb R^N)=SO_N$ by:

$$T_{\varepsilon|E}=S_\varepsilon$$

$$T_{\varepsilon|E^\perp}=T_{|E^\perp}$$

It is clear that we have:

$$||T-T_\varepsilon||\leq||T_{|E}-S_\varepsilon||<\varepsilon$$

Also, we have:

$$Tr(T_\varepsilon)=Tr(S_\varepsilon)+Tr(T_{|E^\perp})\neq 0$$

Thus, we have proved our claim.

Step 5. Our claim is that $\mathbb TO_N$ is the normalizer of $\mathbb TSO_N$ in $U_N$, i.e. is the subgroup of $U_N$ consisting of the unitaries $U$ for which, for all $X\in\mathbb TSO_N$:

$$U^{-1}XU\in\mathbb TSO_N$$

It is clear that the group $\mathbb TO_N$ normalizes $\mathbb TSO_N$, so in order to prove the result, we must show that if $U\in U_N$ normalizes $\mathbb TSO_N$ then $U\in\mathbb TO_N$.

First note that $U$ normalizes $SO_N$. Indeed if $X\in SO_N$ then:

$$U^{-1}XU\in\mathbb TSO_N$$

Thus $U^{-1}XU=\lambda Y$ for some $\lambda\in\mathbb T$ and $Y\in SO_N$.

If $Tr(X)\neq 0$, we have $\lambda\in\mathbb R$ and hence:

$$\lambda Y=U^{-1}XU\in SO_N$$

The set of matrices having nonzero trace being dense in $SO_N$, we conclude that $U^{-1}XU\in SO_N$ for all $X\in SO_N$. Thus, we have:

$$\begin{aligned}X\in SO_N&\implies (UXU^{-1})^t(UXU^{-1})=I_N\\&\implies X^tU^tUX=U^tU\\&\implies U^tU\in SO_N'\end{aligned}$$

It follows that at $N\geq 3$ we have $U^tU=\alpha I_N$, with $\alpha\in\mathbb T$, since $U$ is unitary. Hence we have $U=\alpha^{1/2}(\alpha^{-1/2}U)$ with:

$$\alpha^{-1/2}U\in O_N$$

$$U\in\mathbb TO_N$$

If $N=2$, $(U^tU)^t=U^tU$ gives again that $U^tU=\alpha I_2$, and we conclude as in the previous case.

Step 6. Our claim is that the inclusion $\mathbb TO_N\subset U_N$ is maximal in the category of compact groups.

Suppose indeed that $\mathbb{T}O_N \subset G \subset U_N$ is a compact group such that $G \neq U_N$. It is a well-known fact that the connected component of the identity in $G$ is a normal subgroup, denoted $G_0$. Since we have $\mathbb{T}SO_N \subset G_0 \subset U_N$, we must have:

$$G_0 = \mathbb{T}SO_N$$

But since $G_0$ is normal in $G$, the group $G$ normalizes $\mathbb{T}SO_N$, and hence $G \subset \mathbb{T}O_N$.

Step 7. Our claim is that the inclusion $PO_N \subset PU_N$ is maximal in the category of compact groups.

This follows from the above result. Indeed, if $PO_N \subset G \subset PU_N$ is a proper intermediate subgroup, then its preimage under the quotient map $U_N \to PU_N$ would be a proper intermediate subgroup of $\mathbb{T}O_N \subset U_N$, which is a contradiction. □

In connection now with the "edge question" of classifying the intermediate groups $O_N \subset G \subset U_N$, the above result leads to a dichotomy, coming from:

$$PG \in \{PO_N, PU_N\}$$

In the lack of a classification result here, which is surely known, but that we were unable to find in the literature, here are some basic examples of such intermediate groups:

**Proposition 12.21.** *We have compact groups $O_N \subset G \subset U_N$ as follows:*

(1) *The following groups, depending on a parameter $r \in \mathbb{N} \cup \{\infty\}$,*

$$\mathbb{Z}_r O_N \left\{ wU \Big| w \in \mathbb{Z}_r, U \in O_N \right\}$$

*whose projective versions equal $PO_N$, and the biggest of which is the group $\mathbb{T}O_N$, which appears as affine lift of $PO_N$.*

(2) *The following groups, depending on a parameter $d \in 2\mathbb{N} \cup \{\infty\}$,*

$$U_N^d = \left\{ U \in U_N \Big| \det U \in \mathbb{Z}_d \right\}$$

*interpolating between $U_N^2$ and $U_N^\infty = U_N$, whose projective versions equal $PU_N$.*

*Proof.* All the assertions are elementary, and well-known. □

The above results suggest that the solutions of $O_N \subset G \subset U_N$ should come from $O_N, U_N$, by succesively applying the constructions $G \to \mathbb{Z}_r G$ and $G \to G \cap U_N^d$. These operations do not exactly commute, but normally we should be led in this way to a 2-parameter series, unifying the two 1-parameter series from (1,2) above. However, some other groups like $\mathbb{Z}_N SO_N$ work too, so all this is probably a bit more complicated.

We have as well the following result, also from [18]:

**Theorem 12.22.** *The inclusion of compact quantum groups*

$$O_N \subset O_N^*$$

*is maximal in the category of compact quantum groups.*

*Proof.* The idea is that this follows from the result regarding $PO_N \subset PU_N$, by taking affine lifts, and using algebraic techniques. Consider indeed a sequence of surjective Hopf $*$-algebra maps as follows, whose composition is the canonical surjection:

$$C(O_N^*) \xrightarrow{f} A \xrightarrow{g} C(O_N)$$

This produces a diagram of Hopf algebra maps with pre-exact rows, as follows:

$$\begin{tikzcd}
\mathbb{C} \arrow[r] & C(PO_N^*) \arrow[r] \arrow[d, "f_|"] & C(O_N^*) \arrow[r] \arrow[d, "f"] & C(\mathbb{Z}_2) \arrow[r] \arrow[d, equal] & \mathbb{C} \\
\mathbb{C} \arrow[r] & PA \arrow[r] \arrow[d, "g_|"] & A \arrow[r] \arrow[d, "g"] & C(\mathbb{Z}_2) \arrow[r] \arrow[d, equal] & \mathbb{C} \\
\mathbb{C} \arrow[r] & PC(O_N) \arrow[r] & C(O_N) \arrow[r] & C(\mathbb{Z}_2) \arrow[r] & \mathbb{C}
\end{tikzcd}$$

Consider now the following composition, with the isomorphism on the left being something well-known, coming from [49], that we will explain in section 16 below:

$$C(PU_N) \simeq C(PO_N^*) \xrightarrow{f_|} PA \xrightarrow{g_|} PC(O_N) \simeq C(PO_N)$$

This induces, at the group level, the embedding $PO_N \subset PU_N$. Thus $f_|$ or $g_|$ is an isomorphism. If $f_|$ is an isomorphism we get a commutative diagram of Hopf algebra morphisms with pre-exact rows, as follows:

$$\begin{tikzcd}
\mathbb{C} \arrow[r] & C(PO_N^*) \arrow[r] \arrow[d, equal] & C(O_N^*) \arrow[r] \arrow[d, "f"] & C(\mathbb{Z}_2) \arrow[r] \arrow[d, equal] & \mathbb{C} \\
\mathbb{C} \arrow[r] & C(PO_N^*) \arrow[r] & A \arrow[r] & C(\mathbb{Z}_2) \arrow[r] & \mathbb{C}
\end{tikzcd}$$

Then $f$ is an isomorphism. Similarly if $g_|$ is an isomorphism, then $g$ is an isomorphism. For further details on all this, we refer to [18]. $\square$

Finally, let us discuss twisting results. Let us go back to the standard cube, namely:

$$\begin{tikzcd}
 & K_N^+ \arrow[rr] & & U_N^+ \\
H_N^+ \arrow[rr] \arrow[ur] & & O_N^+ \arrow[ur] & \\
 & K_N \arrow[rr] \arrow[uu] & & U_N \arrow[uu] \\
H_N \arrow[rr] \arrow[uu] \arrow[ur] & & O_N \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

According to the general Schur-Weyl twisting method from section 7 above, all these quantum groups can be twisted. In addition, the continuous twists were explicitely computed in section 7 above, and the discrete objects were shown in section 10 above to be equal to their own twists. Thus, we are led to the following conclusion:

**Theorem 12.23.** *The Schur-Weyl twists of the main quantum groups are*

$$\begin{tikzcd}
 & K_N^+ \arrow[rr] & & U_N^+ \\
H_N^+ \arrow[rr] \arrow[ur] & & O_N^+ \arrow[ur] & \\
 & K_N \arrow[rr] \arrow[uu] & & \bar{U}_N \arrow[uu] \\
H_N \arrow[rr] \arrow[uu] \arrow[ur] & & \bar{O}_N \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

*and we will call this diagram "twisted standard cube".*

*Proof.* This follows indeed from the above discussion. □

This construction raises the perspective of finding the twisted versions of the above classification results. Following [6], in the uniform case, the result here is as follows:

**Theorem 12.24.** *The classical and free uniform twisted easy quantum groups are*

$$\begin{tikzcd}
& & K_N^+ \arrow[r] & K_N^{++} \arrow[r] & U_N^+ \\
& H_N^{s+} \arrow[ur] & & & \\
H_N^+ \arrow[ur] \arrow[rrr] & & & O_N^+ \arrow[uur] & \\
& & K_N \arrow[uuu, dotted] \arrow[rr] & & \bar{U}_N \arrow[uuu, dotted] \\
& H_N^s \arrow[ur] & & & \\
H_N \arrow[ur] \arrow[uuu, dotted] \arrow[rrr] & & & \bar{O}_N \arrow[uuu, dotted] \arrow[uur] &
\end{tikzcd}$$

*where* $H_s = \mathbb{Z}_s \wr S_N$, $H_N^{s+} = \mathbb{Z}_s \wr_* S_N^+$ *with* $s = 4, 6, 8\ldots$, *and where* $K_N^+ = \widetilde{K_N^+}$.

*Proof.* This follows indeed from Theorem 12.1, and from the above discussion. $\square$

It is possible to get beyond this, with further classification results in the twisted case, and also with noncommutative geometry considerations, in the spirit of those mentioned at the end of section 11. We refer here to [4], [5], [6] and related papers.

This above was just an introduction to the classification problems for the compact quantum groups, and for more we refer to [81], [87], [102], [107], [108], [117], [126].

## 13. **Toral subgroups**

We have seen in the previous sections that the group dual subgroups $\widehat{\Lambda}\subset G$ play an important role in the theory. Our purpose here is to understand how the structure of a closed subgroup $G\subset U_N^+$ can be recovered from the knowledge of such subgroups. Let us start with a basic statement, regarding the classical and group dual cases:

**Proposition 13.1.** *Let $G\subset U_N^+$ be a compact quantum group, and consider the group dual subgroups $\widehat{\Lambda}\subset G$, also called toral subgroups, or simply "tori".*

(1) *In the classical case, where $G\subset U_N$ is a compact Lie group, these are the usual tori, where by torus we mean here closed abelian subgroup.*
(2) *In the group dual case, $G=\widehat{\Gamma}$ with $\Gamma=<g_1,\ldots,g_N>$ being a discrete group, these are the duals of the various quotients $\Gamma\to\Lambda$.*

*Proof.* Both these assertions are elementary, as follows:

(1) This follows indeed from the fact that a closed subgroup $H\subset U_N^+$ is at the same time classical, and a group dual, precisely when it is classical and abelian.

(2) This follows from the general propreties of the Pontrjagin duality, and more precisely from the fact that the subgroups $\widehat{\Lambda}\subset\widehat{\Gamma}$ correspond to the quotients $\Gamma\to\Lambda$. □

Based on the above simple facts, regarding the groups and the group duals, we can see that in general, there are two motivations for the study of toral subgroups $\widehat{\Lambda}\subset G$:

(1) It is well-known that the fine structure of a compact Lie group $G\subset U_N$ is partly encoded by its maximal torus. Thus, in view of Proposition 13.1, the various tori $\widehat{\Lambda}\subset G$ encode interesting information about a quantum group $G\subset U_N^+$, both in the classical and the group dual case. We can expect this to hold in general.
(2) Any action $G\curvearrowright X$ on some geometric object, such as a manifold, will produce actions of its tori on the same object, $\widehat{\Lambda}\curvearrowright X$. And, due to the fact that $\Lambda$ are familiar objects, namely discrete groups, these latter actions are easier to study, and this can ultimately lead to results about the action $G\curvearrowright X$ itself.

At a more concrete level now, most of the tori that we met appear as diagonal tori, in the sense of section 2 above. Let us first review this material. We first have:

**Theorem 13.2.** *Given a closed subgroup $G\subset U_N^+$, consider its "diagonal torus", which is the closed subgroup $T\subset G$ constructed as follows:*

$$C(T)=C(G)\Big/\left\langle u_{ij}=0\Big|\forall i\neq j\right\rangle$$

*This torus is then a group dual, $T=\widehat{\Lambda}$, where $\Lambda=<g_1,\ldots,g_N>$ is the discrete group generated by the elements $g_i=u_{ii}$, which are unitaries inside $C(T)$.*

*Proof.* This is something that we already know, from section 2. Indeed, the elements $g_i = u_{ii}$ are unitaries and group-like inside $C(T)$, and this gives the result. $\square$

Alternatively, we have the following construction:

**Proposition 13.3.** *The diagonal torus $T \subset G$ can be defined as well by*

$$T = G \cap \mathbb{T}_N^+$$

*where $\mathbb{T}_N^+ \subset U_N^+$ is the free complex torus, appearing as*

$$\mathbb{T}_N^+ = \widehat{F_N}$$

*with $F_N =< g_1, \ldots, g_N >$ being the free group on $N$ generators.*

*Proof.* We recall from Theorem 12.2 that the diagonal torus is defined via:

$$C(T) = C(G) \Big/ \left\langle u_{ij} = 0 \Big| \forall i \neq j \right\rangle$$

On the other hand, the free complex torus $\mathbb{T}_N^+$ appears as follows:

$$C(\mathbb{T}_N^+) = C(U_N^+) \Big/ \left\langle u_{ij} = 0 \Big| \forall i \neq j \right\rangle$$

Thus, by intersecting with $G$ we obtain the diagonal torus of $G$. $\square$

Most of our computations so far of diagonal tori, that we will recall in a moment, concern various classes of easy quantum groups. In the general easy case, we have:

**Proposition 13.4.** *For an easy quantum group $G \subset U_N^+$, coming from a category of partitions $D \subset P$, the associated diagonal torus is $T = \widehat{\Gamma}$, with:*

$$\Gamma = F_N \Big/ \left\langle g_{i_1} \ldots g_{i_k} = g_{j_1} \ldots g_{j_l} \Big| \forall i, j, k, l, \exists \pi \in D(k,l), \delta_\pi \begin{pmatrix} i \\ j \end{pmatrix} \neq 0 \right\rangle$$

*Moreover, we can just use partitions $\pi$ which generate the category $D$.*

*Proof.* Let $g_i = u_{ii}$ be the standard coordinates on the diagonal torus $T$, and set $g = diag(g_i)$. We have then the following computation:

$$\begin{aligned}
C(T) &= \left[ C(U_N^+) \Big/ \left\langle T_\pi \in Hom(u^{\otimes k}, u^{\otimes l}) \Big| \forall \pi \in D \right\rangle \right] \Big/ \left\langle u_{ij} = 0 \Big| \forall i \neq j \right\rangle \\
&= \left[ C(U_N^+) \Big/ \left\langle u_{ij} = 0 \Big| \forall i \neq j \right\rangle \right] \Big/ \left\langle T_\pi \in Hom(u^{\otimes k}, u^{\otimes l}) \Big| \forall \pi \in D \right\rangle \\
&= C^*(F_N) \Big/ \left\langle T_\pi \in Hom(g^{\otimes k}, g^{\otimes l}) \Big| \forall \pi \in D \right\rangle
\end{aligned}$$

The associated discrete group, $\Gamma = \widehat{T}$, is therefore given by:

$$\Gamma = F_N \Big/ \left\langle T_\pi \in Hom(g^{\otimes k}, g^{\otimes l}) \Big| \forall \pi \in D \right\rangle$$

Now observe that, with $g = diag(g_1, \ldots, g_N)$ as above, we have:

$$T_\pi g^{\otimes k}(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \delta_\pi \begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l} \cdot g_{i_1} \ldots g_{i_k}$$

We have as well the following formula:

$$g^{\otimes l} T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \delta_\pi \begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l} \cdot g_{j_1} \ldots g_{j_l}$$

We conclude that the relation $T_\pi \in Hom(g^{\otimes k}, g^{\otimes l})$ reformulates as follows:

$$\delta_\pi \begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{pmatrix} \neq 0 \implies g_{i_1} \ldots g_{i_k} = g_{j_1} \ldots g_{j_l}$$

Thus, we obtain the formula in the statement. Finally, the last assertion follows from Tannakian duality, because we can replace everywhere $D$ by a generating subset. □

In practice now, in the continuous case we have the following result:

**Theorem 13.5.** *The diagonal tori of the basic unitary quantum groups, namely*

$$\begin{tikzcd} U_N \arrow[r] & U_N^* \arrow[r] & U_N^+ \\ O_N \arrow[r] \arrow[u] & O_N^* \arrow[r] \arrow[u] & O_N^+ \arrow[u] \end{tikzcd}$$

*and of their $q = -1$ twists as well, are the standard cube and torus, namely*

$$T_N = \mathbb{Z}_2^N$$
$$\mathbb{T}_N = \mathbb{T}^N$$

*in the classical case, and their liberations in general, which are as follows:*

$$\begin{tikzcd} \mathbb{T}_N \arrow[r] & \mathbb{T}_N^* \arrow[r] & \mathbb{T}_N^+ \\ T_N \arrow[r] \arrow[u] & T_N^* \arrow[r] \arrow[u] & T_N^+ \arrow[u] \end{tikzcd}$$

*Also, for the quantum groups $B_N, B_N^+, C_N, C_N^+$, the diagonal torus collapses to $\{1\}$.*

*Proof.* We have several assertions here, the idea being as follows:

(1) The main assertion, regarding the basic unitary quantum groups, is something that we already know, from section 2 above, with the various liberations $T_N^\times, \mathbb{T}_N^\times$ of the basic tori $T_N, \mathbb{T}_N$ in the statement being by definition those appearing there.

(2) Regarding the invariance under twisting, this is best seen by using Proposition 13.4. Indeed, the computation in the proof there applies in the same way to the general quizzy case, and shows that the diagonal torus is invariant under twisting.

(3) In the bistochastic case the fundamental corepresentation $g = diag(g_1, \ldots, g_N)$ of the diagonal torus must be bistochastic, and so $g_1 = \ldots = g_N = 1$, as claimed. $\square$

Regarding now the discrete case, the result is as follows:

**Theorem 13.6.** *The diagonal tori of the basic quantum reflection groups, namely*

$$\begin{array}{ccccc} K_N & \longrightarrow & K_N^* & \longrightarrow & K_N^+ \\ \uparrow & & \uparrow & & \uparrow \\ H_N & \longrightarrow & H_N^* & \longrightarrow & H_N^+ \end{array}$$

*are the same as those for $O_N^\times, U_N^\times$ described above, namely:*

$$\begin{array}{ccccc} \mathbb{T}_N & \longrightarrow & \mathbb{T}_N^* & \longrightarrow & \mathbb{T}_N^+ \\ \uparrow & & \uparrow & & \uparrow \\ T_N & \longrightarrow & T_N^* & \longrightarrow & T_N^+ \end{array}$$

*Also, for the quantum permutation groups $S_N, S_N^+$ we have $T = \{1\}$.*

*Proof.* The first assertion follows from the general fact that the diagonal torus of $G_N \subset U_N^+$ equals the diagonal torus of the discrete version, namely:

$$G_N^d = G_N \cap K_N^+$$

Indeed, this fact follows from definitions, for instance via Proposition 13.3. As for the second assertion, this follows from:

$$S_N \subset B_N$$

$$S_N^+ \subset B_N^+$$

Indeed, by using the last assertion in Theorem 13.5, we obtain the result. $\square$

As a conclusion, the diagonal torus $T \subset G$ is usually a quite interesting object, but for certain quantum groups like the bistochastic ones, or the quantum permutation group ones, this torus collapses to $\{1\}$, and so it cannot be of use in the study of $G$.

In order to deal with this issue, the idea, from [11], [34], will be that of using:

**Theorem 13.7.** *Given a closed subgroup $G\subset U_N^+$ and a matrix $Q\in U_N$, we let $T_Q\subset G$ be the diagonal torus of $G$, with fundamental representation spinned by $Q$:*

$$C(T_Q)=C(G)\Big/\left\langle(QuQ^*)_{ij}=0\Big|\forall i\neq j\right\rangle$$

*This torus is then a group dual, given by $T_Q=\widehat{\Lambda}_Q$, where $\Lambda_Q=<g_1,\ldots,g_N>$ is the discrete group generated by the elements*

$$g_i=(QuQ^*)_{ii}$$

*which are unitaries inside the quotient algebra $C(T_Q)$.*

*Proof.* This follows from Theorem 13.2, because, as said in the statement, $T_Q$ is by definition a diagonal torus. Equivalently, since $v=QuQ^*$ is a unitary corepresentation, its diagonal entries $g_i=v_{ii}$, when regarded inside $C(T_Q)$, are unitaries, and satisfy:

$$\Delta(g_i)=g_i\otimes g_i$$

Thus $C(T_Q)$ is a group algebra, and more specifically we have $C(T_Q)=C^*(\Lambda_Q)$, where $\Lambda_Q=<g_1,\ldots,g_N>$ is the group in the statement, and this gives the result. □

Summarizing, associated to any closed subgroup $G\subset U_N^+$ is a whole family of tori, indexed by the unitaries $U\in U_N$. We use the following terminology:

**Definition 13.8.** *Let $G\subset U_N^+$ be a closed subgroup.*
(1) *The tori $T_Q\subset G$ constructed above are called standard tori of $G$.*
(2) *The collection of tori $T=\left\{T_Q\subset G\Big|Q\in U_N\right\}$ is called skeleton of $G$.*

This might seem a bit awkward, but in view of various results, examples and counterexamples, to be presented below, this is perhaps the best terminology. As a first general result now regarding these tori, coming from [148], we have:

**Theorem 13.9.** *Any torus $T\subset G$ appears as follows, for a certain $Q\in U_N$:*

$$T\subset T_Q\subset G$$

*In other words, any torus appears inside a standard torus.*

*Proof.* Given a torus $T\subset G$, we have an inclusion as follows:

$$T\subset G\subset U_N^+$$

On the other hand, we know from section 3 above that each torus $T\subset U_N^+$ has a fundamental corepresentation as follows, with $Q\in U_N$:

$$u=Qdiag(g_1,\ldots,g_N)Q^*$$

But this shows that we have $T\subset T_Q$, and this gives the result. □

Let us do now some computations. In the classical case, the result is as follows:

**Proposition 13.10.** *For a closed subgroup $G\subset U_N$ we have*

$$T_Q=G\cap(Q^*\mathbb T^NQ)$$

*where $\mathbb T^N\subset U_N$ is the group of diagonal unitary matrices.*

*Proof.* This is indeed clear at $Q=1$, where $\Gamma_1$ appears by definition as the dual of the compact abelian group $G\cap\mathbb T^N$. In general, this follows by conjugating by $Q$. □

In the group dual case now, we have the following result:

**Proposition 13.11.** *Given a finitely generated discrete group*

$$\Gamma=<g_1,\ldots,g_N>$$

*consider its dual compact quantum group $G=\widehat{\Gamma}$, diagonally embedded into $U_N^+$. We have*

$$\Lambda_Q=\Gamma/\left\langle g_i=g_j\Big|\exists k,Q_{ki}\neq0,Q_{kj}\neq0\right\rangle$$

*with the embedding $T_Q\subset G=\widehat{\Gamma}$ coming from the quotient map $\Gamma\to\Lambda_Q$.*

*Proof.* Assume indeed that $\Gamma=<g_1,\ldots,g_N>$ is a discrete group, with dual $\widehat{\Gamma}\subset U_N^+$ coming via $u=diag(g_1,\ldots,g_N)$. With $v=QuQ^*$, we have the following computation:

$$\begin{aligned}\sum_s\bar{Q}_{si}v_{sk}&=\sum_{st}\bar{Q}_{si}Q_{st}\bar{Q}_{kt}g_t\\&=\sum_t\delta_{it}\bar{Q}_{kt}g_t\\&=\bar{Q}_{ki}g_i\end{aligned}$$

Thus the condition $v_{ij}=0$ for $i\neq j$ gives:

$$\bar{Q}_{ki}v_{kk}=\bar{Q}_{ki}g_i$$

But this is the same as saying that:

$$Q_{ki}\neq0\implies g_i=v_{kk}$$

Now this latter equality reads:

$$g_i=\sum_j|Q_{kj}|^2g_j$$

We conclude from this that, as desired:

$$Q_{ki}\neq0,Q_{kj}\neq0\implies g_i=g_j$$

As for the converse, this is elementary to establish as well. □

According to the above results, we can expect the skeleton $T$ to encode various algebraic and analytic properties of $G$. We will discuss this in what follows, with a number of results and conjectures. We first have the following result:

**Theorem 13.12.** *The following results hold, both over the category of compact Lie groups, and over the category of duals of finitely generated discrete groups:*

(1) *Injectivity: the construction $G \to T$ is injective, in the sense that $G \neq H$ implies, for some $Q \in U_N$:*
$$T_Q(G) \neq T_Q(H)$$

(2) *Monotony: the construction $G \to T$ is increasing, in the sense that passing to a subgroup $H \subset G$ decreases at least one of the tori $T_Q$:*
$$T_Q(H) \neq T_Q(G)$$

(3) *Generation: any closed quantum subgroup $G \subset U_N^+$ is generated by its tori, or, equivalently, has the following generation property:*
$$G = < T_Q | Q \in U_N >$$

*Proof.* We have two cases to be investigated, as follows:

(1) Assume first that we are in the classical case, $G \subset U_N$. In order to prove the generation property we use the following formula, established above:
$$T_Q = G \cap Q^* \mathbb{T}^N Q$$

Now since any group element $U \in G$ is unitary, and so diagonalizable, we can write, for certain matrices $Q \in U_N$ and $D \in \mathbb{T}^N$:
$$U = Q^* D Q$$

But we have then, for this precise value of the spinning matrix $Q \in U_N$:
$$U \in T_Q$$

Thus we have proved the generation property, and the injectivity and monotony properties follow from this.

(2) Regarding now the group duals, here everything is trivial. Indeed, when the group duals are diagonally embedded we can take $Q = 1$, and when the group duals are embedded by using a spinning matrix $Q \in U_N$, we can use precisely this matrix $Q$. □

As explained in [34], it is possible to go beyond the above verifications, notably with some results regarding the half-classical and the free cases. However, there is no serious idea so far, in order to deal with the general case. See [34].

We will be back to this, in section 14 below.

Let us focus now on the generation property, from Theorem 13.12 (3), which is perhaps the most important. In order to discuss the general case, we will need:

**Proposition 13.13.** *Given a closed subgroup $G \subset U_N^+$ and a matrix $Q \in U_N$, the corresponding standard torus and its Tannakian category are given by*

$$T_Q = G \cap \mathbb{T}_Q$$

$$C_{T_Q} = < C_G, C_{\mathbb{T}_Q} >$$

*where $\mathbb{T}_Q \subset U_N^+$ is the dual of the free group $F_N = < g_1, \ldots, g_N >$, with the fundamental corepresentation of $C(\mathbb{T}_Q)$ being the matrix $u = Qdiag(g_1, \ldots, g_N)Q^*$.*

*Proof.* The first assertion comes from the well-known fact that given two closed subgroups $G, H \subset U_N^+$, the corresponding quotient algebra $C(U_N^+) \to C(G \cap H)$ appears by dividing by the kernels of both the following quotient maps:

$$C(U_N^+) \to C(G)$$

$$C(U_N^+) \to C(H)$$

Indeed, the construction of $T_Q$ from Theorem 13.7 amounts precisely in performing this operation, with $H = \mathbb{T}_Q$, and so we obtain, as claimed:

$$T_Q = G \cap \mathbb{T}_Q$$

As for the Tannakian category formula, this follows from this, and from the following general Tannakian duality formula from section 6 above:

$$C_{G \cap H} = < C_G, C_H >$$

Thus, we are led to the conclusion in the statement. □

We have the following Tannakian reformulation of the toral generation property:

**Theorem 13.14.** *Given a closed subgroup $G \subset U_N^+$, the subgroup*

$$G' = < T_Q | Q \in U_N >$$

*generated by its standard tori has the following Tannakian category:*

$$C_{G'} = \bigcap_{Q \in U_N} < C_G, C_{\mathbb{T}_Q} >$$

*In particular we have $G = G'$ when this intersection reduces to $C_G$.*

*Proof.* Consider indeed the subgroup $G' \subset G$ constructed in the statement. We have:

$$C_{G'} = \bigcap_{Q \in U_N} C_{T_Q}$$

Together with the formula in Proposition 13.13, this gives the result. □

Let us further discuss now the toral generation property, with some modest results, regarding its behaviour with respect to product operations. We first have:

**Proposition 13.15.** *Given two closed subgroups $G, H \subset U_N^+$, and $Q \in U_N$, we have:*

$$< T_Q(G), T_Q(H) > \subset T_Q(< G, H >)$$

*Also, the toral generation property is stable under the operation $<\,,>$.*

*Proof.* The first assertion can be proved either by using Theorem 13.14, or directly. For the direct proof, which is perhaps the simplest, we have:

$$\begin{aligned} T_Q(G) &= G \cap \mathbb{T}_Q \subset < G, H > \cap \mathbb{T}_Q \\ &= T_Q(< G, H >) \end{aligned}$$

We have as well the following computation:

$$\begin{aligned} T_Q(H) &= H \cap \mathbb{T}_Q \subset < G, H > \cap \mathbb{T}_Q \\ &= T_Q(< G, H >) \end{aligned}$$

Now since $A, B \subset C$ implies $< A, B > \subset C$, this gives the result.

Regarding now the second assertion, we have the following computation:

$$\begin{aligned} < G, H > &= << T_Q(G) | Q \in U_N >, < T_Q(H) | Q \in U_N >> \\ &= < T_Q(G), T_Q(H) | Q \in U_N > \\ &= << T_Q(G), T_Q(H) > | Q \in U_N > \\ &\subset < T_Q(< G, H >) | Q \in U_N > \end{aligned}$$

Thus the quantum group $< G, H >$ is generated by its tori, as claimed. □

We have as well the following result:

**Proposition 13.16.** *We have the following formula, for any $G, H$ and $R, S$:*

$$T_{R \otimes S}(G \times H) = T_R(G) \times T_S(H)$$

*Also, the toral generation property is stable under usual products $\times$.*

*Proof.* The product formula in the statement is clear from definitions. Regarding now the second assertion, we have the following computation:

$$\begin{aligned} & < T_Q(G \times H) | Q \in U_{MN} > \\ \supset\ & < T_{R \otimes S}(G \times H) | R \in U_M, S \in U_N > \\ =\ & < T_R(G) \times T_S(H) | R \in U_M, S \in U_N > \\ =\ & < T_R(G) \times \{1\}, \{1\} \times T_S(H) | R \in U_M, S \in U_N > \\ =\ & < T_R(G) | R \in U_M > \times < T_G(H) | H \in U_N > \\ =\ & G \times H \end{aligned}$$

Thus the quantum group $G \times H$ is generated by its tori, as claimed. □

In order to get beyond this, let us discuss now some weaker versions of the generation property, which are partly related to the classification program from section 12:

**Definition 13.17.** *A closed subgroup $G_N\subset U_N^+$, with classical version $G_N^c$, is called:*

(1) *Weakly generated by its tori, when:*

$$G_N=<G_N^c,(T_Q)_{Q\in U_N}>$$

(2) *A diagonal liberation of $G_N^c$, when:*

$$G_N=<G_N^c,T_1>$$

According to our results above, the first property is satisfied for the groups, for the group duals, and is stable under generations, and direct products. Regarding the second property, this is something quite interesting, which takes us away from our original generation questions. The idea here, from [55] and subsequent papers, is that such things can be proved by recurrence on $N\in\mathbb N$. In order to discuss this, let us start with:

**Proposition 13.18.** *Assume that $G=(G_N)$ is weakly uniform, let $n\in\{2,3,\ldots,\infty\}$ be minimal such that $G_n$ is not classical, and consider the following conditions:*

(1) *Strong generation: $G_N=<G_N^c,G_n>$, for any $N>n$.*
(2) *Usual generation: $G_N=<G_N^c,G_{N-1}>$, for any $N>n$.*
(3) *Initial step generation: $G_{n+1}=<G_{n+1}^c,G_n>$.*

*We have then $(1)\iff(2)\implies(3)$, and $(3)$ is in general strictly weaker.*

*Proof.* All the implications and non-implications are elementary, as follows:

$(1)\implies(2)$ This follows from $G_n\subset G_{N-1}$ for $N>n$, coming from uniformity.

$(2)\implies(1)$ By using twice the usual generation, and then the uniformity, we have:

$$\begin{aligned}G_N&=<G_N^c,G_{N-1}>\\&=<G_N^c,G_{N-1}^c,G_{N-2}>\\&=<G_N^c,G_{N-2}>\end{aligned}$$

Thus we have a descent method, and we end up with the strong generation condition.

$(2)\implies(3)$ This is clear, because $(2)$ at $N=n+1$ is precisely $(3)$.

$(3)\not\Longrightarrow(2)$ In order to construct counterexamples here, the simplest is to use group duals. Indeed, with $G_N=\widehat{\Gamma_N}$ and $\Gamma_N=<g_1,\ldots,g_N>$, the uniformity condition tells us that we must be in a projective limit situation, as follows:

$$\Gamma_1\leftarrow\Gamma_2\leftarrow\Gamma_3\leftarrow\Gamma_4\leftarrow\ldots$$

$$\Gamma_{N-1}=\Gamma_N/<g_N=1>$$

Now by assuming for instance that $\Gamma_2$ is given and not abelian, there are many ways of completing the sequence, and so the uniqueness coming from (2) can only fail. □

Let us introduce now a few more notions, as follows:

**Proposition 13.19.** *Assume that $G = (G_N)$ is weakly uniform, let $n \in \{2, 3, \ldots, \infty\}$ be as above, and consider the following conditions, where $I_N \subset G_N$ is the diagonal torus:*

1. *Strong diagonal liberation: $G_N =< G_N^c, I_n >$, for any $N \geq n$.*
2. *Technical condition: $G_N =< G_N^c, I_{N-1} >$ for any $N > n$, and $G_n =< G_n^c, I_n >$.*
3. *Diagonal liberation: $G_N =< G_N^c, I_N >$, for any $N$.*
4. *Initial step diagonal liberation: $G_n =< G_n^c, I_n >$.*

*We have then* $(1) \implies (2) \implies (3) \implies (4)$.

*Proof.* Our claim is that when assuming that $G = (G_N)$ is weakly uniform, so is the family of diagonal tori $I = (I_N)$. Indeed, we have the following computation:

$$\begin{aligned} I_N \cap U_{N-1}^+ &= (G_N \cap \mathbb{T}_N^+) \cap U_{N-1}^+ \\ &= (G_N \cap U_{N-1}^+) \cap (\mathbb{T}_N^+ \cap U_{N-1}^+) \\ &= G_{N-1} \cap \mathbb{T}_{N-1}^+ \\ &= I_{N-1} \end{aligned}$$

Thus our claim is proved, and this gives the various implications in the statement. □

We can now formulate a key theoretical observation, as follows:

**Theorem 13.20.** *Assuming that $G = (G_N)$ is weakly uniform, and with $n \in \{2, 3, \ldots, \infty\}$ being as above, the following conditions are equivalent, modulo their initial steps:*

1. *Generation: $G_N =< G_N^c, G_{N-1} >$, for any $N > n$.*
2. *Strong generation: $G_N =< G_N^c, G_n >$, for any $N > n$.*
3. *Diagonal liberation: $G_N =< G_N^c, I_N >$, for any $N \geq n$.*
4. *Strong diagonal liberation: $G_N =< G_N^c, I_n >$, for any $N \geq n$.*

*Proof.* Our first claim is that generation plus initial step diagonal liberation imply the technical diagonal liberation condition. Indeed, the recurrence step goes as follows:

$$\begin{aligned} G_N &= < G_N^c, G_{N-1} > \\ &= < G_N^c, G_{N-1}^c, I_{N-1} > \\ &= < G_N^c, I_{N-1} > \end{aligned}$$

In order to pass now from the technical diagonal liberation condition to the strong diagonal liberation condition itself, observe that we have:

$$\begin{aligned} G_N &= < G_N^c, G_{N-1} > \\ &= < G_N^c, G_{N-1}^c, I_{N-1} > \\ &= < G_N^c, I_{N-1} > \end{aligned}$$

With this condition in hand, we have then as well:

$$\begin{aligned} G_N &= < G_N^c, G_{N-1} > \\ &= < G_N^c, G_{N-1}^c, I_{N-2} > \\ &= < G_N^c, I_{N-2} > \end{aligned}$$

This procedure can be of course be continued. Thus we have a descent method, and we end up with the strong diagonal liberation condition.

In the other sense now, we want to prove that we have, at $N \geq n$:

$$G_N =< G_N^c, G_{N-1} >$$

At $N = n+1$ this is something that we already have. At $N = n+2$ now, we have:

$$\begin{aligned} G_{n+2} &= < G_{n+2}^c, I_n > \\ &= < G_{n+2}^c, G_{n+1}^c, I_n > \\ &= < G_{n+2}^c, G_{n+1} > \end{aligned}$$

This procedure can be of course be continued. Thus, we have a descent method, and we end up with the strong generation condition. □

It is possible to prove that many interesting quantum groups have the above properties, and hence appear as diagonal liberations, but the whole subject is quite technical. Here is however a statement, collecting most of the known results on the subject:

**Theorem 13.21.** *The basic quantum unitary and reflection groups are as follows:*

(1) $O_N^*, U_N^*$ *appear via diagonal liberation.*
(2) $O_N^+, U_N^+$ *appear via diagonal liberation.*
(3) $H_N^*, K_N^*$ *appear via diagonal liberation.*
(4) $H_N^+, K_N^+$ *do not appear via diagonal liberation.*

*In addition,* $B_N^+, C_N^+, S_N^+$ *do not appear either via diagonal liberation.*

*Proof.* All this is quite technical, the idea being as follows:

(1) The quantum groups $O_N^*, U_N^*$ are not uniform, and cannot be investigated with the above techniques. However, these quantum groups can be studied by using the technology in [16], [47], [49], which will be briefly discussed in section 16 below, and this leads to $O_N^* =< O_N, T_N^* >$, as well as to $U_N^* =< U_N, T_N^* >$, which implies $U_N^* =< U_N, \mathbb{T}_N^* >$.

(2) The quantum groups $O_N^+, U_N^+$ are uniform, and a quite technical computation, from [52], [53], [55], [56], shows that the generation conditions from Theorem 13.20 are satisfied for $O_N^+$. Thus we obtain $O_N^+ =< O_N, T_N^+ >$, and from this we can deduce via the results in [18] that we have $U_N^+ =< U_N, T_N^+ >$, which implies $U_N^+ =< U_N, \mathbb{T}_N^+ >$. See [56].

(3) The situation for $H_N^*, K_N^*$ is quite similar to the one for $O_N^*, U_N^*$, explained above. Indeed, the technology in [16], [47], [49] applies, and this leads to $H_N^* =< H_N, T_N^* >$, as

well as to $K_N^*=<K_N,T_N^*>$, which implies $K_N^*=<K_N,\mathbb{T}_N^*>$. In fact, these results are stronger than the above ones for $O_N^*,U_N^*$, via some standard generation formulae.

(4) This is something subtle as well, coming from the quantum groups $H_N^{[\infty]},K_N^{[\infty]}$ from [117], discussed before. Indeed, since the relations $g_ig_ig_j=g_jg_ig_i$ are trivially satisfied for real reflections, the diagonal tori of these quantum groups coincide with those for $H_N^+,K_N^+$. Thus, the diagonal liberation procedure "stops" at $H_N^{[\infty]},K_N^{[\infty]}$.

Finally, regarding the last assertion, here $B_N^+,C_N^+,S_N^+$ do not appear indeed via diagonal liberation, and this because of a trivial reason, namely $T=\{1\}$. □

Summarizing, all this is quite technical. Now regardless of these difficulties, and of the various positive results on the subject, the notion of diagonal liberation is obviously not the good one. As a conjectural solution to these difficulties, we have the notion of Fourier liberation, that we will discuss now. For this purpose, we will need a lot of preliminaries, in relation with the group dual subgroups $\widehat{\Gamma}\subset G$ of the quantum permutation groups, $G\subset S_N^+$, following the work of Bichon [46] and related papers.

Let us start with the following basic fact, which generalizes the embedding $\widehat{D}_\infty\subset S_4^+$ that we met in section 9 above, when proving that we have $S_4^+\neq S_4$:

**Proposition 13.22.** *Consider a discrete group generated by elements of finite order, written as a quotient group, as follows:*

$$\mathbb{Z}_{N_1}*\ldots*\mathbb{Z}_{N_k}\to\Gamma$$

*We have then an embedding of quantum groups $\widehat{\Gamma}\subset S_N^+$, where $N=N_1+\ldots+N_k$.*

*Proof.* We have a sequence of embeddings and isomorphisms as follows:

$$\begin{aligned}
\widehat{\Gamma}&\subset \widehat{\mathbb{Z}_{N_1}*\ldots*\mathbb{Z}_{N_k}}\\
&=\widehat{\mathbb{Z}_{N_1}}\,\hat{*}\,\ldots\,\hat{*}\,\widehat{\mathbb{Z}_{N_k}}\\
&\simeq \mathbb{Z}_{N_1}\,\hat{*}\,\ldots\,\hat{*}\,\mathbb{Z}_{N_k}\\
&\subset S_{N_1}\,\hat{*}\,\ldots\,\hat{*}\,S_{N_k}\\
&\subset S_{N_1}^+\,\hat{*}\,\ldots\,\hat{*}\,S_{N_k}^+\\
&\subset S_N^+
\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

The above result is quite abstract, and it is worth working out the details, with an explicit formula for the associated magic matrix. Let us start with a study of the simplest situation, where $k=1$, and where $\Gamma=\mathbb{Z}_{N_1}$. The result here is as follows:

**Proposition 13.23.** *The magic matrix for the quantum permutation group*
$$\widehat{\mathbb{Z}}_N \simeq \mathbb{Z}_N \subset S_N \subset S_N^+$$
*with standard Fourier isomorphism on the left, is given by the formula*
$$u = FIF^*$$
*where $F = \frac{1}{\sqrt{N}}(w^{ij})$ with $w = e^{2\pi i/N}$ is the Fourier matrix, and where*
$$I = \begin{pmatrix} 1 &&& \\ & g && \\ && \ddots & \\ &&& g^{N-1} \end{pmatrix}$$
*is the diagonal matrix formed by the elements of $\mathbb{Z}_N$, regarded as elements of $C^*(\mathbb{Z}_N)$.*

*Proof.* The magic matrix for the quantum group $\mathbb{Z}_N \subset S_N \subset S_N^+$ is given by:
$$\begin{aligned} v_{ij} &= \chi\left(\sigma \in \mathbb{Z}_N \middle| \sigma(j) = i\right) \\ &= \delta_{i-j} \end{aligned}$$

Let us apply now the Fourier transform. According to our Pontrjagin duality conventions from section 1 above, we have a pair of inverse isomorphisms, as follows:
$$\Phi : C(\mathbb{Z}_N) \to C^*(\mathbb{Z}_N) \quad , \quad \delta_i \to \frac{1}{N}\sum_k w^{ik} g^k$$
$$\Psi : C^*(\mathbb{Z}_N) \to C(\mathbb{Z}_N) \quad , \quad g^i \to \sum_k w^{-ik}\delta_k$$

Here $w = e^{2\pi i/N}$, and we use the standard Fourier analysis convention that the indices are $0, 1, \ldots, N-1$. With $F = \frac{1}{\sqrt{N}}(w^{ij})$ and $I = diag(g^i)$ as above, we have:
$$\begin{aligned} u_{ij} &= \Phi(v_{ij}) \\ &= \frac{1}{N}\sum_k w^{(i-j)k} g^k \\ &= \frac{1}{N}\sum_k w^{ik} g^k w^{-jk} \\ &= \sum_k F_{ik} I_{kk} (F^*)_{kj} \\ &= (FIF^*)_{ij} \end{aligned}$$

Thus, the magic matrix that we are looking for is $u = FIF^*$, as claimed. □

With the above result in hand, we can complement Proposition 13.22 with:

**Proposition 13.24.** *Given a quotient group $\mathbb{Z}_{N_1} * \ldots * \mathbb{Z}_{N_k} \to \Gamma$, the magic matrix for the subgroup $\widehat{\Gamma} \subset S_N^+$ found in Proposition 13.22, with $N = N_1 + \ldots + N_k$, is given by*

$$u = \begin{pmatrix} F_{N_1} I_1 F_{N_1}^* \\ & \ddots \\ & & F_{N_k} I_k F_{N_k}^* \end{pmatrix}$$

*where $F_N = \frac{1}{\sqrt{N}}(w_N^{ij})$ with $w_N = e^{2\pi i/N}$ are Fourier matrices, and where*

$$I_r = \begin{pmatrix} 1 \\ & g_r \\ & & \ddots \\ & & & g_r^{N_r - 1} \end{pmatrix}$$

*with $g_1, \ldots, g_k$ being the standard generators of $\Gamma$.*

*Proof.* This follows indeed from Proposition 13.22 and Proposition 13.23. □

Following [46], let us prove now that this construction provides us with all the group duals $\widehat{\Gamma} \subset S_N^+$. The idea will be that of using orbit theory, which is as follows:

**Theorem 13.25.** *Given a closed subgroup $G \subset S_N^+$, with standard coordinates denoted $u_{ij} \in C(G)$, the following defines an equivalence relation on $\{1, \ldots, N\}$,*

$$i \sim j \iff u_{ij} \neq 0$$

*that we call orbit decomposition associated to the action $G \curvearrowright \{1, \ldots, N\}$. In the classical case, $G \subset S_N$, this is the usual orbit equivalence coming from the action of $G$.*

*Proof.* We first check the fact that we have indeed an equivalence relation:

(1) The condition $i \sim i$ follows from $\varepsilon(u_{ij}) = \delta_{ij}$, which gives:

$$\varepsilon(u_{ii}) = 1$$

(2) The condition $i \sim j \implies j \sim i$ follows from $S(u_{ij}) = u_{ji}$, which gives:

$$u_{ij} \neq 0 \implies u_{ji} \neq 0$$

(3) The condition $i \sim j, j \sim k \implies i \sim k$ follows from:

$$\Delta(u_{ik}) = \sum_j u_{ij} \otimes u_{jk}$$

Indeed, in this formula, the right-hand side is a sum of projections, so assuming that we have $u_{ij} \neq 0, u_{jk} \neq 0$ for a certain index $j$, we have:

$$u_{ij} \otimes u_{jk} > 0$$

Thus we have $\Delta(u_{ik}) > 0$, which gives $u_{ik} \neq 0$, as desired. Finally, in the classical case, $G \subset S_N$, the standard coordinates are the following characteristic functions:

$$u_{ij} = \chi\left(\sigma \in G \middle| \sigma(j) = i\right)$$

Thus $u_{ij} \neq 0$ is equivalent to the existence of an element $\sigma \in G$ such that $\sigma(j) = i$. But this means precisely that $i, j$ must be in the same orbit of $G$, as claimed. $\square$

Generally speaking, the theory from the classical case extends well to the quantum group setting, and we have in particular the following result, also from [46]:

**Theorem 13.26.** *Given a closed subgroup $G \subset S_N^+$, with magic matrix denoted $u = (u_{ij})$, consider the associated coaction map, on the space $X = \{1, \ldots, N\}$:*

$$\Phi : C(X) \to C(X) \otimes C(G) \quad , \quad e_i \to \sum_j e_j \otimes u_{ji}$$

*The following three subalgebras of $C(X)$ are then equal*

$$Fix(u) = \left\{\xi \in C(X) \middle| u\xi = \xi\right\}$$

$$Fix(\Phi) = \left\{\xi \in C(X) \middle| \Phi(\xi) = \xi \otimes 1\right\}$$

$$F = \left\{\xi \in C(X) \middle| i \sim j \implies \xi(i) = \xi(j)\right\}$$

*where $\sim$ is the orbit equivalence relation constructed in Theorem 13.25.*

*Proof.* The fact that we have $Fix(u) = Fix(\Phi)$ is standard, with this being valid for any corepresentation $u = (u_{ij})$. Regarding now the equality with $F$, we know from Theorem 13.25 that the magic unitary $u = (u_{ij})$ is block-diagonal, with respect to the orbit decomposition there. But this shows that the algebra $Fix(u) = Fix(\Phi)$ decomposes as well with respect to the orbit decomposition, and so in order to prove the result, we are left with a study in the transitive case, where the result is clear. See [46]. $\square$

We have as well the following result, of analytic flavor:

**Proposition 13.27.** *For a closed subgroup $G \subset S_N^+$, the following are equivalent:*

(1) *$G$ is transitive.*
(2) *$Fix(u) = \mathbb{C}\xi$, where $\xi$ is the all-one vector.*
(3) *$\int_G u_{ij} = \frac{1}{N}$, for any $i, j$.*

*Proof.* This is well-known in the classical case. In general, the proof is as follows:

(1) $\iff$ (2) This follows from the identifications in Theorem 13.26.

(2) $\iff$ (3) This is clear from the general properties of the Haar integration. $\square$

As a comment here, we should mention that the whole theory of quantum group orbits and transitivity, originally developed in [46], has an interesting extension into a theory of quantum group orbitals and 2-transitivity, recently developed in [105].

Now back to the tori, we have the following key result, from [46]:

**Theorem 13.28.** *Consider a quotient group as follows, with $N = N_1 + \ldots + N_k$:*

$$\mathbb{Z}_{N_1} * \ldots * \mathbb{Z}_{N_k} \to \Gamma$$

*We have then $\widehat{\Gamma} \subset S_N^+$, and any group dual subgroup of $S_N^+$ appears in this way.*

*Proof.* The fact that we have a subgroup as in the statement is something that we already know. Conversely, assume that we have a group dual subgroup $\widehat{\Gamma} \subset S_N^+$. The corresponding magic unitary must be of the following form, with $U \in U_N$:

$$u = U diag(g_1, \ldots, g_N) U^*$$

Consider now the orbit decomposition for $\widehat{\Gamma} \subset S_N^+$, coming from Theorem 13.25:

$$N = N_1 + \ldots + N_k$$

We conclude that $u$ has a $N = N_1 + \ldots + N_k$ block-diagonal pattern, and so that $U$ has as well this $N = N_1 + \ldots + N_k$ block-diagonal pattern.

But this discussion reduces our problem to its $k = 1$ particular case, with the statement here being that the cyclic group $\mathbb{Z}_N$ is the only transitive group dual $\widehat{\Gamma} \subset S_N^+$. The proof of this latter fact being elementary, we obtain the result. See [46]. □

Here is a related result, from [11], which is useful for our purposes:

**Theorem 13.29.** *For the quantum permutation group $S_N^+$, we have:*

(1) *Given $Q \in U_N$, the quotient $F_N \to \Lambda_Q$ comes from the following relations:*

$$\begin{cases} g_i = 1 & \text{if } \sum_l Q_{il} \neq 0 \\ g_i g_j = 1 & \text{if } \sum_l Q_{il} Q_{jl} \neq 0 \\ g_i g_j g_k = 1 & \text{if } \sum_l Q_{il} Q_{jl} Q_{kl} \neq 0 \end{cases}$$

(2) *Given a decomposition $N = N_1 + \ldots + N_k$, for the matrix $Q = diag(F_{N_1}, \ldots, F_{N_k})$, where $F_N = \frac{1}{\sqrt{N}}(\xi^{ij})_{ij}$ with $\xi = e^{2\pi i/N}$ is the Fourier matrix, we obtain:*

$$\Lambda_Q = \mathbb{Z}_{N_1} * \ldots * \mathbb{Z}_{N_k}$$

(3) *Given an arbitrary matrix $Q \in U_N$, there exists a decomposition $N = N_1 + \ldots + N_k$, such that $\Lambda_Q$ appears as quotient of $\mathbb{Z}_{N_1} * \ldots * \mathbb{Z}_{N_k}$.*

*Proof.* This is something more or less equivalent to Theorem 13.28, and the proof can be deduced either from Theorem 13.28, or from some direct computations. See [11]. □

Summarizing, in the quantum permutation group case, the standard tori parametrized by Fourier matrices play a special role. Now let us recall from section 7 that in what regards the bistochastic groups, which are our second class of examples where the diagonal liberation procedure does not apply, the Fourier matrices appear there as well.

All this discussion suggests formulating the following definition:

**Definition 13.30.** *Consider a closed subgroup $G \subset U_N^+$.*

(1) *Its standard tori $T_F$, with $F = F_{N_1} \otimes \ldots \otimes F_{N_k}$, and $N = N_1 + \ldots + N_k$ being regarded as a partition, are called Fourier tori.*
(2) *In the case where we have $G_N =< G_N^c, (T_F)_F >$, we say that $G_N$ appears as a Fourier liberation of its classical version $G_N^c$.*

We believe that the easy quantum groups should appear as Fourier liberations. With respect to Theorem 13.21 above, the situation in the free case is as follows:

(1) $O_N^+, U_N^+$ are diagonal liberations, so they are Fourier liberations as well.
(2) $B_N^+, C_N^+$ are Fourier liberations too, by using the results in section 7.
(3) $S_N^+$ is a Fourier liberation too, being generated by its tori [52], [56].
(4) $H_N^+, K_N^+$ remain to be investigated, by using the general theory in [117].

Finally, as a word of warning here, observe that an arbitrary classical group $G_N \subset U_N$ is not necessarily generated by its Fourier tori, and nor is an arbitrary discrete group dual, with spinned embedding. Thus, the Fourier tori, and the related notion of Fourier liberation, remain something quite technical, in connection with the easy case.

## 14. Amenability, growth

We have seen so far that the theory of the compact quantum Lie groups, $G \subset U_N^+$, can be developed with inspiration from the theory of compact Lie groups, $G \subset U_N$. In this section we discuss an alternative approach to all this, by looking at the finitely generated discrete quantum groups $\Gamma = \widehat{G}$ which are dual to our objects. Thus, the idea will be that of developing the theory of the finitely generated discrete quantum groups, $\widehat{U_N^+} \to \Gamma$, with inspiration from the theory of finitely generated discrete groups, $F_N \to \Gamma$.

As a first observation, the theory is already there, as developed in the previous sections, which equally concern the compact quantum group $G$ and its discrete dual $\Gamma = \widehat{G}$. However, from the discrete group viewpoint, what has been worked out so far looks more like specialized mathematics, and there are still a lot of basic things, to be developed. In short, what we will be doing here will be a "complement" to the material from the previous sections, obtained by using a different, and somehow opposite, philosophy.

Let us begin with a reminder regarding the cocommutative Woronowicz algebras, which will be our "main objects" in this section, coming before the commutative ones, that we are so used to have in the #1 spot. As explained in section 3 above, we have:

**Theorem 14.1.** *For a Woronowicz algebra $A$, the following are equivalent:*

(1) *$A$ is cocommutative, $\Sigma\Delta = \Delta$.*
(2) *The irreducible corepresentations of $A$ are all 1-dimensional.*
(3) *$A = C^*(\Gamma)$, for some group $\Gamma =< g_1, \ldots, g_N >$, up to equivalence.*

*Proof.* This follows from the Peter-Weyl theory, as follows:

(1) $\implies$ (2) The assumption $\Sigma\Delta = \Delta$ tells us that the inclusion $\mathcal{A}_{central} \subset \mathcal{A}$ is an isomorphism, and by using Peter-Weyl theory we conclude that any irreducible corepresentation of $A$ must be equal to its character, and so must be 1-dimensional.

(2) $\implies$ (3) This follows once again from Peter-Weyl, because if we denote by $\Gamma$ the group formed by the 1-dimensional corepresentations, then we have $\mathcal{A} = \mathbb{C}[\Gamma]$, and so $A = C^*(\Gamma)$ up to the standard equivalence relation for Woronowicz algebras.

(3) $\implies$ (1) This is something trivial, that we already know from section 2. □

The above result is not the end of the story, because one can still ask what are the cocommutative Woronowicz algebras, without reference to the equivalence relation.

More generally, we are led in this way into the question, that we have usually avoided so far, as being not part of the "compact" philosophy, of computing the equivalence class of a given Woronowicz algebra $A$. We first have here the following construction:

**Theorem 14.2.** *Given a Woronowicz algebra $(A,u)$, the enveloping $C^*$-algebra $A_{full}$ of the algebra of "smooth functions" $\mathcal{A}=<u_{ij}>$ has morphisms*

$$\Delta : A_{full} \to A_{full} \otimes A_{full}$$

$$\varepsilon : A_{full} \to \mathbb{C}$$

$$S : A_{full} \to A_{full}^{opp}$$

*which make it a Woronowicz algebra, which is equivalent to $A$. In the cocommutative case, where $A \sim C^*(\Gamma)$, we obtain in this way the full group algebra $C^*(\Gamma)$.*

*Proof.* There are several assertions here, the idea being as follows:

(1) Consider indeed the algebra $A_{full}$, obtained by completing the $*$-algebra $\mathcal{A} \subset A$ with respect to its maximal $C^*$-norm. We have then a quotient map, as follows:

$$\pi : A_{full} \to A$$

By universality of $A_{full}$, the comultiplication, counit and antipode of $A$ lift into morphisms $\Delta, \varepsilon, S$ as in the statement, and the Woronowicz algebra axioms are satisfied.

(2) The fact that we have an equivalence $A_{full} \sim A$ is clear from definitions, because at the level of $*$-algebras of coefficients, the above quotient map $\pi$ is an isomorphism.

(3) Finally, in the cocommutative case, where $A \sim C^*(\Gamma)$, the coefficient algebra is $\mathcal{A} = \mathbb{C}[\Gamma]$, and the corresponding enveloping $C^*$-algebra is $A_{full} = C^*(\Gamma)$. □

Summarizing, in connection with our equivalence class question, we already have an advance, with the construction of a biggest object in each equivalence class:

$$A_{full} \to A$$

We could of course stop our study here, by formulating the following statement, which apparently terminates any further discussion about equivalence classes:

**Proposition 14.3.** *Let us call a Woronowicz algebra "full" when the following canonical quotient map is an isomorphism:*

$$\pi : A_{full} \to A$$

*Then any Woronowicz algebra is equivalent to a full Woronowicz algebra, and when restricting the attention to the full algebras, we have 1 object per equivalence class.*

*Proof.* The first assertion is clear from Theorem 14.2, which tells us that we have $A \sim A_{full}$, and the second assertion holds as well, for exactly the same reason. □

As a first observation, restricting the attention to the full Woronowicz algebras is more or less what we have being doing so far in this book, with all the algebras that we introduced and studied being full by definition. However, there are several good reasons for not leaving things like this, and for further getting into the subject, one problem for instance coming from the fact that for the non-amenable groups $\Gamma$, we have:

$$C^*(\Gamma) \not\subset L(\Gamma)$$

To be more precise, on the right we have the group von Neumann algebra $L(\Gamma)$, appearing by definition as the weak closure of $\mathbb{C}[\Gamma]$, in the left regular representation. It is known that the above non-inclusion happens indeed in the non-amenable case, and in terms of the quantum group $G = \widehat{\Gamma}$, we are led to the following bizarre conclusion:

$$C(G) \not\subset L^\infty(G)$$

In other words, we have noncommutative continuous functions which are not measurable! This is something that we must clarify. Welcome to functional analysis.

Before anything, we must warn the reader that a lot of modesty and faith is needed, in order to deal with such questions. We are basically doing quantum mechanics here, where the moving objects don't have clear positions, or clear speeds, and where the precise laws of motion are not known, and where any piece of extra data costs a few billion dollars. Thus, the fact that we have $C(G) \not\subset L^\infty(G)$ is just one problem, among many other.

With this discussion made, let us go back now to Theorem 14.2. As a next step in our study, we can attempt to construct a smallest object $A_{red}$ in each equivalence class. The situation here is more tricky, and we have the following statement:

**Theorem 14.4.** *Given a Woronowicz algebra $(A, u)$, its quotient $A \to A_{red}$ by the null ideal of the Haar integration $tr : A \to \mathbb{C}$ has morphisms as follows,*

$$\Delta : A_{red} \to A_{red} \times A_{red}$$

$$\varepsilon : \mathcal{A}_{red} \to \mathbb{C}$$

$$S : A_{red} \to A_{red}^{opp}$$

*where $\times$ is the spatial tensor product of $C^*$-algebras, and where $\mathcal{A}_{red} =< u_{ij} >$. In the case where these morphisms lift into morphisms*

$$\Delta : A_{red} \to A_{red} \otimes A_{red}$$

$$\varepsilon : A_{red} \to \mathbb{C}$$

$$S : A_{red} \to A_{red}^{opp}$$

*we have a Woronowicz algebra, which is equivalent to $A$. Also, in the cocommutative case, where $A \sim C^*(\Gamma)$, we obtain in this way the reduced group algebra $C^*_{red}(\Gamma)$.*

*Proof.* We have several assertions here, the idea being as follows:

(1) Consider indeed the algebra $A_{red}$, obtained by dividing $A$ by the null ideal of the Haar integration $tr : A \to \mathbb{C}$. We have then a quotient map, as follows:

$$\pi : A \to A_{red}$$

Also, by GNS construction, we have an embedding as follows:

$$i : A_{red} \subset B(L^2(A))$$

By using these morphisms $\pi, i$, we can see that the comultiplication, counit and antipode of the $*$-algebra $\mathcal{A}$ lift into morphisms $\Delta, \varepsilon, S$ as in the statement, or, equivalently, that the comultiplication, counit and antipode of the $C^*$-algebra $A$ factorize into morphisms $\Delta, \varepsilon, S$ as in the statement. Thus, we have our morphisms, as claimed.

(2) In the case where the morphisms $\Delta, \varepsilon, S$ that we just constructed lift, as indicated in the statement, the Woronowicz algebra axioms are clearly satisfied, and so the algebra $A_{red}$, together with the matrix $u = (u_{ij})$, is a Woronowicz algebra, in our sense.

(3) The fact that we have an equivalence $A_{red} \sim A$ is clear from definitions, because at the level of $*$-algebras of coefficients, the above quotient map $\pi$ is an isomorphism.

(4) Finally, in the cocommutative case, where $A \sim C^*(\Gamma)$, the above embedding $i$ is the left regular representation, and so we have $A_{red} = C^*_{red}(\Gamma)$, as claimed. $\square$

With the above result in hand, which is complementary to Theorem 14.2, we can now answer some of our philosophical questions, the idea being as follows:

(1) In the group dual case we have $C^*_{red}(\Gamma) \subset L(\Gamma)$, as subalgebras of $B(l^2(\Gamma))$, and so in terms of the compact quantum group $G = \widehat{\Gamma}$, the conclusion is that we have $C(G) \subset L^\infty(G)$, as we should, with the convention $C(G) = C^*_{red}(\Gamma)$.

(2) In view of this, it is tempting to modify our Woronowicz algebra axioms, with $\Delta, \varepsilon, S$ being redefined as in the first part of Theorem 14.4, as to include the reduced group algebras $C^*_{red}(\Gamma)$, and more generally, all the algebras $A_{red}$.

(3) With such a modification done, we could call then a Woronowicz algebra "reduced" when the quotient map $A \to A_{red}$ is an isomorphism. This would lead to a nice situation like in Proposition 14.3, with 1 object per equivalence class.

(4) However, we will not do this, simply because the bulk of the present book, which is behind us, is full of interesting examples of Woronowicz algebras constructed with generators and relations, which are full by definition.

In short, nevermind for the philosophy, we will keep our axioms which are nice, simple and powerful, keeping however in mind the fact that the full picture is as follows:

**Theorem 14.5.** *Given a Woronowicz algebra $A$, we have morphisms*

$$A_{full} \to A \to A_{red} \subset A_{red}''$$

*which in terms of the associated compact quantum group $G$ read*

$$C_{full}(G) \to A \to C_{red}(G) \subset L^\infty(G)$$

*and in terms of the associated discrete quantum group $\Gamma$ read*

$$C^*(\Gamma) \to A \to C^*_{red}(\Gamma) \subset L(\Gamma)$$

*with Woronowicz algebras at left, and with von Neumann algebras at right.*

*Proof.* This is something rather philosophical, coming by putting together the results that we have, namely Theorem 14.2 and Theorem 14.4. □

With this discussion made, and with the reiterated warning that a lot of modesty and basic common sense is needed, in order to deal with such questions, let us get now into the real thing, namely the understanding of the following projection map:

$$\pi : A_{full} \to A_{red}$$

As already mentioned before, on numerous occasions, when the algebra $A$ is cocommutative, $A \sim C^*(\Gamma)$, and with the underlying group $\Gamma$ being assumed amenable, this projection map is an isomorphism. And the contrary happens when $\Gamma$ is not amenable.

This leads us into the amenability question for the general Woronowicz algebras $A$. We have seen the basic theory here in section 3 above, in the form of a list of equivalent conditions, which altogether are called amenability. The theory presented there, worked out now in more detail, and with a few items added, is as follows:

**Theorem 14.6.** *Let $A_{full}$ be the enveloping $C^*$-algebra of $\mathcal{A}$, and let $A_{red}$ be the quotient of $A$ by the null ideal of the Haar integration. The following are then equivalent:*

1. *The Haar functional of $A_{full}$ is faithful.*
2. *The projection map $A_{full} \to A_{red}$ is an isomorphism.*
3. *The counit map $\varepsilon : A \to \mathbb{C}$ factorizes through $A_{red}$.*
4. *We have $N \in \sigma(Re(\chi_u))$, the spectrum being taken inside $A_{red}$.*
5. $||ax_k - \varepsilon(a)x_k|| \to 0$ *for any $a \in \mathcal{A}$, for certain norm 1 vectors $x_k \in L^2(A)$.*

*If this is the case, we say that the underlying discrete quantum group $\Gamma$ is amenable.*

*Proof.* Before starting, we should mention that amenability and the present result are a bit like the Spectral Theorem, in the sense that knowing that the result formally holds does not help much, and in practice, one needs to remember the proof as well. For this reason, we will work out explicitely all the possible implications between (1-5), whenever possible, adding to the global formal proof, which will be linear, as follows:

$$(1) \implies (2) \implies (3) \implies (4) \implies (5) \implies (1)$$

In order to prove these implications, and the other ones too, the general idea is that this is is well-known in the group dual case, $A = C^*(\Gamma)$, with $\Gamma$ being a usual discrete group, and in general, the result follows by adapting the group dual case proof.

(1) $\iff$ (2) This follows from the fact that the GNS construction for the algebra $A_{full}$ with respect to the Haar functional produces the algebra $A_{red}$.

(2) $\implies$ (3) This is trivial, because we have quotient maps $A_{full} \to A \to A_{red}$, and so our assumption $A_{full} = A_{red}$ implies that we have $A = A_{red}$.

(3) $\implies$ (2) Assume indeed that we have a counit map $\varepsilon : A_{red} \to \mathbb{C}$. In order to prove $A_{full} = A_{red}$, we can use the right regular corepresentation. Indeed, as explained in [114], we can define such a corepresentation by the following formula:

$$W(a \otimes x) = \Delta(a)(1 \otimes x)$$

This corepresentation is unitary, so we can define a morphism as follows:

$$\Delta' : A_{red} \to A_{red} \otimes A_{full}$$

$$a \to W(a \otimes 1)W^*$$

Now by composing with $\varepsilon \otimes id$, we obtain a morphism as follows:

$$(\varepsilon \otimes id)\Delta' : A_{red} \to A_{full}$$

$$u_{ij} \to u_{ij}$$

Thus, we have our inverse for the canonical projection $A_{full} \to A_{red}$, as desired.

(3) $\implies$ (4) This implication is clear, because we have:

$$\begin{aligned}
\varepsilon(Re(\chi_u)) &= \frac{1}{2}\left(\sum_{i=1}^N \varepsilon(u_{ii}) + \sum_{i=1}^N \varepsilon(u_{ii}^*)\right) \\
&= \frac{1}{2}(N+N) \\
&= N
\end{aligned}$$

Thus the element $N - Re(\chi_u)$ is not invertible in $A_{red}$, as claimed.

(4) $\implies$ (3) In terms of the corepresentation $v = u + \bar{u}$, whose dimension is $2N$ and whose character is $2Re(\chi_u)$, our assumption $N \in \sigma(Re(\chi_u))$ reads:

$$\dim v \in \sigma(\chi_v)$$

By functional calculus the same must hold for $w = v + 1$, and then once again by functional calculus, the same must hold for any tensor power of $w$:

$$w_k = w^{\otimes k}$$

Now choose for each $k \in \mathbb{N}$ a state $\varepsilon_k \in A_{red}^*$ having the following property:

$$\varepsilon_k(w_k) = \dim w_k$$

By Peter-Weyl we must have $\varepsilon_k(r) = \dim r$ for any $r \leq w_k$, and since any irreducible corepresentation appears in this way, the sequence $\varepsilon_k$ converges to a counit map:

$$\varepsilon : A_{red} \to \mathbb{C}$$

(4) $\implies$ (5) Consider the following elements of $A_{red}$, which are positive:

$$a_i = 1 - Re(u_{ii})$$

Our assumption $N \in \sigma(Re(\chi_u))$ tells us that $a = \sum a_i$ is not invertible, and so there exists a sequence $x_k$ of norm one vectors in $L^2(A)$ such that:

$$< ax_k, x_k > \to 0$$

Since the summands $< a_i x_k, x_k >$ are all positive, we must have, for any $i$:

$$< a_i x_k, x_k > \to 0$$

We can go back to the variables $u_{ii}$ by using the following general formula:

$$||vx - x||^2 = ||vx||^2 + 2 < (1 - Re(v))x, x > -1$$

Indeed, with $v = u_{ii}$ and $x = x_k$ the middle term on the right goes to 0, and so the whole term on the right becomes asymptotically negative, and so we must have:

$$||u_{ii}x_k - x_k|| \to 0$$

Now let $M_n(A_{red})$ act on $\mathbb{C}^n \otimes L^2(A)$. Since $u$ is unitary we have:

$$\sum_i ||u_{ij}x_k||^2 = ||u(e_j \otimes x_k)|| = 1$$

From $||u_{ii}x_k|| \to 1$ we obtain $||u_{ij}x_k|| \to 0$ for $i \neq j$. Thus we have, for any $i, j$:

$$||u_{ij}x_k - \delta_{ij}x_k|| \to 0$$

Now by remembering that we have $\varepsilon(u_{ij}) = \delta_{ij}$, this formula reads:

$$||u_{ij}x_k - \varepsilon(u_{ij})x_k|| \to 0$$

By linearity, multiplicativity and continuity, we must have, for any $a \in \mathcal{A}$, as desired:

$$||ax_k - \varepsilon(a)x_k|| \to 0$$

(5) $\implies$ (1) This is something well-known, which follows via some standard functional analysis arguments, worked out in Blanchard's paper [51].

(1) $\implies$ (5) Once again this is something well-known, which follows via some standard functional analysis arguments, worked out in Blanchard's paper [51]. □

This was for the basic amenability theory. We will be back to this on several occasions, with more specialized amenability conditions, which will add to the above list.

As a first application of the above result, we can now advance on a problem left before, in section 3 above, and then in the beginning of the present section as well:

**Theorem 14.7.** *The cocommutative Woronowicz algebras are the intermediate quotients of the following type, with $\Gamma =< g_1, \ldots, g_N >$ being a discrete group,*

$$C^*(\Gamma) \to C^*_\pi(\Gamma) \to C^*_{red}(\Gamma)$$

*and with $\pi$ being a unitary representation of $\Gamma$, subject to weak containment conditions of type $\pi \otimes \pi \subset \pi$ and $1 \subset \pi$, which guarantee the existence of $\Delta, \varepsilon$.*

*Proof.* We use Theorem 14.1 above, combined with Theorem 14.5 and then with Theorem 14.6, the idea being to proceed in several steps, as follows:

(1) Theorem 14.1 and standard functional analysis arguments show that the cocommutative Woronowicz algebras should appear as intermediate quotients, as follows:

$$C^*(\Gamma) \to A \to C^*_{red}(\Gamma)$$

(2) The existence of $\Delta : A \to A \otimes A$ requires our intermediate quotient to appear as follows, with $\pi$ being a unitary representation of $\Gamma$, satisfying the condition $\pi \otimes \pi \subset \pi$, taken in a weak containment sense, and with the tensor product $\otimes$ being taken here to be compatible with our usual maximal tensor product $\otimes$ for the $C^*$-algebras:

$$C^*(\Gamma) \to C^*_\pi(\Gamma) \to C^*_{red}(\Gamma)$$

(3) With this condition imposed, the existence of the antipode $S : A \to A^{opp}$ is then automatic, coming from the group antirepresentation $g \to g^{-1}$.

(4) The existence of the counit $\varepsilon : A \to \mathbb{C}$, however, is something non-trivial, related to amenability, and leading to a condition of type $1 \subset \pi$, as in the statement. □

The above result is of course not the end of the story, because as formulated, with the above highly abstract conditions on $\pi$, it comes along with 0 non-trivial examples. We refer to Woronowicz's paper [148] for more on these topics, and to [101] for a more advanced discussion, dealing with the non-cocommutative case as well.

Let us get back now to real life, and concrete mathematics, and focus on the Kesten amenability criterion, from Theorem 14.6 (4) above, which brings connections with interesting mathematics and physics, and which in practice will be our main amenability criterion. In order to discuss this, we will need the following standard fact:

**Proposition 14.8.** *Given a Woronowicz algebra $(A, u)$, with $u \in M_N(A)$, the moments of the main character $\chi = \sum_i u_{ii}$ are given by:*

$$\int_G \chi^k = \dim\left(Fix(u^{\otimes k})\right)$$

*In the case $u \sim \bar{u}$ the law of $\chi$ is a usual probability measure, supported on $[-N, N]$.*

*Proof.* There are two assertions here, the proof being as follows:

(1) The first assertion follows from the Peter-Weyl theory, which tells us that we have the following formula, valid for any corepresentation $v \in M_n(A)$:

$$\int_G \chi_v = \dim(Fix(v))$$

Indeed, with $v = u^{\otimes k}$ the corresponding character is:

$$\chi_v = \chi^k$$

Thus, we obtain the result, as a consequence of the above formula.

(2) As for the second assertion, if we assume $u \sim \bar{u}$ then we have $\chi = \chi^*$, and so the general theory, explained above, tells us that $law(\chi)$ is in this case a real probability measure, supported by the spectrum of $\chi$. But, since $u \in M_N(A)$ is unitary, we have:

$$\begin{aligned} uu^* = 1 \quad &\implies \quad ||u_{ij}|| \leq 1, \forall i, j \\ &\implies \quad ||\chi|| \leq N \end{aligned}$$

Thus the spectrum of the character satisfies:

$$\sigma(\chi) \subset [-N, N]$$

Thus, we are led to the conclusion in the statement. □

In relation now with the notion of amenability, we have:

**Theorem 14.9.** *A Woronowicz algebra $(A, u)$, with $u \in M_N(A)$, is amenable when*

$$N \in supp\Big(law(Re(\chi))\Big)$$

*and the support on the right depends only on $law(\chi)$.*

*Proof.* There are two assertions here, the proof being as follows:

(1) According to the Kesten amenability criterion, from Theorem 14.6 (4) above, the algebra $A$ is amenable when the following condition is satisfied:

$$N \in \sigma(Re(\chi))$$

Now since $Re(\chi)$ is self-adjoint, we know from spectral theory that the support of its spectral measure $law(Re(\chi))$ is precisely its spectrum $\sigma(Re(\chi))$, as desired:

$$supp(law(Re(\chi))) = \sigma(Re(\chi))$$

(2) Regarding the second assertion, once again the variable $Re(\chi)$ being self-adjoint, its law depends only on the moments $\int_G Re(\chi)^p$, with $p \in \mathbb{N}$. But, we have:

$$\begin{aligned}\int_G Re(\chi)^p &= \int_G \left(\frac{\chi+\chi^*}{2}\right)^p \\ &= \frac{1}{2^p}\sum_{|k|=p}\int_G \chi^k\end{aligned}$$

Thus $law(Re(\chi))$ depends only on $law(\chi)$, and this gives the result. $\square$

Let us work out now in detail the group dual case. Here we obtain a very interesting measure, called Kesten measure of the group [99], as follows:

**Proposition 14.10.** *In the case $A = C^*(\Gamma)$ and $u = diag(g_1, \ldots, g_N)$, and with the following normalization made,*

$$1 \in u = \bar{u}$$

*the moments of the main character are given by the formula*

$$\int_{\widehat{\Gamma}} \chi^p = \#\left\{i_1, \ldots, i_p \Big| g_{i_1} \ldots g_{i_p} = 1\right\}$$

*counting the loops based at 1, having lenght $p$, on the corresponding Cayley graph.*

*Proof.* Consider indeed a discrete group $\Gamma =< g_1, \ldots, g_N >$. The main character of $A = C^*(\Gamma)$, with fundamental corepresentation $u = diag(g_1, \ldots, g_N)$, is then:

$$\chi = g_1 + \ldots + g_N$$

Given a colored integer $k = e_1 \ldots e_p$, the corresponding moment is given by:

$$\begin{aligned}\int_{\widehat{\Gamma}} \chi^k &= \int_{\widehat{\Gamma}} (g_1 + \ldots + g_N)^k \\ &= \#\left\{i_1, \ldots, i_p \Big| g_{i_1}^{e_1} \ldots g_{i_p}^{e_p} = 1\right\}\end{aligned}$$

In the self-adjoint case, $u \sim \bar{u}$, we are only interested in the moments with respect to usual integers, $p \in \mathbb{N}$, and the above formula becomes:

$$\int_{\widehat{\Gamma}} \chi^p = \#\left\{i_1, \ldots, i_p \Big| g_{i_1} \ldots g_{i_p} = 1\right\}$$

Assume now that we have in addition $1 \in u$, so that the condition $1 \in u = \bar{u}$ in the statement is satisfied. At the level of the generating set $S = \{g_1, \ldots, g_N\}$ this means:

$$1 \in S = S^{-1}$$

Thus the corresponding Cayley graph is well-defined, with the elements of $\Gamma$ as vertices, and with the edges $g - h$ appearing when the following condition is satisfied:

$$gh^{-1} \in S$$

A loop on this graph based at 1, having lenght $p$, is then a sequence as follows:

$$(1) - (g_{i_1}) - (g_{i_1}g_{i_2}) - \ldots - (g_{i_1}\ldots g_{i_{p-1}}) - (g_{i_1}\ldots g_{i_p} = 1)$$

Thus the moments of $\chi$ count indeed such loops, as claimed. $\square$

In order to generalize the above result to arbitrary Woronowicz algebras, we can use the discrete quantum group philosophy. The fundamental result here is as follows:

**Theorem 14.11.** *Let $(A,u)$ be a Woronowicz algebra, and assume, by enlarging if necessary $u$, that we have $1 \in u = \bar{u}$. The following formula*

$$d(v,w) = \min\left\{k \in \mathbb{N}\Big| 1 \subset \bar{v} \otimes w \otimes u^{\otimes k}\right\}$$

*defines then a distance on $Irr(A)$, which coincides with the geodesic distance on the associated Cayley graph. In the group dual case we obtain the usual distance.*

*Proof.* The fact that the lengths are finite follows from Woronowicz's analogue of Peter-Weyl theory, and the other verifications are as follows:

(1) The symmetry axiom is clear.

(2) The triangle inequality is elementary to establish as well.

(3) Finally, the last assertion is elementary as well.

In the group dual case now, where our Woronowicz algebra is of the form $A = C^*(\Gamma)$, with $\Gamma =< S >$ being a finitely generated discrete group, our normalization condition $1 \in u = \bar{u}$ means that the generating set must satisfy:

$$1 \in S = S^{-1}$$

But this is precisely the normalization condition for the discrete groups, and the fact that we obtain the same metric space is clear. $\square$

Summarizing, we have a good understanding of what a discrete quantum group is. We can now formulate a generalization of Proposition 14.10, as follows:

**Theorem 14.12.** *Let $(A,u)$ be a Woronowicz algebra, with the normalization assumption $1 \in u = \bar{u}$ made. The moments of the main character,*

$$\int_G \chi^p = \dim\left(Fix(u^{\otimes p})\right)$$

*count then the loops based at 1, having lenght $p$, on the corresponding Cayley graph.*

*Proof.* Here the formula of the moments, with $p \in \mathbb{N}$, is the one coming from Proposition 14.8 above, and the Cayley graph interpretation comes from Theorem 14.11. $\square$

Here is a related useful result, in relation with the notion of amenability:

**Theorem 14.13.** *A Woronowicz algebra $(A, u)$ is amenable precisely when*

$$||X|| = N$$

*where $X$ is the principal graph of the associated planar algebra*

$$P_k = End(u^{\otimes k})$$

*obtained by deleting the reflections in the Bratteli diagram of $P = (P_k)$.*

*Proof.* This is something which might look quite complicated, but the idea is very simple, namely that, via some standard identifications and rescalings, we have:

$$\begin{aligned} ||X|| &= ||M_{\chi_u}||_{A_{central}} \\ &= ||\chi_u||_{A_{central}} \\ &= ||\chi_u||_{A_{red}} \end{aligned}$$

Thus, the result follows from the Kesten amenability criterion. □

There are many concrete illustrations for the above results, and we will be back to this. As an application of this, we can introduce the notion of growth, as follows:

**Definition 14.14.** *Given a closed subgroup $G \subset U_N^+$, with $1 \in u = \bar{u}$, consider the series whose coefficients are the ball volumes on the corresponding Cayley graph,*

$$f(z) = \sum_k b_k z^k$$

$$b_k = \sum_{l(v) \leq k} \dim(v)^2$$

*and call it growth series of the discrete quantum group $\widehat{G}$. In the group dual case, $G = \widehat{\Gamma}$, we obtain in this way the usual growth series of $\Gamma$.*

There are many things that can be said about the growth, and we will be back to this in a moment, with explicit examples, and some general theory as well. As a first result, in relation with the notion of amenability, we have:

**Theorem 14.15.** *Polynomial growth implies amenability.*

*Proof.* We recall from Theorem 14.11 above that the Cayley graph of $\widehat{G}$ has by definition the elements of $Irr(G)$ as vertices, and the distance is as follows:

$$d(v, w) = \min\left\{k \in \mathbb{N} \Big| 1 \subset \bar{v} \otimes w \otimes u^{\otimes k}\right\}$$

By taking $w = 1$ and by using Frobenius reciprocity, the lenghts are given by:

$$l(v) = \min\left\{k \in \mathbb{N} \Big| v \subset u^{\otimes k}\right\}$$

By Peter-Weyl we have a decomposition as follows, where $B_k$ is the ball of radius $k$, and $m_k(v) \in \mathbb{N}$ are certain multiplicities:

$$u^{\otimes k} = \sum_{v \in B_k} m_k(v) \cdot v$$

By using now Cauchy-Schwarz, we obtain the following inequality:

$$\begin{aligned} m_{2k}(1)b_k &= \sum_{v \in B_k} m_k(v)^2 \sum_{v \in B_k} \dim(v)^2 \\ &\geq \left( \sum_{v \in B_k} m_k(v) \dim(v) \right)^2 \\ &= N^{2k} \end{aligned}$$

But shows that if $b_k$ has polynomial growth, then the following happens:

$$\limsup_{k \to \infty} m_{2k}(1)^{1/2k} \geq N$$

Thus, the Kesten type criterion applies, and gives the result. $\square$

Let us discuss now as well, as a continuation of all this, the notions of connectedness for $G$, and no torsion for $\widehat{\Gamma}$. These two notions are in fact related, as follows:

**Theorem 14.16.** *For a closed subgroup $G \subset U_N^+$ the following conditions are equivalent, and if they are satisfied, we call $G$ connected:*

1. *There is no finite quantum group quotient $G \to F \neq \{1\}$.*
2. *The algebra $< v_{ij} >$ is infinite dimensional, for any corepresentation $v \neq 1$.*

*In the classical case, $G \subset U_N$, we recover in this way the usual notion of connectedness. For the group duals, $G = \widehat{\Gamma}$, this is the same as asking for $\Gamma$ to have no torsion.*

*Proof.* The above equivalence comes from the fact that a quotient $G \to F$ must correspond to an embedding $C(F) \subset C(G)$, which must be of the form:

$$C(F) = < v_{ij} >$$

Regarding now the last two assertions, the situation here is as follows:

(1) In the classical case, $G \subset U_N$, it is well-known that $F = G/G_1$ is a finite group, where $G_1$ is the connected component of the identity $1 \in G$, and this gives the result.

(2) As for the group dual case, $G = \widehat{\Gamma}$, here the irreducible corepresentations are 1-dimensional, corresponding to the group elements $g \in \Gamma$, and this gives the result. $\square$

Along the same lines, and at a more specialized level, we can talk as well about the connected component of the identity, as follows:

**Theorem 14.17.** *Associated to any compact quantum group $G$ is the connected component of the identity*

$$G_0 \subset G$$

*which is a connected compact quantum group, in the above sense.*

*Proof.* This is something well-known, and for more on these topics, and on the Lie theory in general, in the present quantum group setting, we refer to [57], [66], [141]. □

Finally, once again in connection with all this, we can talk about normal subgroups, and about simple compact quantum groups, as follows:

**Definition 14.18.** *Given a quantum subgroup $H \subset G$, coming from a quotient map $\pi : C(G) \to C(H)$, the following are equivalent:*

(1) $A = \{a \in C(G) | (id \otimes \pi)\Delta(a) = a \otimes 1\}$ *satisfies* $\Delta(A) \subset A \otimes A$.
(2) $B = \{a \in C(G) | (\pi \otimes id)\Delta(a) = 1 \otimes a\}$ *satisfies* $\Delta(B) \subset B \otimes B$.
(3) *We have $A = B$, as subalgebras of $C(G)$.*

*If these conditions are satisfied, we say that $H \subset G$ is a normal subgroup.*

*Proof.* This is something well-known, the idea being as follows:

(1) The conditions in the statement are indeed equivalent, and in the classical case we obtain the usual normality notion for the subgroups.

(2) In the group dual case the normality of any subgroup, which must be a group dual subgroup, is then automatic, with this being something trivial.

(3) For more on these topics, and on the basic compact group theory in general, extended to the present quantum group setting, we refer to [57], [66], [141]. □

Let us discuss now some further questions, in relation with the theory of toral subgroups, developed in section 13 above. We have the following result, from [34]:

**Theorem 14.19.** *The following results hold, both over the category of compact Lie groups, and over the category of duals of finitely generated discrete groups:*

(1) *Characters: if $G$ is connected, for any nonzero $P \in C(G)_{central}$ there exists $Q \in U_N$ such that $P$ becomes nonzero, when mapped into $C(T_Q)$.*
(2) *Amenability: a closed subgroup $G \subset U_N^+$ is coamenable if and only if each of the tori $T_Q$ is coamenable, in the usual discrete group sense.*
(3) *Growth: assuming $G \subset U_N^+$, the discrete quantum group $\widehat{G}$ has polynomial growth if and only if each the discrete groups $\widehat{T_Q}$ has polynomial growth.*

*Proof.* In the classical case, where $G \subset U_N$, the proof goes as follows:

(1) Characters. We can take here $Q \in U_N$ to be such that $QTQ^* \subset \mathbb{T}^N$, where $T \subset U_N$ is a maximal torus for $G$, and this gives the result.

(2) Amenability. This conjecture holds trivially in the classical case, $G \subset U_N$, due to the fact that these latter quantum groups are all coamenable.

(3) Growth. This is something nontrivial, well-known from the theory of compact Lie groups, and we refer here for instance to [66].

Regarding now the group duals, here everything is trivial. Indeed, when the group duals are diagonally embedded we can take $Q = 1$, and when the group duals are embedded by using a spinning matrix $Q \in U_N$, we can use precisely this matrix $Q$. □

As in the previous section with the general results regarding the tori there, it is conjectures that the properties in Theorem 14.19 should hold in general. Following [34], we have the following result, regarding the free quantum groups:

**Theorem 14.20.** *The character, amenability and growth conjectures hold for the free quantum groups $G = O_N^+, U_N^+, S_N^+, H_N^+$.*

*Proof.* We have $3 \times 4 = 12$ assertions to be proved, and the idea in each case will be that of using certain special group dual subgroups. We will mostly use the group dual subgroups coming at $Q = 1$, which are well-known to be as follows:

$$G = O_N^+, U_N^+, S_N^+, H_N^+ \implies \Gamma_1 = \mathbb{Z}_2^{*N}, F_N, \{1\}, \mathbb{Z}_2^{*N}$$

However, for some of our 12 questions, using these subgroups will not be enough, and we will use as well some carefully chosen subgroups of type $\Gamma_Q$, with $Q \neq 1$.

As a last ingredient, we will need some specialized structure results for $G$, in the cases where $G$ is coamenable. Once again, the theory here is well-known, and the situations where $G = O_N^+, U_N^+, S_N^+, H_N^+$ is coamenable, along with the values of $G$, are as follows:

$$\begin{cases} O_2^+ = SU_2^{-1} \\ S_2^+ = S_2, S_3^+ = S_3, S_4^+ = SO_3^{-1} \\ H_2^+ = O_2^{-1} \end{cases}$$

To be more precise, the equalities $S_N^+ = S_N$ at $N \leq 3$ are known since Wang's paper [140], and the twisting results are all well-known, and we refer here to [11], [50].

With these ingredients in hand, we can now go ahead with the proof. It is technically convenient to split the discussion over the 3 conjectures, as follows:

(1) Characters. For $G = O_N^+, U_N^+$, it is known that the algebra $C(G)_{central}$ is polynomial, respectively $*$-polynomial, on the variable $\chi = \sum_i u_{ii}$. Thus, it is enough to show that the variable $\rho = \sum_i g_i$ generates a polynomial, respectively $*$-polynomial algebra, inside the group algebra of the discrete groups $\mathbb{Z}_2^{*N}, F_N$. But for $\mathbb{Z}_2^{*N}$ this is clear, and by using a multiplication by a unitary free from $\mathbb{Z}_2^{*N}$, the result holds as well for $F_N$.

Regarding now $G = S_N^+$, we have three cases to be discussed, as follows:

– At $N=2,3$ this quantum group collapses to the usual permutation group $S_N$, and the character conjecture holds indeed.

– At $N=4$ we have $S_4^+=SO_3^{-1}$, the fusion rules are well-known to be the Clebsch-Gordan ones, and the algebra $C(G)_{central}$ is therefore polynomial on $\chi=\sum_i u_{ii}$. Now observe that the spinned torus, with $Q=diag(F_2,F_2)$, is the following discrete group:

$$\Gamma_Q=\mathbb{Z}_2*\mathbb{Z}_2=D_\infty$$

Since $Tr(u)=Tr(Q^*uQ)$, the image of $\chi=\sum_i u_{ii}$ in the quotient $C^*(\Gamma_Q)$ is the variable $\rho=2+g+h$, where $g,h$ are the generators of the two copies of $\mathbb{Z}_2$. Now since this latter variable generates a polynomial algebra, we obtain the result.

– At $N\geq 5$ the fusion rules are once again known to be the Clebsch-Gordan ones, the algebra $C(G)_{central}$ is, as before, polynomial on $\chi=\sum_i u_{ii}$, and the result follows by functoriality from the result at $N=4$, by using the embedding $S_4^+\subset S_N^+$.

Regarding now $G=H_N^+$, here it is known, from the computations in [40], that the algebra $C(G)_{central}$ is polynomial on the following two variables:

$$\chi=\sum_i u_{ii}\quad,\quad\chi'=\sum_i u_{ii}^2$$

We have two cases to be discussed, as follows:

– At $N=2$ we have $H_2^+=O_2^{-1}$, and, as explained in [11], with $Q=F_2$ we have $\Gamma_Q=D_\infty$. Let us compute now the images $\rho,\rho'$ of the variables $\chi,\chi'$ in the group algebra of $D_\infty$. As before, from $Tr(u)=Tr(Q^*uQ)$ we obtain $\rho=g+h$, where $g,h$ are the generators of the two copies of $\mathbb{Z}_2$. Regarding now $\rho'$, let us first recall that the quotient map $C(H_2^+)\to C^*(D_\infty)$ is constructed as follows:

$$\frac{1}{2}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}u_{11}&u_{12}\\u_{21}&u_{22}\end{pmatrix}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\to\begin{pmatrix}g&0\\0&h\end{pmatrix}$$

Equivalently, this quotient map is constructed as follows:

$$\begin{aligned}\begin{pmatrix}u_{11}&u_{12}\\u_{21}&u_{22}\end{pmatrix}&\to\frac{1}{2}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}g&0\\0&h\end{pmatrix}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\\&=\frac{1}{2}\begin{pmatrix}g+h&g-h\\g-h&g+h\end{pmatrix}\end{aligned}$$

We can now compute the image of our character, as follows:

$$\rho'=\frac{1}{2}(g+h)^2=\frac{1}{2}(2+2gh)=1+gh$$

By using now the elementary fact that the variables $\rho=g+h$ and $\rho'=1+gh$ generate a polynomial algebra inside $C^*(D_\infty)$, this gives the result.

– Finally, at $N\geq 3$ the result follows by functoriality, via the standard diagonal inclusion $H_2^+\subset H_N^+$, from the result at $N=2$, that we established above.

(2) Amenability. Here the cases where $G$ is not coamenable are those of $O_N^+$ with $N\geq 3$, $U_N^+$ with $N\geq 2$, $S_N^+$ with $N\geq 5$, and $H_N^+$ with $N\geq 3$. For $G=O_N^+,H_N^+$ with $N\geq 3$ the result is clear, because $\Gamma_1=\mathbb Z_2^{*N}$ is not amenable. Clear as well is the result for $U_N^+$ with $N\geq 2$, because $\Gamma_1=F_N$ is not amenable. Finally, for $S_N^+$ with $N\geq 5$ the result holds as well, because of the presence of Bichon's group dual subgroup $\widehat{\mathbb Z_2*\mathbb Z_3}$.

(3) Growth. Here the growth is polynomial precisely in the situations where $G$ is infinite and coamenable, the precise cases being $O_2^+=SU_2^{-1}$, $S_4^+=SO_3^{-1}$, $H_2^+=O_2^{-1}$, and the result follows from the fact that the growth invariants are stable by twisting. □

We will prove now that the 3 conjectures hold for any half-classical quantum group. In order to do so, we can use the approach from [49], which is as follows:

**Theorem 14.21.** *Given a conjugation-stable closed subgroup $H\subset U_N$, consider the algebra $C([H])\subset M_2(C(H))$ generated by the following variables:*

$$u_{ij}=\begin{pmatrix}0&v_{ij}\\ \bar v_{ij}&0\end{pmatrix}$$

*Then $[H]$ is a compact quantum group, we have $[H]\subset O_N^*$, and any non-classical subgroup $G\subset O_N^*$ appears in this way, with $G=O_N^*$ itself appearing from $H=U_N$.*

*Proof.* The $2\times 2$ matrices in the statement are self-adjoint, half-commute, and the $N\times N$ matrix $u=(u_{ij})$ that they form is orthogonal, so we have an embedding $[H]\subset O_N^*$. The quantum group property of $[H]$ is also elementary to check, by using an alternative, equivalent construction, with a quantum group embedding as follows:

$$C([H])\subset C(H)\rtimes\mathbb Z_2$$

The surjectivity part is non-trivial, and we refer here to [49]. □

Regarding now the maximal tori, the situation is very simple, as follows:

**Proposition 14.22.** *The group dual subgroups $\widehat{[\Gamma]}_Q\subset[H]$ appear via*

$$[\Gamma]_Q=[\Gamma_Q]$$

*from the group dual subgroups $\widehat\Gamma_Q\subset H$ associated to $H\subset U_N$.*

*Proof.* Let us first discuss the case $Q=1$. Consider the diagonal subgroup $\widehat\Gamma_1\subset H$, with the associated quotient map $C(H)\to C(\widehat\Gamma_1)$ denoted:

$$v_{ij}\to\delta_{ij}h_i$$

At the level of the algebras of $2\times 2$ matrices, this map induces a quotient map:

$$M_2(C(H))\to M_2(C(\widehat\Gamma_1))$$

Our claim is that we have a factorization, as follows:

$$\begin{matrix} C([H]) & \subset & M_2(C(H)) \\ \downarrow & & \downarrow \\ C([\widehat{\Gamma}_1]) & \subset & M_2(C(\widehat{\Gamma}_1)) \end{matrix}$$

Indeed, it is enough to show that the standard generators of $C([H])$ and of $C([\widehat{\Gamma}_1])$ map to the same elements of $M_2(C(\widehat{\Gamma}_1))$. But these generators map indeed as follows:

$$\begin{matrix} u_{ij} & \to & \begin{pmatrix} 0 & v_{ij} \\ \bar{v}_{ij} & 0 \end{pmatrix} \\ & & \downarrow \\ \delta_{ij} v_{ij} & \to & \begin{pmatrix} 0 & \delta_{ij} h_i \\ \delta_{ij} h_i^{-1} & 0 \end{pmatrix} \end{matrix}$$

Thus we have the above factorization, and since the map on the left is obtained by imposing the relations $u_{ij} = 0$ with $i \neq j$, we obtain, as desired:

$$[\Gamma]_1 = [\Gamma_1]$$

In the general case now, $Q \in U_N$, the result follows by applying the above $Q = 1$ result to the quantum group $[H]$, with fundamental corepresentation $w = QuQ^*$. $\square$

Now back to our conjectures, we have the following result:

**Theorem 14.23.** *The* 3 *conjectures hold for any half-classical quantum group of the form* $[H] \subset O_N^*$*, with* $H \subset U_N$ *being connected.*

*Proof.* We know that the conjectures hold for $H \subset U_N$. The idea will be that of "transporting" these results, via $H \to [H]$:

(1) Characters. We can pick here a maximal torus $T = \Gamma_Q$ for the compact group $H \subset U_N$, and by using the formula $[\Gamma]_Q = [\Gamma_Q] = [T]$ from Proposition 14.22 above, we obtain the result, via the identification in Theorem 14.21.

(2) Amenability. There is nothing to be proved here, because $O_N^*$ is coamenable, and so are all its quantum subgroups. Note however, in relation with the comments made in section 3 above, that in the connected case, the Kesten measures of $G, [T]$ are intimately related. For some explicit formulae here, for $G = O_N^*$ itself, see [38].

(3) Growth. Here the situation is similar to the one for the amenability conjecture, because $[H]$ has polynomial growth. $\square$

## 15. Homogeneous spaces

We have seen that the closed subgroups $G \subset U_N^+$ can be investigated with a variety of techniques, for the most belonging to algebraic geometry and probability theory. Our purpose here is to extend some of these results to certain classes of "quantum homogeneous spaces". This is somehow the first step into extending what we have into a theory of noncommutative geometry, of algebraic and probabilistic nature.

This can be done at several levels of generality, and there has been quite some work here, starting with [32], [37], then going further with [6], and even further with [8]. In what follows we discuss the formalism in [6], which is quite broad, while remaining not very abstract. We will study the spaces of the following type:

$$X = (G_M \times G_N)\big/(G_L \times G_{M-L} \times G_{N-L})$$

These spaces cover indeed the quantum groups and the spheres. And also, they are quite concrete and useful objects, consisting of certain classes of "partial isometries". Our main result will be a verification of the Bercovici-Pata liberation criterion, for certain variables associated $\chi \in C(X)$, in a suitable $L, M, N \to \infty$ limit.

We begin with a study in the classical case. Our starting point will be:

**Definition 15.1.** *Associated to any integers $L \leq M, N$ are the spaces*

$$O_{MN}^L = \left\{ T : E \to F \text{ isometry} \Big| E \subset \mathbb{R}^N, F \subset \mathbb{R}^M, \dim_{\mathbb{R}} E = L \right\}$$

$$U_{MN}^L = \left\{ T : E \to F \text{ isometry} \Big| E \subset \mathbb{C}^N, F \subset \mathbb{C}^M, \dim_{\mathbb{C}} E = L \right\}$$

*where the notion of isometry is with respect to the usual real/complex scalar products.*

As a first observation, at $L = M = N$ we obtain the groups $O_N, U_N$:

$$O_{NN}^N = O_N \quad , \quad U_{NN}^N = U_N$$

Another interesting specialization is $L = M = 1$. Here the elements of $O_{1N}^1$ are the isometries $T : E \to \mathbb{R}$, with $E \subset \mathbb{R}^N$ one-dimensional. But such an isometry is uniquely determined by $T^{-1}(1) \in \mathbb{R}^N$, which must belong to $S_{\mathbb{R}}^{N-1}$. Thus, we have $O_{1N}^1 = S_{\mathbb{R}}^{N-1}$. Similarly, in the complex case we have $U_{1N}^1 = S_{\mathbb{C}}^{N-1}$, and so our results here are:

$$O_{1N}^1 = S_{\mathbb{R}}^{N-1} \quad , \quad U_{1N}^1 = S_{\mathbb{C}}^{N-1}$$

Yet another interesting specialization is $L = N = 1$. Here the elements of $O_{1N}^1$ are the isometries $T : \mathbb{R} \to F$, with $F \subset \mathbb{R}^M$ one-dimensional. But such an isometry is uniquely determined by $T(1) \in \mathbb{R}^M$, whichmust belong to $S_{\mathbb{R}}^{M-1}$. Thus, we have $O_{M1}^1 = S_{\mathbb{R}}^{M-1}$. Similarly, in the complex case we have $U_{M1}^1 = S_{\mathbb{C}}^{M-1}$, and so our results here are:

$$O_{M1}^1 = S_{\mathbb{R}}^{M-1} \quad , \quad U_{M1}^1 = S_{\mathbb{C}}^{M-1}$$

In general, the most convenient is to view the elements of $O_{MN}^L, U_{MN}^L$ as rectangular matrices, and to use matrix calculus for their study. We have indeed:

**Proposition 15.2.** *We have identifications of compact spaces*

$$O_{MN}^L \simeq \left\{ U \in M_{M\times N}(\mathbb{R}) \Big| UU^t = \text{projection of trace } L \right\}$$

$$U_{MN}^L \simeq \left\{ U \in M_{M\times N}(\mathbb{C}) \Big| UU^* = \text{projection of trace } L \right\}$$

*with each partial isometry being identified with the corresponding rectangular matrix.*

*Proof.* We can indeed identify the partial isometries $T : E \to F$ with their corresponding extensions $U : \mathbb{R}^N \to \mathbb{R}^M$, $U : \mathbb{C}^N \to \mathbb{C}^M$, obtained by setting $U_{E^\perp} = 0$. Then, we can identify these latter linear maps $U$ with the corresponding rectangular matrices. □

As an illustration, at $L = M = N$ we recover in this way the usual matrix description of $O_N, U_N$. Also, at $L = M = 1$ we obtain the usual description of $S_\mathbb{R}^{N-1}, S_\mathbb{C}^{N-1}$, as row spaces over the corresponding groups $O_N, U_N$. Finally, at $L = N = 1$ we obtain the usual description of $S_\mathbb{R}^{N-1}, S_\mathbb{C}^{N-1}$, as column spaces over the corresponding groups $O_N, U_N$.

Now back to the general case, observe that the isometries $T : E \to F$, or rather their extensions $U : \mathbb{K}^N \to \mathbb{K}^M$, with $\mathbb{K} = \mathbb{R}, \mathbb{C}$, obtained by setting $U_{E^\perp} = 0$, can be composed with the isometries of $\mathbb{K}^M, \mathbb{K}^N$, according to the following scheme:

$$\begin{tikzcd}
\mathbb{K}^N \arrow[r, "B^*"] & \mathbb{K}^N \arrow[r, dotted, "U"] & \mathbb{K}^M \arrow[r, "A"] & \mathbb{K}^M \\
B(E) \arrow[u] \arrow[r, dotted] & E \arrow[u] \arrow[r, "T"] & F \arrow[u] \arrow[r, dotted] & A(F) \arrow[u]
\end{tikzcd}$$

With the identifications in Proposition 15.2 made, the precise statement here is:

**Proposition 15.3.** *We have an action map as follows, which is transitive,*

$$O_M \times O_N \curvearrowright O_{MN}^L$$

$$(A, B)U = AUB^t$$

*as well as an action map as follows, transitive as well,*

$$U_M \times U_N \curvearrowright U_{MN}^L$$

$$(A, B)U = AUB^*$$

*whose stabilizers are respectively:*

$$O_L \times O_{M-L} \times O_{N-L}$$

$$U_L \times U_{M-L} \times U_{N-L}$$

*Proof.* We have indeed action maps as in the statement, which are transitive. Let us compute now the stabilizer $G$ of the following point:

$$U = \begin{pmatrix}1&0\\0&0\end{pmatrix}$$

Since $(A,B) \in G$ satisfy $AU = UB$, their components must be of the following form:

$$A = \begin{pmatrix}x&*\\0&a\end{pmatrix} \quad , \quad B = \begin{pmatrix}x&0\\*&b\end{pmatrix}$$

Now since $A, B$ are both unitaries, these matrices follow to be block-diagonal, and so:

$$G = \left\{(A,B)\Big|A = \begin{pmatrix}x&0\\0&a\end{pmatrix}, B = \begin{pmatrix}x&0\\0&b\end{pmatrix}\right\}$$

The stabilizer of $U$ is then parametrized by triples $(x, a, b)$ belonging respectively to:

$$O_L \times O_{M-L} \times O_{N-L}$$

$$U_L \times U_{M-L} \times U_{N-L}$$

Thus, we are led to the conclusion in the statement. $\square$

Finally, let us work out the quotient space description of $O_{MN}^L, U_{MN}^L$. We have here:

**Theorem 15.4.** *We have isomorphisms of homogeneous spaces as follows,*

$$\begin{aligned} O_{MN}^L &= (O_M \times O_N)/(O_L \times O_{M-L} \times O_{N-L}) \\ U_{MN}^L &= (U_M \times U_N)/(U_L \times U_{M-L} \times U_{N-L}) \end{aligned}$$

*with the quotient maps being given by $(A,B) \to AUB^*$, where:*

$$U = \begin{pmatrix}1&0\\0&0\end{pmatrix}$$

*Proof.* This is just a reformulation of Proposition 15.3 above, by taking into account the fact that the fixed point used in the proof there was $U = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$. $\square$

Once again, the basic examples here come from the cases $L = M = N$ and $L = M = 1$. At $L = M = N$ the quotient spaces at right are respectively:

$$O_N, U_N$$

At $L = M = 1$ the quotient spaces at right are respectively:

$$O_N/O_{N-1} \quad , \quad U_N/U_{N-1}$$

In fact, in the general orthogonal $L = M$ case we obtain the following spaces:

$$\begin{aligned} O_{MN}^M &= (O_M \times O_N)/(O_M \times O_{N-M}) \\ &= O_N/O_{N-M} \end{aligned}$$

Also, in the general unitary $L=M$ case we obtain the following spaces:

$$\begin{aligned} U_{MN}^M &= (U_M\times U_N)/(U_M\times U_{N-M})\\ &= U_N/U_{N-M} \end{aligned}$$

Similarly, the examples coming from the cases $L=M=N$ and $L=N=1$ are particular cases of the general $L=N$ case, where we obtain the following spaces:

$$\begin{aligned} O_{MN}^N &= (O_M\times O_N)/(O_M\times O_{M-N})\\ &= O_N/O_{M-N} \end{aligned}$$

In the unitary case, we obtain the following spaces:

$$\begin{aligned} U_{MN}^N &= (U_M\times U_N)/(U_M\times U_{M-N})\\ &= U_N/U_{M-N} \end{aligned}$$

We can liberate the spaces $O_{MN}^L,U_{MN}^L$, as follows:

**Definition 15.5.** *Associated to any integers $L\leq M,N$ are the algebras*

$$\begin{aligned} C(O_{MN}^{L+}) &= C^*\left((u_{ij})_{i=1,\ldots,M,j=1,\ldots,N}\Big|u=\bar{u},uu^t=\text{projection of trace }L\right)\\ C(U_{MN}^{L+}) &= C^*\left((u_{ij})_{i=1,\ldots,M,j=1,\ldots,N}\Big|uu^*,\bar{u}u^t=\text{projections of trace }L\right) \end{aligned}$$

*with the trace being by definition the sum of the diagonal entries.*

Observe that the above universal algebras are indeed well-defined, as it was previously the case for the free spheres, and this due to the trace conditions, which read:

$$\sum_{ij}u_{ij}u_{ij}^*=\sum_{ij}u_{ij}^*u_{ij}=L$$

We have inclusions between the various spaces constructed so far, as follows:

$$\begin{array}{ccc} O_{MN}^{L+} & \longrightarrow & U_{MN}^{L+}\\ \uparrow & & \uparrow\\ O_{MN}^{L} & \longrightarrow & U_{MN}^{L} \end{array}$$

At the level of basic examples now, we first have the following result:

**Proposition 15.6.** *At $L=M=1$ we obtain the diagram*

$$\begin{tikzcd} S^{N-1}_{\mathbb{R},+} \arrow[r] & S^{N-1}_{\mathbb{C},+} \\ S^{N-1}_{\mathbb{R}} \arrow[r] \arrow[u] & S^{N-1}_{\mathbb{C}} \arrow[u] \end{tikzcd}$$

*and at $L=N=1$ we obtain the diagram:*

$$\begin{tikzcd} S^{M-1}_{\mathbb{R},+} \arrow[r] & S^{M-1}_{\mathbb{C},+} \\ S^{M-1}_{\mathbb{R}} \arrow[r] \arrow[u] & S^{M-1}_{\mathbb{C}} \arrow[u] \end{tikzcd}$$

*Proof.* Both the assertions are clear from definitions. □

We have as well the following result:

**Proposition 15.7.** *At $L=M=N$ we obtain the diagram*

$$\begin{tikzcd} O_N^+ \arrow[r] & U_N^+ \\ O_N \arrow[r] \arrow[u] & U_N \arrow[u] \end{tikzcd}$$

*consisting of the groups $O_N,U_N$, and their liberations.*

*Proof.* We recall that the various quantum groups in the statement are constructed as follows, with the symbol $\times$ standing once again for "commutative" and "free":

$$\begin{aligned} C(O_N^\times) &= C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\bar{u},uu^t=u^tu=1\right)\\ C(U_N^\times) &= C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big|uu^*=u^*u=1,\bar{u}u^t=u^t\bar{u}=1\right)\end{aligned}$$

On the other hand, according to Proposition 15.2 and to Definition 15.5 above, we have the following presentation results:

$$\begin{aligned} C(O_{NN}^{N\times}) &= C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\bar{u},uu^t=\text{projection of trace }N\right)\\ C(U_{NN}^{N\times}) &= C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big|uu^*,\bar{u}u^t=\text{projections of trace }N\right)\end{aligned}$$

We use now the standard fact that if $p=aa^*$ is a projection then $q=a^*a$ is a projection too. We use as well the following formulae:

$$Tr(uu^*)=Tr(u^t\bar{u})$$

$$Tr(\bar{u}u^t)=Tr(u^*u)$$

We therefore obtain the following formulae:

$$\begin{aligned}
C(O_{NN}^{N\times}) &= C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=\bar{u},\ uu^t,u^tu=\text{projections of trace }N\right)\\
C(U_{NN}^{N\times}) &= C^*_\times\left((u_{ij})_{i,j=1,\ldots,N}\Big|uu^*,u^*u,\bar{u}u^t,u^t\bar{u}=\text{projections of trace }N\right)
\end{aligned}$$

Now observe that, in tensor product notation, and by using the normalized trace, the conditions at right are all of the form:

$$(tr\otimes id)p=1$$

To be more precise, $p$ is a follows, for the above conditions:

$$p=uu^*,u^*u,\bar{u}u^t,u^t\bar{u}$$

We therefore obtain, for any faithful state $\varphi$:

$$(tr\otimes\varphi)(1-p)=0$$

It follows from this that the projections $p=uu^*,u^*u,\bar{u}u^t,u^t\bar{u}$ must be all equal to the identity, as desired, and this finishes the proof. $\square$

Regarding now the homogeneous space structure of $O_{MN}^{L\times},U_{MN}^{L\times}$, the situation here is more complicated in the free case than in the classical case. We have:

**Proposition 15.8.** *The spaces $U_{MN}^{L\times}$ have the following properties:*

(1) *We have an action $U_M^\times\times U_N^\times\curvearrowright U_{MN}^{L\times}$, given by:*

$$u_{ij}\to\sum_{kl}u_{kl}\otimes a_{ki}\otimes b_{lj}^*$$

(2) *We have a map $U_M^\times\times U_N^\times\to U_{MN}^{L\times}$, given by:*

$$u_{ij}\to\sum_{r\leq L}a_{ri}\otimes b_{rj}^*$$

*Similar results hold for the spaces $O_{MN}^{L\times}$, with all the $*$ exponents removed.*

*Proof.* In the classical case, consider the action and quotient maps:

$$U_M\times U_N\curvearrowright U_{MN}^L$$

$$U_M\times U_N\to U_{MN}^L$$

The transposes of these two maps are as follows, where $J = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$:

$$\begin{aligned}\varphi &\to ((U,A,B)\to\varphi(AUB^*))\\ \varphi &\to ((A,B)\to\varphi(AJB^*))\end{aligned}$$

But with $\varphi = u_{ij}$ we obtain precisely the formulae in the statement. The proof in the orthogonal case is similar. Regarding now the free case, the proof goes as follows:

(1) Assuming $uu^*u = u$, let us set:

$$U_{ij} = \sum_{kl} u_{kl}\otimes a_{ki}\otimes b_{lj}^*$$

We have then:

$$\begin{aligned}(UU^*U)_{ij} &= \sum_{pq}\sum_{klmnst} u_{kl}u_{mn}^*u_{st}\otimes a_{ki}a_{mq}^*a_{sq}\otimes b_{lp}^*b_{np}b_{tj}^*\\ &= \sum_{klmt} u_{kl}u_{ml}^*u_{mt}\otimes a_{ki}\otimes b_{tj}^*\\ &= \sum_{kt} u_{kt}\otimes a_{ki}\otimes b_{tj}^*\\ &= U_{ij}\end{aligned}$$

Also, assuming that we have $\sum_{ij} u_{ij}u_{ij}^* = L$, we obtain:

$$\begin{aligned}\sum_{ij} U_{ij}U_{ij}^* &= \sum_{ij}\sum_{klst} u_{kl}u_{st}^*\otimes a_{ki}a_{si}^*\otimes b_{lj}^*b_{tj}\\ &= \sum_{kl} u_{kl}u_{kl}^*\otimes 1\otimes 1\\ &= L\end{aligned}$$

(2) Assuming $uu^*u = u$, let us set:

$$V_{ij} = \sum_{r\leq L} a_{ri}\otimes b_{rj}^*$$

We have then:

$$\begin{aligned}(VV^*V)_{ij} &= \sum_{pq}\sum_{x,y,z\leq L} a_{xi}a_{yq}^*a_{zq}\otimes b_{xp}^*b_{yp}b_{zj}^*\\ &= \sum_{x\leq L} a_{xi}\otimes b_{xj}^*\\ &= V_{ij}\end{aligned}$$

Also, assuming that we have $\sum_{ij}u_{ij}u_{ij}^*=L$, we obtain:

$$\begin{aligned}\sum_{ij}V_{ij}V_{ij}^* &= \sum_{ij}\sum_{r,s\leq L}a_{ri}a_{si}^*\otimes b_{rj}^*b_{sj}\\ &= \sum_{l\leq L}1\\ &= L\end{aligned}$$

By removing all the $*$ exponents, we obtain as well the orthogonal results. $\square$

Let us examine now the relation between the above maps. In the classical case, given a quotient space $X=G/H$, the associated action and quotient maps are given by:

$$\begin{cases}a:X\times G\to X & : \quad (Hg,h)\to Hgh\\ p:G\to X & : \quad g\to Hg\end{cases}$$

Thus we have $a(p(g),h)=p(gh)$. In our context, a similar result holds:

**Theorem 15.9.** *With $G=G_M\times G_N$ and $X=G_{MN}^L$, where $G_N=O_N^\times,U_N^\times$, we have*

$$\begin{tikzcd} G\times G \arrow[r, "m"] \arrow[d, "p\times id"'] & G \arrow[d, "p"] \\ X\times G \arrow[r, "a"] & X \end{tikzcd}$$

*where $a,p$ are the action map and the map constructed in Proposition 15.8.*

*Proof.* At the level of the associated algebras of functions, we must prove that the following diagram commutes, where $\Phi,\alpha$ are morphisms of algebras induced by $a,p$:

$$\begin{tikzcd} C(X) \arrow[r, "\Phi"] \arrow[d, "\alpha"'] & C(X\times G) \arrow[d, "\alpha\otimes id"] \\ C(G) \arrow[r, "\Delta"] & C(G\times G) \end{tikzcd}$$

When going right, and then down, the composition is as follows:

$$\begin{aligned}(\alpha\otimes id)\Phi(u_{ij}) &= (\alpha\otimes id)\sum_{kl}u_{kl}\otimes a_{ki}\otimes b_{lj}^*\\ &= \sum_{kl}\sum_{r\leq L}a_{rk}\otimes b_{rl}^*\otimes a_{ki}\otimes b_{lj}^*\end{aligned}$$

On the other hand, when going down, and then right, the composition is as follows, where $F_{23}$ is the flip between the second and the third components:

$$\begin{aligned}\Delta\pi(u_{ij}) &= F_{23}(\Delta\otimes\Delta)\sum_{r\leq L} a_{ri}\otimes b_{rj}^* \\ &= F_{23}\left(\sum_{r\leq L}\sum_{kl} a_{rk}\otimes a_{ki}\otimes b_{rl}^*\otimes b_{lj}^*\right)\end{aligned}$$

Thus the above diagram commutes indeed, and this gives the result. □

Let us discuss now some extensions of the above constructions. We will be mostly interested in the quantum reflection groups, so let us first discuss, with full details, the case of the quantum groups $H_N^s, H_N^{s+}$. We use the following notion:

**Definition 15.10.** *Associated to any partial permutation, $\sigma : I \simeq J$ with $I\subset\{1,\ldots,N\}$ and $J\subset\{1,\ldots,M\}$, is the real/complex partial isometry*

$$T_\sigma : span\left(e_i\Big|i\in I\right)\to span\left(e_j\Big|j\in J\right)$$

*given on the standard basis elements by $T_\sigma(e_i)=e_{\sigma(i)}$.*

We denote by $S_{MN}^L$ the set of partial permutations $\sigma : I\simeq J$ as above, with range $I\subset\{1,\ldots,N\}$ and target $J\subset\{1,\ldots,M\}$, and with $L=|I|=|J|$. In analogy with the decomposition result $H_N^s=\mathbb{Z}_s\wr S_N$, we have:

**Proposition 15.11.** *The space of partial permutations signed by elements of $\mathbb{Z}_s$,*

$$H_{MN}^{sL}=\left\{T(e_i)=w_ie_{\sigma(i)}\Big|\sigma\in S_{MN}^L, w_i\in\mathbb{Z}_s\right\}$$

*is isomorphic to the quotient space*

$$(H_M^s\times H_N^s)/(H_L^s\times H_{M-L}^s\times H_{N-L}^s)$$

*via a standard isomorphism.*

*Proof.* This follows by adapting the computations in the proof of Proposition 15.3 above. Indeed, we have an action map as follows, which is transitive:

$$H_M^s\times H_N^s\to H_{MN}^{sL}$$

$$(A,B)U=AUB^*$$

Consider now the following point:

$$U=\begin{pmatrix}1&0\\0&0\end{pmatrix}$$

The stabilizer of this point follows to be the following group:

$$H_L^s\times H_{M-L}^s\times H_{N-L}^s$$

To be more precise, this group is embedded via:

$$(x,a,b)\to\left[\begin{pmatrix}x&0\\0&a\end{pmatrix},\begin{pmatrix}x&0\\0&b\end{pmatrix}\right]$$

But this gives the result. □

In the free case now, the idea is similar, by using inspiration from the construction of the quantum group $H_N^{s+}=\mathbb{Z}_s\wr_* S_N^+$ in [10]. The result here is as follows:

**Proposition 15.12.** *The compact quantum space $H_{MN}^{sL+}$ associated to the algebra*

$$C(H_{MN}^{sL+})=C(U_{MN}^{L+})\Big/\left\langle u_{ij}u_{ij}^*=u_{ij}^*u_{ij}=p_{ij}=\text{projections},u_{ij}^s=p_{ij}\right\rangle$$

*has an action map, and is the target of a quotient map, as in Theorem 15.9 above.*

*Proof.* We must show that if the variables $u_{ij}$ satisfy the relations in the statement, then these relations are satisfied as well for the following variables:

$$U_{ij}=\sum_{kl}u_{kl}\otimes a_{ki}\otimes b_{lj}^*$$

$$V_{ij}=\sum_{r\leq L}a_{ri}\otimes b_{rj}^*$$

We use the fact that the standard coordinates $a_{ij},b_{ij}$ on the quantum groups $H_M^{s+},H_N^{s+}$ satisfy the following relations, for any $x\neq y$ on the same row or column of $a,b$:

$$xy=xy^*=0$$

We obtain, by using these relations:

$$\begin{aligned}U_{ij}U_{ij}^*&=\sum_{klmn}u_{kl}u_{mn}^*\otimes a_{ki}a_{mi}^*\otimes b_{lj}^*b_{mj}\\&=\sum_{kl}u_{kl}u_{kl}^*\otimes a_{ki}a_{ki}^*\otimes b_{lj}^*b_{lj}\end{aligned}$$

We have as well the following formula:

$$\begin{aligned}V_{ij}V_{ij}^*&=\sum_{r,t\leq L}a_{ri}a_{ti}^*\otimes b_{rj}^*b_{tj}\\&=\sum_{r\leq L}a_{ri}a_{ri}^*\otimes b_{rj}^*b_{rj}\end{aligned}$$

Consider now the following projections:

$$x_{ij}=a_{ij}a_{ij}^*$$

$$y_{ij}=b_{ij}b_{ij}^*$$

$$p_{ij}=u_{ij}u_{ij}^*$$

In terms of these projections, we have:

$$U_{ij}U_{ij}^* = \sum_{kl} p_{kl} \otimes x_{ki} \otimes y_{lj}$$

$$V_{ij}V_{ij}^* = \sum_{r\leq L} x_{ri} \otimes y_{rj}$$

By repeating the computation, we conclude that these elements are projections. Also, a similar computation shows that $U_{ij}^*U_{ij}, V_{ij}^*V_{ij}$ are given by the same formulae.

Finally, once again by using the relations of type $xy = xy^* = 0$, we have:

$$\begin{aligned} U_{ij}^s &= \sum_{k_r l_r} u_{k_1 l_1} \ldots u_{k_s l_s} \otimes a_{k_1 i} \ldots a_{k_s i} \otimes b_{l_1 j}^* \ldots b_{l_s j}^* \\ &= \sum_{kl} u_{kl}^s \otimes a_{ki}^s \otimes (b_{lj}^*)^s \end{aligned}$$

We have as well the following formula:

$$\begin{aligned} V_{ij}^s &= \sum_{r_l \leq L} a_{r_1 i} \ldots a_{r_s i} \otimes b_{r_1 j}^* \ldots b_{r_s j}^* \\ &= \sum_{r\leq L} a_{ri}^s \otimes (b_{rj}^*)^s \end{aligned}$$

Thus the conditions of type $u_{ij}^s = p_{ij}$ are satisfied as well, and we are done. □

Let us discuss now the general case. We have the following result:

**Proposition 15.13.** *The various spaces $G_{MN}^L$ constructed so far appear by imposing to the standard coordinates of $U_{MN}^{L+}$ the relations*

$$\sum_{i_1\ldots i_s} \sum_{j_1\ldots j_s} \delta_\pi(i)\delta_\sigma(j) u_{i_1 j_1}^{e_1} \ldots u_{i_s j_s}^{e_s} = L^{|\pi\vee\sigma|}$$

*with $s = (e_1, \ldots, e_s)$ ranging over all the colored integers, and with $\pi, \sigma \in D(0, s)$.*

*Proof.* According to the various constructions above, the relations defining $G_{MN}^L$ can be written as follows, with $\sigma$ ranging over a family of generators, with no upper legs, of the corresponding category of partitions $D$:

$$\sum_{j_1\ldots j_s} \delta_\sigma(j) u_{i_1 j_1}^{e_1} \ldots u_{i_s j_s}^{e_s} = \delta_\sigma(i)$$

We therefore obtain the relations in the statement, as follows:

$$\begin{aligned}\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\delta_\pi(i)\delta_\sigma(j)u_{i_1j_1}^{e_1}\ldots u_{i_sj_s}^{e_s} &= \sum_{i_1\ldots i_s}\delta_\pi(i)\sum_{j_1\ldots j_s}\delta_\sigma(j)u_{i_1j_1}^{e_1}\ldots u_{i_sj_s}^{e_s}\\ &= \sum_{i_1\ldots i_s}\delta_\pi(i)\delta_\sigma(i)\\ &= L^{|\pi\vee\sigma|}\end{aligned}$$

As for the converse, this follows by using the relations in the statement, by keeping $\pi$ fixed, and by making $\sigma$ vary over all the partitions in the category. □

In the general case now, where $G=(G_N)$ is an arbitary uniform easy quantum group, we can construct spaces $G_{MN}^L$ by using the above relations, and we have:

**Theorem 15.14.** *The spaces $G_{MN}^L\subset U_{MN}^{L+}$ constructed by imposing the relations*

$$\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\delta_\pi(i)\delta_\sigma(j)u_{i_1j_1}^{e_1}\ldots u_{i_sj_s}^{e_s}=L^{|\pi\vee\sigma|}$$

*with $\pi,\sigma$ ranging over all the partitions in the associated category, having no upper legs, are subject to an action map/quotient map diagram, as in Theorem 15.9.*

*Proof.* We proceed as in the proof of Proposition 15.8. We must prove that, if the variables $u_{ij}$ satisfy the relations in the statement, then so do the following variables:

$$U_{ij}=\sum_{kl}u_{kl}\otimes a_{ki}\otimes b_{lj}^*$$

$$V_{ij}=\sum_{r\leq L}a_{ri}\otimes b_{rj}^*$$

Regarding the variables $U_{ij}$, the computation here goes as follows:

$$\begin{aligned}&\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\delta_\pi(i)\delta_\sigma(j)U_{i_1j_1}^{e_1}\ldots U_{i_sj_s}^{e_s}\\ =&\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\sum_{k_1\ldots k_s}\sum_{l_1\ldots l_s}u_{k_1l_1}^{e_1}\ldots u_{k_sl_s}^{e_s}\otimes\delta_\pi(i)\delta_\sigma(j)a_{k_1i_1}^{e_1}\ldots a_{k_si_s}^{e_s}\otimes(b_{l_sj_s}^{e_s}\ldots b_{l_1j_1}^{e_1})^*\\ =&\sum_{k_1\ldots k_s}\sum_{l_1\ldots l_s}\delta_\pi(k)\delta_\sigma(l)u_{k_1l_1}^{e_1}\ldots u_{k_sl_s}^{e_s}=L^{|\pi\vee\sigma|}\end{aligned}$$

For the variables $V_{ij}$ the proof is similar, as follows:

$$\begin{aligned}
&\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\delta_\pi(i)\delta_\sigma(j)V_{i_1j_1}^{e_1}\ldots V_{i_sj_s}^{e_s}\\
=&\sum_{i_1\ldots i_s}\sum_{j_1\ldots j_s}\sum_{l_1,\ldots,l_s\leq L}\delta_\pi(i)\delta_\sigma(j)a_{l_1i_1}^{e_1}\ldots a_{l_si_s}^{e_s}\otimes(b_{l_sj_s}^{e_s}\ldots b_{l_1j_1}^{e_1})^*\\
=&\sum_{l_1,\ldots,l_s\leq L}\delta_\pi(l)\delta_\sigma(l)=L^{|\pi\vee\sigma|}
\end{aligned}$$

Thus we have constructed an action map, and a quotient map, as in Proposition 15.8 above, and the commutation of the diagram in Theorem 15.9 is then trivial. □

Let us discuss now the integration over $G_{MN}^L$. We have:

**Definition 15.15.** *The integration functional of $G_{MN}^L$ is the composition*

$$\int_{G_{MN}^L}:C(G_{MN}^L)\to C(G_M\times G_N)\to\mathbb{C}$$

*of the representation $u_{ij}\to\sum_{r\leq L}a_{ri}\otimes b_{rj}^*$ with the Haar functional of $G_M\times G_N$.*

Observe that in the case $L=M=N$ we obtain the integration over $G_N$. Also, at $L=M=1$, or at $L=N=1$, we obtain the integration over the sphere. In the general case now, we first have the following result:

**Proposition 15.16.** *The integration functional of $G_{MN}^L$ has the invariance property*

$$\left(\int_{G_{MN}^L}\otimes\, id\right)\Phi(x)=\int_{G_{MN}^L}x$$

*with respect to the coaction map:*

$$\Phi(u_{ij})=\sum_{kl}u_{kl}\otimes a_{ki}\otimes b_{lj}^*$$

*Proof.* We restrict the attention to the orthogonal case, the proof in the unitary case being similar. We must check the following formula:

$$\left(\int_{G_{MN}^L}\otimes\, id\right)\Phi(u_{i_1j_1}\ldots u_{i_sj_s})=\int_{G_{MN}^L}u_{i_1j_1}\ldots u_{i_sj_s}$$

Let us compute the left term. This is given by:

$$\begin{aligned}
X &= \left(\int_{G_{MN}^L}\otimes id\right)\sum_{k_xl_x}u_{k_1l_1}\ldots u_{k_sl_s}\otimes a_{k_1i_1}\ldots a_{k_si_s}\otimes b_{l_1j_1}^*\ldots b_{l_sj_s}^*\\
&= \sum_{k_xl_x}\sum_{r_x\leq L}a_{k_1i_1}\ldots a_{k_si_s}\otimes b_{l_1j_1}^*\ldots b_{l_sj_s}^*\int_{G_M}a_{r_1k_1}\ldots a_{r_sk_s}\int_{G_N}b_{r_1l_1}^*\ldots b_{r_sl_s}^*\\
&= \sum_{r_x\leq L}\sum_{k_x}a_{k_1i_1}\ldots a_{k_si_s}\int_{G_M}a_{r_1k_1}\ldots a_{r_sk_s}\otimes\sum_{l_x}b_{l_1j_1}^*\ldots b_{l_sj_s}^*\int_{G_N}b_{r_1l_1}^*\ldots b_{r_sl_s}^*
\end{aligned}$$

By using now the invariance property of the Haar functionals of $G_M,G_N$, we obtain:

$$\begin{aligned}
X &= \sum_{r_x\leq L}\left(\int_{G_M}\otimes id\right)\Delta(a_{r_1i_1}\ldots a_{r_si_s})\otimes\left(\int_{G_N}\otimes id\right)\Delta(b_{r_1j_1}^*\ldots b_{r_sj_s}^*)\\
&= \sum_{r_x\leq L}\int_{G_M}a_{r_1i_1}\ldots a_{r_si_s}\int_{G_N}b_{r_1j_1}^*\ldots b_{r_sj_s}^*\\
&= \left(\int_{G_M}\otimes\int_{G_N}\right)\sum_{r_x\leq L}a_{r_1i_1}\ldots a_{r_si_s}\otimes b_{r_1j_1}^*\ldots b_{r_sj_s}^*
\end{aligned}$$

But this gives the formula in the statement, and we are done. $\square$

We will prove now that the above functional is in fact the unique positive unital invariant trace on $C(G_{MN}^L)$. For this purpose, we will need the Weingarten formula:

**Theorem 15.17.** *We have the Weingarten type formula*

$$\int_{G_{MN}^L}u_{i_1j_1}\ldots u_{i_sj_s}=\sum_{\pi\sigma\tau\nu}L^{|\pi\vee\tau|}\delta_\sigma(i)\delta_\nu(j)W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu)$$

*where the matrices on the right are given by $W_{sM}=G_{sM}^{-1}$, with $G_{sM}(\pi,\sigma)=M^{|\pi\vee\sigma|}$.*

*Proof.* We make use of the usual quantum group Weingarten formula, for which we refer to [22], [38]. By using this formula for $G_M,G_N$, we obtain:

$$\begin{aligned}
&\int_{G_{MN}^L}u_{i_1j_1}\ldots u_{i_sj_s}\\
&= \sum_{l_1\ldots l_s\leq L}\int_{G_M}a_{l_1i_1}\ldots a_{l_si_s}\int_{G_N}b_{l_1j_1}^*\ldots b_{l_sj_s}^*\\
&= \sum_{l_1\ldots l_s\leq L}\sum_{\pi\sigma}\delta_\pi(l)\delta_\sigma(i)W_{sM}(\pi,\sigma)\sum_{\tau\nu}\delta_\tau(l)\delta_\nu(j)W_{sN}(\tau,\nu)\\
&= \sum_{\pi\sigma\tau\nu}\left(\sum_{l_1\ldots l_s\leq L}\delta_\pi(l)\delta_\tau(l)\right)\delta_\sigma(i)\delta_\nu(j)W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu)
\end{aligned}$$

The coefficient being $L^{|\pi\vee\tau|}$, we obtain the formula in the statement. □

We can now derive an abstract characterization of the integration, as follows:

**Theorem 15.18.** *The integration of $G_{MN}^L$ is the unique positive unital trace*
$$C(G_{MN}^L)\to\mathbb C$$
*which is invariant under the action of the quantum group $G_M\times G_N$.*

*Proof.* We use a standard method, from [32], [37], the point being to show that we have the following ergodicity formula:
$$\left(id\otimes\int_{G_M}\otimes\int_{G_N}\right)\Phi(x)=\int_{G_{MN}^L}x$$

We restrict the attention to the orthogonal case, the proof in the unitary case being similar. We must verify that the following holds:
$$\left(id\otimes\int_{G_M}\otimes\int_{G_N}\right)\Phi(u_{i_1j_1}\ldots u_{i_sj_s})=\int_{G_{MN}^L}u_{i_1j_1}\ldots u_{i_sj_s}$$

By using the Weingarten formula, the left term can be written as follows:
$$\begin{aligned}
X &= \sum_{k_1\ldots k_s}\sum_{l_1\ldots l_s}u_{k_1l_1}\ldots u_{k_sl_s}\int_{G_M}a_{k_1i_1}\ldots a_{k_si_s}\int_{G_N}b_{l_1j_1}^*\ldots b_{l_sj_s}^*\\
&= \sum_{k_1\ldots k_s}\sum_{l_1\ldots l_s}u_{k_1l_1}\ldots u_{k_sl_s}\sum_{\pi\sigma}\delta_\pi(k)\delta_\sigma(i)W_{sM}(\pi,\sigma)\sum_{\tau\nu}\delta_\tau(l)\delta_\nu(j)W_{sN}(\tau,\nu)\\
&= \sum_{\pi\sigma\tau\nu}\delta_\sigma(i)\delta_\nu(j)W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu)\sum_{k_1\ldots k_s}\sum_{l_1\ldots l_s}\delta_\pi(k)\delta_\tau(l)u_{k_1l_1}\ldots u_{k_sl_s}
\end{aligned}$$

By using now the summation formula in Theorem 15.14, we obtain:
$$X=\sum_{\pi\sigma\tau\nu}L^{|\pi\vee\tau|}\delta_\sigma(i)\delta_\nu(j)W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu)$$

Now by comparing with the Weingarten formula for $G_{MN}^L$, this proves our claim.

Assume now that $\tau:C(G_{MN}^L)\to\mathbb C$ satisfies the invariance condition. We have:
$$\begin{aligned}
\tau\left(id\otimes\int_{G_M}\otimes\int_{G_N}\right)\Phi(x) &= \left(\tau\otimes\int_{G_M}\otimes\int_{G_N}\right)\Phi(x)\\
&= \left(\int_{G_M}\otimes\int_{G_N}\right)(\tau\otimes id)\Phi(x)\\
&= \left(\int_{G_M}\otimes\int_{G_N}\right)(\tau(x)1)\\
&= \tau(x)
\end{aligned}$$

On the other hand, according to the formula established above, we have as well:

$$\begin{aligned}\tau\left(id\otimes\int_{G_M}\otimes\int_{G_N}\right)\Phi(x) &= \tau(tr(x)1)\\ &= tr(x)\end{aligned}$$

Thus we obtain $\tau=tr$, and this finishes the proof. □

As a main application, we have:

**Proposition 15.19.** *For a sum of coordinates*

$$\chi_E=\sum_{(ij)\in E}u_{ij}$$

*which do not overlap on rows and columns we have*

$$\int_{G_{MN}^L}\chi_E^s=\sum_{\pi\sigma\tau\nu}K^{|\pi\vee\tau|}L^{|\sigma\vee\nu|}W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu)$$

*where $K=|E|$ is the cardinality of the indexing set.*

*Proof.* With $K=|E|$, we can write $E=\{(\alpha(i),\beta(i))\}$, for certain embeddings:

$$\alpha:\{1,\ldots,K\}\subset\{1,\ldots,M\}$$

$$\beta:\{1,\ldots,K\}\subset\{1,\ldots,N\}$$

In terms of these maps $\alpha,\beta$, the moment in the statement is given by:

$$M_s=\int_{G_{MN}^L}\left(\sum_{i\leq K}u_{\alpha(i)\beta(i)}\right)^s$$

By using the Weingarten formula, we can write this quantity as follows:

$$\begin{aligned}&M_s\\ =&\int_{G_{MN}^L}\sum_{i_1\ldots i_s\leq K}u_{\alpha(i_1)\beta(i_1)}\ldots u_{\alpha(i_s)\beta(i_s)}\\ =&\sum_{i_1\ldots i_s\leq K}\sum_{\pi\sigma\tau\nu}L^{|\sigma\vee\nu|}\delta_\pi(\alpha(i_1),\ldots,\alpha(i_s))\delta_\tau(\beta(i_1),\ldots,\beta(i_s))W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu)\\ =&\sum_{\pi\sigma\tau\nu}\left(\sum_{i_1\ldots i_s\leq K}\delta_\pi(i)\delta_\tau(i)\right)L^{|\sigma\vee\nu|}W_{sM}(\pi,\sigma)W_{sN}(\tau,\nu)\end{aligned}$$

But, as explained before, the coefficient on the left in the last formula is:

$$C=K^{|\pi\vee\tau|}$$

We therefore obtain the formula in the statement. □

We can further advance in the classical/twisted and free cases, where the Weingarten theory for the corresponding quantum groups is available from [10], [22], [38]:

**Theorem 15.20.** *In the context of the liberation operations*

$$O_{MN}^L \to O_{MN}^{L+}$$
$$U_{MN}^L \to U_{MN}^{L+}$$
$$H_{MN}^{sL} \to H_{MN}^{sL+}$$

*the laws of the sums of non-overlapping coordinates,*

$$\chi_E = \sum_{(ij)\in E} u_{ij}$$

*are in Bercovici-Pata bijection, in the*

$$|E| = \kappa N, L = \lambda N, M = \mu N$$

*regime and $N \to \infty$ limit.*

*Proof.* We use the general theory in [10], [22], [22], [38]. According to Proposition 15.19, in terms of $K = |E|$, the moments of the variables in the statement are given by:

$$M_s = \sum_{\pi\sigma\tau\nu} K^{|\pi\vee\tau|} L^{|\sigma\vee\nu|} W_{sM}(\pi,\sigma) W_{sN}(\tau,\nu)$$

We use now two standard facts, namely:

(1) The fact that in the $N \to \infty$ limit the Weingarten matrix $W_{sN}$ is concentrated on the diagonal.

(2) The fact that we have an inequality as follows, with equality precisely when $\pi = \sigma$:

$$|\pi \vee \sigma| \leq \frac{|\pi| + |\sigma|}{2}$$

For details on all this, we refer to [22].

Let us discuss now what happens in the regime from the statement, namely:

$$K = \kappa N, L = \lambda N, M = \mu N, N \to \infty$$

In this regime, we obtain:

$$\begin{aligned} M_s &\simeq \sum_{\pi\tau} K^{|\pi\vee\tau|} L^{|\pi\vee\tau|} M^{-|\pi|} N^{-|\tau|} \\ &\simeq \sum_{\pi} K^{|\pi|} L^{|\pi|} M^{-|\pi|} N^{-|\pi|} \\ &= \sum_{\pi} \left(\frac{\kappa\lambda}{\mu}\right)^{|\pi|} \end{aligned}$$

In order to interpret this formula, we use general theory from [10], [22], [22]:

(1) For $G_N = O_N, \bar{O}_N/O_N^+$, the above variables $\chi_E$ follow to be asymptotically Gaussian/semicircular, of parameter $\frac{\kappa\lambda}{\mu}$, and hence in Bercovici-Pata bijection.

(2) For $G_N = U_N, \bar{U}_N/U_N^+$ the situation is similar, with $\chi_E$ being asymptotically complex Gaussian/circular, of parameter $\frac{\kappa\lambda}{\mu}$, and in Bercovici-Pata bijection.

(3) Finally, for $G_N = H_N^s/H_N^{s+}$, the variables $\chi_E$ are asymptotically Bessel/free Bessel of parameter $\frac{\kappa\lambda}{\mu}$, and once again in Bercovici-Pata bijection. $\square$

The convergence in the above result is of course in moments, and we do not know whether some stronger convergence results can be formulated. Nor do we know whether one can use linear combinations of coordinates which are more general than the sums $\chi_E$ that we consider. These are interesting questions, that we would like to raise here.

Also, there are several possible extensions of the above result, for instance by using quantum reflection groups instead of unitary quantum groups, and by using twisting operations as well. We refer here to [8], and to [37] as well, for a number of supplementary results, which can be obtained by using the stronger formalism there.

Finally, there are many interesting questions in relation with Connes' noncommutative geometry [59], [60], [61], and more specifically with the quantum extension of the Nash embedding theorem [113]. We refer here to [67], [68], [69], [86], [122], [137].

## 16. Modelling questions

One interesting method for the study of the closed subgroups $G \subset U_N^+$, that we have not tried yet, consists in modelling the standard coordinates $u_{ij} \in C(G)$ by concrete variables $U_{ij} \in B$. Indeed, assuming that the model is faithful in some suitable sense, that the algebra $B$ is something quite familiar, and that the variables $U_{ij}$ are not too complicated, all questions about $G$ would correspond in this way to routine questions inside $B$.

Regarding the choice of $B$, some very convenient algebras are the random matrix ones, $B = M_K(C(T))$, with $K \in \mathbb{N}$, and with $T$ being a compact space. These algebras generalize indeed the most familiar algebras that we know, namely the matrix ones $M_K(\mathbb{C})$, and the commutative ones $C(T)$. We are led in this way to:

**Definition 16.1.** *A matrix model for $G \subset U_N^+$ is a morphism of $C^*$-algebras*

$$\pi : C(G) \to M_K(C(T))$$

*where $T$ is a compact space, and $K \geq 1$ is an integer.*

Let us introduce as well the following related definition:

**Definition 16.2.** *A matrix model $\pi : C(G) \to M_K(C(T))$ is called stationary when*

$$\int_G = \left(tr \otimes \int_T\right)\pi$$

*where $\int_T$ is the integration with respect to a given probability measure on $T$.*

Here the term "stationary" comes from a functional analytic interpretation of all this, with a certain Cesàro limit being needed to be stationary, and this will be explained later on. Yet another explanation comes from a certain relation with the lattice models, but this relation is rather something folklore, not axiomatized yet. We will be back to this later. As a first result now, the stationarity property implies the faithfulness:

**Theorem 16.3.** *Assuming that $G \subset U_N^+$ has a stationary model,*

$$\pi : C(G) \to M_K(C(T))$$

$$\int_G = \left(tr \otimes \int_T\right)\pi$$

*it follows that $G$ must be coamenable, and that the model is faithful.*

*Proof.* Assume that we have a stationary model, as in the statement. By performing the GNS construction with respect to $\int_G$, we obtain a factorization as follows, which commutes with the respective canonical integration functionals:

$$\pi : C(G) \to C(G)_{red} \subset M_K(C(T))$$

Thus, in what regards the coamenability question, we can assume that $\pi$ is faithful. With this assumption made, observe that we have embeddings as follows:

$$C^\infty(G)\subset C(G)\subset M_K(C(T))$$

The point now is that the GNS construction gives a better embedding, as follows:

$$L^\infty(G)\subset M_K(L^\infty(T))$$

Now since the von Neumann algebra on the right is of type I, so must be its subalgebra $A=L^\infty(G)$. This means that, when writing the center of this latter algebra as $Z(A)=L^\infty(X)$, the whole algebra decomposes over $X$, as an integral of type I factors:

$$L^\infty(G)=\int_X M_{K_x}(\mathbb C)\,dx$$

In particular, we can see from this that $C^\infty(G)\subset L^\infty(G)$ has a unique $C^*$-norm, and so $G$ is coamenable. Thus we have proved our first assertion, and the second assertion follows as well, because our factorization of $\pi$ consists of the identity, and of an inclusion. □

Regarding now the examples of stationary models, we first have:

**Proposition 16.4.** *The following have stationary models:*

1. *The compact Lie groups.*
2. *The finite quantum groups.*

*Proof.* Both these assertions are elementary, with the proofs being as follows:

(1) This is clear, because we can use the identity $id:C(G)\to M_1(C(G))$.

(2) Here we can use the regular representation $\lambda:C(G)\to M_{|G|}(\mathbb C)$. Indeed, let us endow the linear space $H=C(G)$ with the scalar product $<a,b>=\int_G ab^*$. We have then a representation $\lambda:C(G)\to B(H)$ given by $a\to[b\to ab]$. Now since we have $H\simeq\mathbb C^{|G|}$ with $|G|=\dim A$, we can view $\lambda$ as a matrix model map, as above, and the stationarity axiom $\int_G=tr\circ\lambda$ is satisfied, as desired. □

In order to discuss the group duals, consider a model $\pi:C^*(\Gamma)\to M_K(C(T))$. According to the general theory of group algebras, these matrix models must come from group representations $\rho:\Gamma\to C(T,U_K)$. With this identification made, we have:

**Proposition 16.5.** *An matrix model $\rho:\Gamma\subset C(T,U_K)$ is stationary when:*

$$\int_T tr(g^x)dx=0,\forall g\neq 1$$

*Moreover, the examples include all the abelian groups, and all finite groups.*

*Proof.* Consider indeed a group embedding $\rho : \Gamma \subset C(T, U_K)$, which produces by linearity a matrix model, as follows:

$$\pi : C^*(\Gamma) \to M_K(C(T))$$

It is enough to formulate the stationarity condition on the group elements $g \in C^*(\Gamma)$. Let us set $\rho(g) = (x \to g^x)$. With this notation, the stationarity condition reads:

$$\int_T tr(g^x)dx = \delta_{g,1}$$

Since this equality is trivially satisfied at $g = 1$, where by unitality of our representation we must have $g^x = 1$ for any $x \in T$, we are led to the condition in the statement. Regarding now the examples, these are both clear. More precisely:

(1) When $\Gamma$ is abelian we can use the following trivial embedding:

$$\Gamma \subset C(\widehat{\Gamma}, U_1)$$

$$g \to [\chi \to \chi(g)]$$

(2) When $\Gamma$ is finite we can use the left regular representation:

$$\Gamma \subset \mathcal{L}(\mathbb{C}\Gamma)$$

$$g \to [h \to gh]$$

Indeed, in both cases, the stationarity condition is trivially satisfied. □

In order to further advance, and to come up with some tools for discussing the non-stationary case as well, let us keep looking at the group duals $G = \widehat{\Gamma}$. We know that a model $\pi : C^*(\Gamma) \to M_K(C(T))$ must come from a group representation $\rho : \Gamma \to C(T, U_K)$. Now observe that when $\rho$ is faithful, the representation $\pi$ is in general not faithful, for instance because when $T = \{.\}$ its target algebra is finite dimensional. On the other hand, this representation "reminds" $\Gamma$, and so can be used in order to fully understand $\Gamma$.

Summarizing, we have a new idea here, basically saying that, for practical purposes, the faithfuless property can be replaced with something much weaker. This weaker notion is called "inner faithfulness", and the theory here, from [14], is as follows:

**Definition 16.6.** *Let $\pi : C(G) \to M_K(C(T))$ be a matrix model.*

(1) *The Hopf image of $\pi$ is the smallest quotient Hopf $C^*$-algebra $C(G) \to C(H)$ producing a factorization of type $\pi : C(G) \to C(H) \to M_K(C(T))$.*
(2) *When the inclusion $H \subset G$ is an isomorphism, i.e. when there is no non-trivial factorization as above, we say that $\pi$ is inner faithful.*

In the case where $G = \widehat{\Gamma}$ is a group dual, $\pi$ must come from a group representation:

$$\rho : \Gamma \to C(T, U_K)$$

The above factorization is simply the one obtained by taking the image:

$$\rho:\Gamma\to\Lambda\subset C(T,U_K)$$

Thus $\pi$ is inner faithful when $\Gamma\subset C(T,U_K)$. Also, given a compact group $G$, and elements $g_1,\ldots,g_K\in G$, we have a representation $\pi:C(G)\to\mathbb C^K$, given by:

$$f\to(f(g_1),\ldots,f(g_K))$$

The minimal factorization of $\pi$ is then via $C(H)$, with:

$$H=\overline{<g_1,\ldots,g_K>}$$

Also, $\pi$ is inner faithful when $G=H$. We will see many other examples.

In general, the existence and uniqueness of the Hopf image comes from dividing $C(G)$ by a suitable ideal, as explained in [14]. In Tannakian terms, we have:

**Theorem 16.7.** *Assuming $G\subset U_N^+$, with fundamental corepresentation $u=(u_{ij})$, the Hopf image of*

$$\pi:C(G)\to M_K(C(T))$$

*comes from the Tannakian category*

$$C_{kl}=Hom(U^{\otimes k},U^{\otimes l})$$

*where $U_{ij}=\pi(u_{ij})$, and where the spaces on the right are taken in a formal sense.*

*Proof.* Since the morphisms increase the intertwining spaces, when defined either in a representation theory sense, or just formally, we have inclusions as follows:

$$Hom(u^{\otimes k},u^{\otimes l})\subset Hom(U^{\otimes k},U^{\otimes l})$$

More generally, we have such inclusions when replacing $(G,u)$ with any pair producing a factorization of $\pi$. Thus, by Tannakian duality, the Hopf image must be given by the fact that the intertwining spaces must be the biggest, subject to the above inclusions.

On the other hand, since $u$ is biunitary, so is $U$, and it follows that the spaces on the right form a Tannakian category. Thus, we have a quantum group $(H,v)$ given by:

$$Hom(v^{\otimes k},v^{\otimes l})=Hom(U^{\otimes k},U^{\otimes l})$$

By the above discussion, $C(H)$ follows to be the Hopf image of $\pi$, as claimed. □

The inner faithful models $\pi:C(G)\to M_K(C(T))$ are a very interesting notion, because they are not subject to the coamenability condition on $G$, as it was the case with the stationary models, as explained in Theorem 16.3. In fact, there are no known restrictions on the class of closed subgroups $G\subset U_N^+$ which can be modelled in an inner faithful way. Thus, our modelling theory applies a priori to any compact quantum group. Regarding now the study of the inner faithful models, a key problem is that of computing the Haar integration functional. The result here, from [30], [142], is as follows:

**Theorem 16.8.** *Given an inner faithful model $\pi : C(G) \to M_K(C(T))$, we have*

$$\int_G = \lim_{k\to\infty}\frac{1}{k}\sum_{r=1}^k \int_G^r$$

*where $\int_G^r = (\varphi\circ\pi)^{*r}$, with $\varphi = tr\otimes\int_T$ being the random matrix trace.*

*Proof.* As a first observation, there is an obvious similarity here with the Woronowicz construction of the Haar measure, explained in section 1 above. In fact, the above result holds more generally for any model $\pi : C(G) \to B$, with $\varphi \in B^*$ being a faithful trace. With this picture in hand, the Woronowicz construction simply corresponds to the case $\pi = id$, and the result itself is therefore a generalization of Woronowicz's result. In order to prove now the result, we can proceed as in section 1. If we denote by $\int_G'$ the limit in the statement, we must prove that this limit converges, and that we have:

$$\int_G' = \int_G$$

It is enough to check this on the coefficients of corepresentations, and if we let $v = u^{\otimes k}$ be one of the Peter-Weyl corepresentations, we must prove that we have:

$$\left(id\otimes\int_G'\right)v = \left(id\otimes\int_G\right)v$$

We know from section 1 that the matrix on the right is the orthogonal projection onto $Fix(v)$. Regarding now the matrix on the left, this is the orthogonal projection onto the 1-eigenspace of $(id\otimes\varphi\pi)v$. Now observe that, if we set $V_{ij} = \pi(v_{ij})$, we have:

$$(id\otimes\varphi\pi)v = (id\otimes\varphi)V$$

Thus, as in section 1, we conclude that the 1-eigenspace that we are interested in equals $Fix(V)$. But, according to Theorem 16.7, we have:

$$Fix(V) = Fix(v)$$

Thus, we have proved that we have $\int_G' = \int_G$, as desired. □

Getting back now to the stationary models, we have the following result, from [7]:

**Theorem 16.9.** *For $\pi : C(G) \to M_K(C(T))$, the following are equivalent:*

(1) *$Im(\pi)$ is a Hopf algebra, and $(tr\otimes\int_T)\pi$ is the Haar integration on it.*
(2) *$\psi = (tr\otimes\int_X)\pi$ satisfies the idempotent state property $\psi*\psi = \psi$.*
(3) *$T_e^2 = T_e$, $\forall p\in\mathbb{N}$, $\forall e\in\{1,*\}^p$, where:*

$$(T_e)_{i_1\ldots i_p,j_1\ldots j_p} = \left(tr\otimes\int_T\right)(U_{i_1j_1}^{e_1}\ldots U_{i_pj_p}^{e_p})$$

*If these conditions are satisfied, we say that $\pi$ is stationary on its image.*

*Proof.* Given a matrix model $\pi : C(G) \to M_K(C(T))$ as in the statement, we can factorize it via its Hopf image, as in Definition 16.6 above:

$$\pi : C(G) \to C(H) \to M_K(C(T))$$

Now observe that the conditions (1,2,3) in the statement depend only on the factorized representation:

$$\nu : C(H) \to M_K(C(T))$$

Thus, we can assume in practice that we have $G = H$, which means that we can assume that $\pi$ is inner faithful. With this assumption made, the general integration formula from Theorem 16.8 applies to our situation, and the proof of the equivalences goes as follows:

(1) $\implies$ (2) This is clear from definitions, because the Haar integration on any compact quantum group satisfies the idempotent state equation:

$$\psi * \psi = \psi$$

(2) $\implies$ (1) Assuming $\psi * \psi = \psi$, we have, for any $r \in \mathbb{N}$:

$$\psi^{*r} = \psi$$

Thus Theorem 16.8 gives $\int_G = \psi$, and by using Theorem 16.3, we obtain the result.

In order to establish now (2) $\iff$ (3), we use the following elementary formula, which comes from the definition of the convolution operation:

$$\psi^{*r}(u^{e_1}_{i_1 j_1} \ldots u^{e_p}_{i_p j_p}) = (T_e^r)_{i_1 \ldots i_p, j_1 \ldots j_p}$$

(2) $\implies$ (3) Assuming $\psi * \psi = \psi$, by using the above formula at $r = 1, 2$ we obtain that the matrices $T_e$ and $T_e^2$ have the same coefficients, and so they are equal.

(3) $\implies$ (2) Assuming $T_e^2 = T_e$, by using the above formula at $r = 1, 2$ we obtain that the linear forms $\psi$ and $\psi * \psi$ coincide on any product of coefficients $u^{e_1}_{i_1 j_1} \ldots u^{e_p}_{i_p j_p}$. Now since these coefficients span a dense subalgebra of $C(G)$, this gives the result. $\square$

As a first illustration, we will apply this criterion to certain models for the quantum groups $O_N^*, U_N^*$. We first have the following result:

**Proposition 16.10.** *We have a matrix model as follows,*

$$C(O_N^*) \to M_2(C(U_N))$$

$$u_{ij} \to \begin{pmatrix} 0 & v_{ij} \\ \bar{v}_{ij} & 0 \end{pmatrix}$$

*where $v$ is the fundamental corepresentation of $C(U_N)$, as well as a model as follows,*

$$C(U_N^*) \to M_2(C(U_N \times U_N))$$

$$u_{ij} \to \begin{pmatrix} 0 & v_{ij} \\ w_{ij} & 0 \end{pmatrix}$$

*where $v, w$ are the fundamental corepresentations of the two copies of $C(U_N)$.*

*Proof.* It is routine to check that the matrices on the right are indeed biunitaries, and since the first matrix is also self-adjoint, we obtain in this way models as follows:

$$C(O_N^+)\to M_2(C(U_N))$$

$$C(U_N^+)\to M_2(C(U_N\times U_N))$$

Regarding now the half-commutation relations, this comes from something general, regarding the antidiagonal $2\times 2$ matrices. Consider indeed matrices as follows:

$$X_i=\begin{pmatrix}0&x_i\\ y_i&0\end{pmatrix}$$

We have then the following computation:

$$X_iX_jX_k=\begin{pmatrix}0&x_i\\ y_i&0\end{pmatrix}\begin{pmatrix}0&x_j\\ y_j&0\end{pmatrix}\begin{pmatrix}0&x_k\\ y_k&0\end{pmatrix}=\begin{pmatrix}0&x_iy_jx_k\\ y_ix_jy_k&0\end{pmatrix}$$

Since this quantity is symmetric in $i,k$, we obtain $X_iX_jX_k=X_kX_jX_i$. Thus, the antidiagonal $2\times 2$ matrices half-commute, and so our models factorize as claimed. $\square$

We can now formulate our first concrete modelling theorem, as folllows:

**Theorem 16.11.** *The above antidiagonal models, namely*

$$C(O_N^*)\to M_2(C(U_N))$$

$$C(U_N^*)\to M_2(C(U_N\times U_N))$$

*are both stationary.*

*Proof.* We first discuss the case of $O_N^*$. We use Theorem 16.9 (3). Since the fundamental representation is self-adjoint, the matrices $T_e$ with $e\in\{1,*\}^p$ are all equal. We denote this common matrix by $T_p$. According to the definition of $T_p$, this matrix is given by:

$$(T_p)_{i_1\ldots i_p,j_1\ldots j_p}=\left(tr\otimes\int_H\right)\left[\begin{pmatrix}0&v_{i_1j_1}\\ \bar{v}_{i_1j_1}&0\end{pmatrix}\ldots\ldots\begin{pmatrix}0&v_{i_pj_p}\\ \bar{v}_{i_pj_p}&0\end{pmatrix}\right]$$

Since when multipliying an odd number of antidiagonal matrices we obtain an atidiagonal matrix, we have $T_p=0$ for $p$ odd. Also, when $p$ is even, we have:

$$\begin{aligned}(T_p)_{i_1\ldots i_p,j_1\ldots j_p}&=\left(tr\otimes\int_H\right)\begin{pmatrix}v_{i_1j_1}\ldots\bar{v}_{i_pj_p}&0\\ 0&\bar{v}_{i_1j_1}\ldots v_{i_pj_p}\end{pmatrix}\\ &=\frac{1}{2}\left(\int_H v_{i_1j_1}\ldots\bar{v}_{i_pj_p}+\int_H\bar{v}_{i_1j_1}\ldots v_{i_pj_p}\right)\\ &=\int_H Re(v_{i_1j_1}\ldots\bar{v}_{i_pj_p})\end{aligned}$$

We have $T_p^2=T_p=0$ when $p$ is odd, so we are left with proving that we have $T_p^2=T_p$, when $p$ is even. For this purpose, we use the following formula:

$$Re(x)Re(y)=\frac{1}{2}\left(Re(xy)+Re(x\bar{y})\right)$$

By using this identity for each of the terms which appear in the product, and multi-index notations in order to simplify the writing, we obtain:

$$\begin{aligned}
&(T_p^2)_{ij}\\
=&\sum_{k_1\ldots k_p}(T_p)_{i_1\ldots i_p,k_1\ldots k_p}(T_p)_{k_1\ldots k_p,j_1\ldots j_p}\\
=&\int_H\int_H\sum_{k_1\ldots k_p}Re(v_{i_1k_1}\ldots\bar{v}_{i_pk_p})Re(w_{k_1j_1}\ldots\bar{w}_{k_pj_p})dvdw\\
=&\frac{1}{2}\int_H\int_H\sum_{k_1\ldots k_p}Re(v_{i_1k_1}w_{k_1j_1}\ldots\bar{v}_{i_pk_p}\bar{w}_{k_pj_p})+Re(v_{i_1k_1}\bar{w}_{k_1j_1}\ldots\bar{v}_{i_pk_p}w_{k_pj_p})dvdw\\
=&\frac{1}{2}\int_H\int_H Re((vw)_{i_1j_1}\ldots(\bar{v}\bar{w})_{i_pj_p})+Re((v\bar{w})_{i_1j_1}\ldots(\bar{v}w)_{i_pj_p})dvdw
\end{aligned}$$

Now since $vw\in H$ is uniformly distributed when $v,w\in H$ are uniformly distributed, the quantity on the left integrates up to $(T_p)_{ij}$. Also, since $H$ is conjugation-stable, $\bar{w}\in H$ is uniformly distributed when $w\in H$ is uniformly distributed, so the quantity on the right integrates up to the same quantity, namely $(T_p)_{ij}$. Thus, we have:

$$\begin{aligned}
(T_p^2)_{ij}&=\frac{1}{2}\Big((T_p)_{ij}+(T_p)_{ij}\Big)\\
&=(T_p)_{ij}
\end{aligned}$$

Summarizing, we have obtained that for any $p$, the condition $T_p^2=T_p$ is satisfied. Thus Theorem 16.9 applies, and shows that our model is stationary, as claimed.

As for the proof of the stationarity for the model for $U_N^*$, this is similar. See [23]. □

Following [33], let us discuss now some more subtle examples of stationary models, related to the Pauli matrices, and Weyl matrices, and physics. We first have:

**Definition 16.12.** *Given a finite abelian group $H$, the associated Weyl matrices are*

$$W_{ia}:e_b\to<i,b>e_{a+b}$$

*where $i\in H$, $a,b\in\widehat{H}$, and where $(i,b)\to<i,b>$ is the Fourier coupling $H\times\widehat{H}\to\mathbb{T}$.*

As a basic example, consider the simplest cyclic group, namely:

$$H=\mathbb{Z}_2=\{0,1\}$$

Here the Fourier coupling is $< i, b >= (-1)^{ib}$, and the Weyl matrices act as follows:

$$W_{00} : e_b \to e_b \quad , \quad W_{10} : e_b \to (-1)^b e_b$$
$$W_{11} : e_b \to (-1)^b e_{b+1} \quad , \quad W_{01} : e_b \to e_{b+1}$$

Thus, we have the following formulae for the Weyl matrices:

$$W_{00} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad W_{10} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$W_{11} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad , \quad W_{01} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

We recognize here, up to some multiplicative factors, the four Pauli matrices. Now back to the general case, we have the following well-known result:

**Proposition 16.13.** *The Weyl matrices are unitaries, and satisfy:*
(1) $W_{ia}^* =< i, a > W_{-i,-a}$.
(2) $W_{ia}W_{jb} =< i, b > W_{i+j,a+b}$.
(3) $W_{ia}W_{jb}^* =< j-i, b > W_{i-j,a-b}$.
(4) $W_{ia}^*W_{jb} =< i, a-b > W_{j-i,b-a}$.

*Proof.* The unitary follows from (3,4), and the rest of the proof goes as follows:

(1) We have indeed the following computation:

$$\begin{aligned} W_{ia}^* &= \left(\sum_b < i, b > E_{a+b,b}\right)^* \\ &= \sum_b < -i, b > E_{b,a+b} \\ &= \sum_b < -i, b-a > E_{b-a,b} \\ &= < i, a > W_{-i,-a} \end{aligned}$$

(2) Here the verification goes as follows:

$$\begin{aligned} W_{ia}W_{jb} &= \left(\sum_d < i, b+d > E_{a+b+d,b+d}\right)\left(\sum_d < j, d > E_{b+d,d}\right) \\ &= \sum_d < i, b >< i+j, d > E_{a+b+d,d} \\ &= < i, b > W_{i+j,a+b} \end{aligned}$$

(3,4) By combining the above two formulae, we obtain:

$$\begin{aligned} W_{ia}W_{jb}^* &= < j, b > W_{ia}W_{-j,-b} \\ &= < j, b >< i, -b > W_{i-j,a-b} \end{aligned}$$

We obtain as well the following formula:

$$\begin{aligned} W_{ia}^*W_{jb} &= < i,a > W_{-i,-a}W_{jb} \\ &= < i,a >< -i,b > W_{j-i,b-a} \end{aligned}$$

But this gives the formulae in the statement, and we are done. □

Observe that, with $n = |H|$, we can use an isomorphism $l^2(\widehat{H}) \simeq \mathbb{C}^n$ as to view each $W_{ia}$ as a usual matrix, $W_{ia} \in M_n(\mathbb{C})$, and hence as a usual unitary, $W_{ia} \in U_n$. Given a vector $\xi$, we denote by $Proj(\xi)$ the orthogonal projection onto $\mathbb{C}\xi$. We have:

**Proposition 16.14.** *Given a closed subgroup $E \subset U_n$, we have a representation*

$$\pi_H : C(S_N^+) \to M_N(C(E))$$

$$w_{ia,jb} \to [U \to Proj(W_{ia}UW_{jb}^*)]$$

*where $n = |H|, N = n^2$, and where $W_{ia}$ are the Weyl matrices associated to $H$.*

*Proof.* The Weyl matrices being given by $W_{ia} : e_b \to < i,b > e_{a+b}$, we have:

$$tr(W_{ia}) = \begin{cases} 1 & \text{if } (i,a) = (0,0) \\ 0 & \text{if } (i,a) \neq (0,0) \end{cases}$$

Together with the formulae in Proposition 16.13, this shows that the Weyl matrices are pairwise orthogonal with respect to the following scalar product on $M_n(\mathbb{C})$:

$$< x,y >= tr(x^*y)$$

Thus, these matrices form an orthogonal basis of $M_n(\mathbb{C})$, consisting of unitaries:

$$W = \left\{ W_{ia} \Big| i \in H, a \in \widehat{H} \right\}$$

Thus, each row and each column of the matrix $\xi_{ia,jb} = W_{ia}UW_{jb}^*$ is an orthogonal basis of $M_n(\mathbb{C})$, and so the corresponding projections form a magic unitary, as claimed. □

We will need the following well-known result:

**Proposition 16.15.** *With $T = Proj(x_1) \ldots Proj(x_p)$ and $||x_i|| = 1$ we have*

$$< T\xi,\eta >=< \xi,x_p >< x_p,x_{p-1} > \ldots < x_2,x_1 >< x_1,\eta >$$

*for any $\xi,\eta$. In particular, we have:*

$$Tr(T) =< x_1,x_p >< x_p,x_{p-1} > \ldots < x_2,x_1 >$$

*Proof.* For $||x||=1$ we have $Proj(x)\xi=<\xi,x>x$. This gives:

$$\begin{aligned}T\xi &= Proj(x_1)\ldots Proj(x_p)\xi\\ &= Proj(x_1)\ldots Proj(x_{p-1})<\xi,x_p>x_p\\ &= Proj(x_1)\ldots Proj(x_{p-2})<\xi,x_p><x_p,x_{p-1}>x_{p-1}\\ &= \ldots\\ &= <\xi,x_p><x_p,x_{p-1}>\ldots<x_2,x_1>x_1\end{aligned}$$

Now by taking the scalar product with $\eta$, this gives the first assertion. As for the second assertion, this follows from the first assertion, by summing over $\xi=\eta=e_i$. □

Now back to the Weyl matrix models, let us first compute $T_p$. We have:

**Proposition 16.16.** *We have the formula*

$$\begin{aligned}&(T_p)_{ia,jb}\\ =\ &\frac{1}{N}<i_1,a_1-a_p>\ldots<i_p,a_p-a_{p-1}><j_1,b_1-b_2>\ldots<j_p,b_p-b_1>\\ &\int_E tr(W_{i_1-i_2,a_1-a_2}UW_{j_2-j_1,b_2-b_1}U^*)\ldots tr(W_{i_p-i_1,a_p-a_1}UW_{j_1-j_p,b_1-b_p}U^*)dU\end{aligned}$$

*with all the indices varying in a cyclic way.*

*Proof.* By using the trace formula in Proposition 16.15 above, we obtain:

$$\begin{aligned}&(T_p)_{ia,jb}\\ =\ &\left(tr\otimes\int_E\right)\left(Proj(W_{i_1a_1}UW^*_{j_1b_1})\ldots Proj(W_{i_pa_p}UW^*_{j_pb_p})\right)\\ =\ &\frac{1}{N}\int_E<W_{i_1a_1}UW^*_{j_1b_1},W_{i_pa_p}UW^*_{j_pb_p}>\ldots<W_{i_2a_2}UW^*_{j_2b_2},W_{i_1a_1}UW^*_{j_1b_1}>dU\end{aligned}$$

In order to compute now the scalar products, observe that we have:

$$\begin{aligned}<W_{ia}UW^*_{jb},W_{kc}UW^*_{ld}> &= tr(W_{jb}U^*W^*_{ia}W_{kc}UW^*_{ld})\\ &= tr(W^*_{ia}W_{kc}UW^*_{ld}W_{jb}U^*)\\ &= <i,a-c><l,d-b>tr(W_{k-i,c-a}UW_{j-l,b-d}U^*)\end{aligned}$$

By plugging these quantities into the formula of $T_p$, we obtain the result. □

Consider now the Weyl group $W=\{W_{ia}\}\subset U_n$, that we already met in the proof of Proposition 16.14 above. We have the following result, from [33]:

**Theorem 16.17.** *For any compact group $W\subset E\subset U_n$, the model*

$$\pi_H:C(S_N^+)\to M_N(C(E))$$
$$w_{ia,jb}\to[U\to Proj(W_{ia}UW^*_{jb})]$$

*constructed above is stationary on its image.*

*Proof.* We must prove that we have $T_p^2=T_p$. We have:

$$\begin{aligned}
&(T_p^2)_{ia,jb}\\
=&\sum_{kc}(T_p)_{ia,kc}(T_p)_{kc,jb}\\
=&\frac{1}{N^2}\sum_{kc}<i_1,a_1-a_p>\ldots<i_p,a_p-a_{p-1}><k_1,c_1-c_2>\ldots<k_p,c_p-c_1>\\
&<k_1,c_1-c_p>\ldots<k_p,c_p-c_{p-1}><j_1,b_1-b_2>\ldots<j_p,b_p-b_1>\\
&\int_E tr(W_{i_1-i_2,a_1-a_2}UW_{k_2-k_1,c_2-c_1}U^*)\ldots tr(W_{i_p-i_1,a_p-a_1}UW_{k_1-k_p,c_1-c_p}U^*)dU\\
&\int_E tr(W_{k_1-k_2,c_1-c_2}VW_{j_2-j_1,b_2-b_1}V^*)\ldots tr(W_{k_p-k_1,c_p-c_1}VW_{j_1-j_p,b_1-b_p}V^*)dV
\end{aligned}$$

By rearranging the terms, this formula becomes:

$$\begin{aligned}
&(T_p^2)_{ia,jb}\\
=&\frac{1}{N^2}<i_1,a_1-a_p>\ldots<i_p,a_p-a_{p-1}><j_1,b_1-b_2>\ldots<j_p,b_p-b_1>\\
&\int_E\int_E\sum_{kc}<k_1-k_p,c_1-c_p>\ldots<k_p-k_{p-1},c_p-c_{p-1}>\\
&tr(W_{i_1-i_2,a_1-a_2}UW_{k_2-k_1,c_2-c_1}U^*)tr(W_{k_1-k_2,c_1-c_2}VW_{j_2-j_1,b_2-b_1}V^*)\\
&\ldots\ldots\\
&tr(W_{i_p-i_1,a_p-a_1}UW_{k_1-k_p,c_1-c_p}U^*)tr(W_{k_p-k_1,c_p-c_1}VW_{j_1-j_p,b_1-b_p}V^*)dUdV
\end{aligned}$$

Let us denote by $I$ the above double integral. By using $W_{kc}^*=<k,c>W_{-k,-c}$ for each of the couplings, and by moving as well all the $U^*$ variables to the left, we obtain:

$$\begin{aligned}
I\ =\ &\int_E\int_E\sum_{kc}tr(U^*W_{i_1-i_2,a_1-a_2}UW_{k_2-k_1,c_2-c_1})tr(W^*_{k_2-k_1,c_2-c_1}VW_{j_2-j_1,b_2-b_1}V^*)\\
&\ldots\ldots\\
&tr(U^*W_{i_p-i_1,a_p-a_1}UW_{k_1-k_p,c_1-c_p})tr(W^*_{k_1-k_p,c_1-c_p}VW_{j_1-j_p,b_1-b_p}V^*)dUdV
\end{aligned}$$

In order to perform now the sums, we use the following formula:

$$\begin{aligned}
tr(AW_{kc})tr(W_{kc}^*B)&=\frac{1}{N}\sum_{qrst}A_{qr}(W_{kc})_{rq}(W_{kc}^*)_{st}B_{ts}\\
&=\frac{1}{N}\sum_{qrst}A_{qr}<k,q>\delta_{r-q,c}<k,-s>\delta_{t-s,c}B_{ts}\\
&=\frac{1}{N}\sum_{qs}<k,q-s>A_{q,q+c}B_{s+c,s}
\end{aligned}$$

If we denote by $A_x, B_x$ the variables which appear in the formula of $I$, we have:

$$\begin{aligned} & I \\ = & \frac{1}{N^p}\int_E\int_E\sum_{kcqs} < k_2-k_1, q_1-s_1 > \ldots < k_1-k_p, q_p-s_p > \\ & (A_1)_{q_1,q_1+c_2-c_1}(B_1)_{s_1+c_2-c_1,s_1}\ldots(A_p)_{q_p,q_p+c_1-c_p}(B_p)_{s_p+c_1-c_p,s_p} \\ = & \frac{1}{N^p}\int_E\int_E\sum_{kcqs} < k_1, q_p-s_p-q_1+s_1 > \ldots < k_p, q_{p-1}-s_{p-1}-q_p+s_p > \\ & (A_1)_{q_1,q_1+c_2-c_1}(B_1)_{s_1+c_2-c_1,s_1}\ldots(A_p)_{q_p,q_p+c_1-c_p}(B_p)_{s_p+c_1-c_p,s_p} \end{aligned}$$

Now observe that we can perform the sums over $k_1,\ldots,k_p$. We obtain in this way a multiplicative factor $n^p$, along with the condition:

$$q_1-s_1=\ldots=q_p-s_p$$

Thus we must have $q_x=s_x+a$ for a certain $a$, and the above formula becomes:

$$I=\frac{1}{n^p}\int_E\int_E\sum_{csa}(A_1)_{s_1+a,s_1+c_2-c_1+a}(B_1)_{s_1+c_2-c_1,s_1}\ldots(A_p)_{s_p+a,s_p+c_1-c_p+a}(B_p)_{s_p+c_1-c_p,s_p}$$

Consider now the variables $r_x=c_{x+1}-c_x$, which altogether range over the set $Z$ of multi-indices having sum 0. By replacing the sum over $c_x$ with the sum over $r_x$, which creates a multiplicative $n$ factor, we obtain the following formula:

$$I=\frac{1}{n^{p-1}}\int_E\int_E\sum_{r\in Z}\sum_{sa}(A_1)_{s_1+a,s_1+r_1+a}(B_1)_{s_1+r_1,s_1}\ldots(A_p)_{s_p+a,s_p+r_p+a}(B_p)_{s_p+r_p,s_p}$$

For an arbitrary multi-index $r$ we have:

$$\delta_{\sum_i r_i,0}=\frac{1}{n}\sum_i < i,r_1 > \ldots < i,r_p >$$

Thus, we can replace the sum over $r\in Z$ by a full sum, as follows:

$$\begin{aligned} I \quad = \quad & \frac{1}{n^p}\int_E\int_E\sum_{rsia} < i,r_1 > (A_1)_{s_1+a,s_1+r_1+a}(B_1)_{s_1+r_1,s_1} \\ & \ldots\ldots \\ & < i,r_p > (A_p)_{s_p+a,s_p+r_p+a}(B_p)_{s_p+r_p,s_p} \end{aligned}$$

In order to "absorb" now the indices $i,a$, we can use the following formula:

$$\begin{aligned} & W_{ia}^*AW_{ia} \\ = & \left(\sum_b < i,-b > E_{b,a+b}\right)\left(\sum_{bc}E_{a+b,a+c}A_{a+b,a+c}\right)\left(\sum_c < i,c > E_{a+c,c}\right) \\ = & \sum_{bc} < i,c-b > E_{bc}A_{a+b,a+c} \end{aligned}$$

Thus we have:

$$(W_{ia}^*AW_{ia})_{bc}=< i,c-b > A_{a+b,a+c}$$

Our formula becomes:

$$\begin{aligned}
& I \\
= & \frac{1}{n^p}\int_E\int_E\sum_{rsia}(W_{ia}^*A_1W_{ia})_{s_1,s_1+r_1}(B_1)_{s_1+r_1,s_1}\ldots(W_{ia}^*A_pW_{ia})_{s_p,s_p+r_p}(B_p)_{s_p+r_p,s_p}\\
= & \int_E\int_E\sum_{ia}tr(W_{ia}^*A_1W_{ia}B_1)\ldots\ldots tr(W_{ia}^*A_pW_{ia}B_p)
\end{aligned}$$

Now by replacing $A_x,B_x$ with their respective values, we obtain:

$$\begin{aligned}
I &= \int_E\int_E\sum_{ia}tr(W_{ia}^*U^*W_{i_1-i_2,a_1-a_2}UW_{ia}VW_{j_2-j_1,b_2-b_1}V^*)\\
&\qquad\ldots\ldots\\
&\qquad tr(W_{ia}^*U^*W_{i_p-i_1,a_p-a_1}UW_{ia}VW_{j_1-j_p,b_1-b_p}V^*)dUdV
\end{aligned}$$

By moving the $W_{ia}^*U^*$ variables at right, we obtain, with $S_{ia}=UW_{ia}V$:

$$\begin{aligned}
I &= \sum_{ia}\int_E\int_E tr(W_{i_1-i_2,a_1-a_2}S_{ia}W_{j_2-j_1,b_2-b_1}S_{ia}^*)\\
&\qquad\ldots\ldots\\
&\qquad tr(W_{i_p-i_1,a_p-a_1}S_{ia}W_{j_1-j_p,b_1-b_p}S_{ia}^*)dUdV
\end{aligned}$$

Now since $S_{ia}$ is Haar distributed when $U,V$ are Haar distributed, we obtain:

$$I=N\int_E\int_E tr(W_{i_1-i_2,a_1-a_2}UW_{j_2-j_1,b_2-b_1}U^*)\ldots tr(W_{i_p-i_1,a_p-a_1}UW_{j_1-j_p,b_1-b_p}U^*)dU$$

But this is exactly $N$ times the integral in the formula of $(T_p)_{ia,jb}$, from Proposition 16.16 above. Since the $N$ factor cancels with one of the two $N$ factors that we found in the beginning of the proof, when first computing $(T_p^2)_{ia,jb}$, we are done. □

As an illustration for the above result, going back to [24], we have:

**Theorem 16.18.** *We have a stationary matrix model*

$$\pi:C(S_4^+)\subset M_4(C(SU_2))$$

*given on the standard coordinates by the formula*

$$\pi(u_{ij})=[x\to Proj(c_ixc_j)]$$

*where $x\in SU_2$, and $c_1,c_2,c_3,c_4$ are the Pauli matrices.*

*Proof.* As already explained in the comments following Definition 16.12, the Pauli matrices appear as particular cases of the Weyl matrices. By working out the details, we conclude that Theorem 16.17 produces in this case the model in the statement. □

Observe that, since the matrix $Proj(c_ixc_j)$ depends only on the image of $x$ in the quotient $SU_2 \to SO_3$, we can replace the model space $SU_2$ by the smaller space $SO_3$, if we want to. This is something that can be used in conjunction with the isomorphism $S_4^+ \simeq SO_3^{-1}$ from section 9 above, and as explained in [14], our model becomes in this way something quite conceptual, algebrically speaking, as follows:

$$\pi : C(SO_3^{-1}) \subset M_4(C(SO_3))$$

In general, going beyond stationarity is a difficult task, and among the results here, let us mention the universal modelling questions for quantum permutations and quantum reflections [33], [52], various results on the flat models for the discrete groups [21], [31], questions regarding the Hadamard matrix models [14], [20], and the related fine analytic study on the compact and discrete quantum groups [53], [82], [129], [131].

In what follows we will only discuss the Hadamard models, which are of particular importance. Let us start with the following well-known definition:

**Definition 16.19.** *A complex Hadamard matrix is a square matrix*

$$H \in M_N(\mathbb{C})$$

*whose entries are on the unit circle, and whose rows are pairwise orthogonal.*

Observe that the orthogonality condition tells us that the rescaled matrix $U = H/\sqrt{N}$ must be unitary. Thus, these matrices form a real algebraic manifold, given by:

$$X_N = M_N(\mathbb{T}) \cap \sqrt{N} U_N$$

The basic example is the Fourier matrix, $F_N = (w^{ij})$ with $w = e^{2\pi i/N}$. More generally, we have as example the Fourier coupling of any finite abelian group $G$, regarded via the isomorphism $G \simeq \widehat{G}$ as a square matrix, $F_G \in M_G(\mathbb{C})$:

$$F_G = < i, j >_{i \in G, j \in \widehat{G}}$$

Observe that for the cyclic group $G = \mathbb{Z}_N$ we obtain in this way the above standard Fourier matrix $F_N$. In general, we obtain a tensor product of Fourier matrices $F_N$.

To be more precise here, we have the following result:

**Theorem 16.20.** *Given a finite abelian group $G$, with dual group $\widehat{G} = \{\chi : G \to \mathbb{T}\}$, consider the Fourier coupling $\mathcal{F}_G : G \times \widehat{G} \to \mathbb{T}$, given by $(i, \chi) \to \chi(i)$.*

1. *Via the standard isomorphism $G \simeq \widehat{G}$, this Fourier coupling can be regarded as a square matrix, $F_G \in M_G(\mathbb{T})$, which is a complex Hadamard matrix.*
2. *In the case of the cyclic group $G = \mathbb{Z}_N$ we obtain in this way, via the standard identification $\mathbb{Z}_N = \{1, \ldots, N\}$, the Fourier matrix $F_N$.*
3. *In general, when using a decomposition $G = \mathbb{Z}_{N_1} \times \ldots \times \mathbb{Z}_{N_k}$, the corresponding Fourier matrix is given by $F_G = F_{N_1} \otimes \ldots \otimes F_{N_k}$.*

*Proof.* This follows indeed from some basic facts from group theory:

(1) With the identification $G \simeq \widehat{G}$ made our matrix is given by $(F_G)_{i\chi} = \chi(i)$, and the scalar products between the rows are computed as follows:

$$< R_i, R_j >= \sum_\chi \chi(i)\overline{\chi(j)} = \sum_\chi \chi(i-j) = |G| \cdot \delta_{ij}$$

Thus, we obtain indeed a complex Hadamard matrix.

(2) This follows from the well-known and elementary fact that, via the identifications $\mathbb{Z}_N = \widehat{\mathbb{Z}_N} = \{1, \ldots, N\}$, the Fourier coupling here is as follows, with $w = e^{2\pi i/N}$:

$$(i, j) \to w^{ij}$$

(3) We use here the following well-known formula, for the duals of products:

$$\widehat{H \times K} = \widehat{H} \times \widehat{K}$$

At the level of the corresponding Fourier couplings, we obtain from this:

$$F_{H \times K} = F_H \otimes F_K$$

Now by decomposing $G$ into cyclic groups, as in the statement, and by using (2) for the cyclic components, we obtain the formula in the statement. $\square$

There are many other examples of Hadamard matrices, with some being fairly exotic, appearing in various branches of mathematics and physics. The idea is that the complex Hadamard matrices can be though of as being "generalized Fourier matrices", and this is where the interest in these matrices comes from. See [75], [88], [125].

In relation with the quantum groups, the starting observation is as follows:

**Proposition 16.21.** *If $H \in M_N(\mathbb{C})$ is Hadamard, the rank one projections*

$$P_{ij} = Proj\left(\frac{H_i}{H_j}\right)$$

*where $H_1, \ldots, H_N \in \mathbb{T}^N$ are the rows of $H$, form a magic unitary.*

*Proof.* This is clear, the verification for the rows being as follows:

$$\left\langle \frac{H_i}{H_j}, \frac{H_i}{H_k} \right\rangle = \sum_l \frac{H_{il}}{H_{jl}} \cdot \frac{H_{kl}}{H_{il}} = \sum_l \frac{H_{kl}}{H_{jl}} = N\delta_{jk}$$

The verification for the columns is similar, as follows:

$$\left\langle \frac{H_i}{H_j}, \frac{H_k}{H_j} \right\rangle = \sum_l \frac{H_{il}}{H_{jl}} \cdot \frac{H_{jl}}{H_{kl}} = \sum_l \frac{H_{il}}{H_{kl}} = N\delta_{ik}$$

Thus, we obtain the result. $\square$

We can proceed now exactly in the same way as we did with the Weyl matrices, namely by constructing a model of $C(S_N^+)$, and performing the Hopf image construction. We are led in this way to the following definition:

**Definition 16.22.** *To any Hadamard matrix $H \in M_N(\mathbb{C})$ we associate the quantum permutation group $G \subset S_N^+$ given by the fact that $C(G)$ is the Hopf image of*

$$\pi : C(S_N^+) \to M_N(\mathbb{C})$$

$$u_{ij} \to Proj\left(\frac{H_i}{H_j}\right)$$

*where $H_1, \ldots, H_N \in \mathbb{T}^N$ are the rows of $H$.*

Summarizing, we have a construction $H \to G$, and our claim is that this construction is something really useful, with $G$ encoding the combinatorics of $H$. To be more precise, our claim is that "$H$ can be thought of as being a kind of Fourier matrix for $G$".

There are several results supporting this claim, with the main evidence coming from the following result, which collects the basic results regarding the construction $H \to G$:

**Theorem 16.23.** *The construction $H \to G$ has the following properties:*

1. *For a Fourier matrix $H = F_G$ we obtain the group $G$ itself, acting on itself.*
2. *For $H \notin \{F_G\}$, the quantum group $G$ is not classical, nor a group dual.*
3. *For a tensor product $H = H' \otimes H''$ we obtain a product, $G = G' \times G''$.*

*Proof.* All this material is standard, and elementary, as follows:

(1) Let us first discuss the cyclic group case, where:

$$H = F_N$$

Here the rows of $H$ are given by $H_i = \rho^i$, where:

$$\rho = (1, w, w^2, \ldots, w^{N-1})$$

Thus, we have the following formula:

$$\frac{H_i}{H_j} = \rho^{i-j}$$

It follows that the corresponding rank 1 projections $P_{ij} = Proj(H_i/H_j)$ form a circulant matrix, all whose entries commute. Since the entries commute, the corresponding quantum group must satisfy $G \subset S_N$. Now by taking into account the circulant property of $P = (P_{ij})$ as well, we are led to the conclusion that we have $G = \mathbb{Z}_N$.

In the general case now, where $H = F_G$, with $G$ being an arbitrary finite abelian group, the result can be proved either by extending the above proof, of by decomposing $G = \mathbb{Z}_{N_1} \times \ldots \times \mathbb{Z}_{N_k}$ and using (3) below, whose proof is independent from the rest.

(2) This is something more tricky, needing some general study of the representations whose Hopf images are commutative, or cocommutative. For details here, along with a number of supplementary facts on the construction $H\to G$, we refer to [20].

(3) Assume that we have a tensor product $H=H'\otimes H''$, and let $G,G',G''$ be the associated quantum permutation groups. We have then a diagram as follows:

$$\begin{array}{ccccc}
C(S_{N'}^+)\otimes C(S_{N''}^+) & \rightarrow & C(G')\otimes C(G'') & \rightarrow & M_{N'}(\mathbb C)\otimes M_{N''}(\mathbb C)\\
\uparrow & & & & \downarrow\\
C(S_N^+) & \longrightarrow & C(G) & \longrightarrow & M_N(\mathbb C)
\end{array}$$

Here all the maps are the canonical ones, with those on the left and on the right coming from $N=N'N''$. At the level of standard generators, the diagram is as follows:

$$\begin{array}{ccccc}
u_{ij}'\otimes u_{ab}'' & \longrightarrow & w_{ij}'\otimes w_{ab}'' & \longrightarrow & P_{ij}'\otimes P_{ab}''\\
\uparrow & & & & \downarrow\\
u_{ia,jb} & \longrightarrow & w_{ia,jb} & \longrightarrow & P_{ia,jb}
\end{array}$$

Now observe that this diagram commutes. We conclude that the representation associated to $H$ factorizes indeed through $C(G')\otimes C(G'')$, and this gives the result. $\square$

Going beyond the above result is an interesting question, and we refer here to [15], and to follow-up papers. There are several computations available here, for the most regarding the deformations of the Fourier models. We believe that the unification of all this with the Weyl matrix models is a very good question, related to many interesting things.

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