Noname manuscript No.
(will be inserted by the editor)

A Definitive Proof of The *abc* Conjecture

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Received: date / Accepted: date

Abstract In this paper, we consider the *abc* conjecture. Firstly, we give an elementary proof the conjecture $c < rad^2(abc)$. Secondly, the proof of the *abc* conjecture is given for $\epsilon \geq 1$, then for $\epsilon \in]0,1[$. We choose the constant $K(\epsilon)$ as $K(\epsilon) = e^{\left(\dfrac{1}{\epsilon^2}\right)}$. Some numerical examples are presented.

Keywords Elementary number theory · real functions of one variable.

Mathematics Subject Classification (2010) 11AXX · 26AXX

To the memory of my Father who taught me arithmetic
To the memory of my colleague and friend Jamel Zaiem (1956-2019)

1 Introduction and notations

Let a positive integer $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as :

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{1}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{2}$$

The *abc* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [1]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

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Conjecture 1 (***abc*** **Conjecture**): Let a, b, c positive integers relatively prime with $c = a + b$, then for each $\epsilon > 0$, there exists a constant $K(\epsilon)$ such that :

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \tag{3}$$

$K(\epsilon)$ depending only of ϵ.

The idea to try to write a paper about this conjecture was born after the publication of an article in Quanta magazine about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [2]. The difficulty to find a proof of the *abc* conjecture is due to the incomprehensibility how the prime factors are organized in c giving a, b with $c = a + b$. So, I will give a simple proof that can be understood by undergraduate students.

We know that numerically, $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [1]. A conjecture was proposed that $c < rad^2(abc)$ [3]. It is the key to resolve the *abc* conjecture. In my paper, I propose an elementary proof of it, it facilitates the proof of the *abc* conjecture. In the second section, we give the proof that $c < rad^2(abc)$. We present the proof of the *abc* conjecture in section three. The numerical examples are discussed in sections four and five.

2 The Proof of the Conjecture $c < rad^2(abc)$

Below is given the definition of the conjecture $c < rad^2(abc)$:

Conjecture 2 Let a, b, c positive integers relatively prime with $c = a + b, a > b, b \geq 2$, then:

$$c < rad^2(abc) \Longrightarrow \frac{Logc}{Log(rad(abc))} < 2 \tag{4}$$

We note $R = rad(abc)$ in the case $c = a+b$ or $R = rad(ac)$ in the case $c = a+1$.

As c is bounded, it exists an unique couple $(m, n) \in \mathbb{Z}^+ \times \mathbb{N}$, $n \geq m + 1 \geq 1$, so that:

$$m.R^2 < c < n.R^2 \tag{5}$$

We can write:

$$c = mR^2 + r, \quad 1 \leq r < R^2 \tag{6}$$

$$c = nR^2 - r', \quad 1 \leq r' < R^2 \tag{7}$$

But $mR^2 + r = nR^2 - r' \Longrightarrow 2 \leq r + r' = (n-m)R^2 < 2R^2 \Longrightarrow n - m < 2$, we deduce $n = m$ or $n = m + 1$. The case $n = m$ presents a contradiction. Hence $n = m + 1$. The equation (5) becomes:

$$m.R^2 < c < (m+1).R^2, \quad m \geq 0 \tag{8}$$

2.1 Proof that $c < R^2$

Proof :

** Case $c < R$: $c < R < R^2$ and the condition (4) is verified.

** Case $c = R$: case to reject.

** Case $c > R$: from (8), we obtain:

$$mR^2 < c < (m+1)R^2 \tag{9}$$

If $m = 0$, we deduce that :

$$0 < c < R^2 \tag{10}$$

and the condition (4) is verified.

We suppose now that $m > 0$. Let $c = a + b$ or $c = a + 1$ so that:

$$mR^2 < c < (m+1)R^2$$

As $c > mR^2$, we can write :

$$c = mR^2 + m', \quad m' < R^2 \tag{11}$$

But $c > R \Longrightarrow c^2 > R^2$, we obtain also:

$$c^2 = lR^2 + l', \quad l' < R^2 \tag{12}$$

From the above equations, we can write:

$$(mR^2 + m')^2 = lR^2 + l' \Longrightarrow m^2R^4 + (2mm' - l)R^2 + m'^2 - l' = 0 \tag{13}$$

From the last equation above, R^2 is the positive root of the polynomial of the second degree:

$$F(T) = m^2T^2 + (2mm' - l)T + m'^2 - l' = 0 \tag{14}$$

The discriminant of $F(T)$ is:

$$\Delta = (2mm' - l)^2 - 4m^2(m'^2 - l') \tag{15}$$

As a real root of $F(T)$ exists, and it is an integer, Δ is written as :

$$\Delta = t^2 \geq 0, t \in \mathbb{Z}^+ \tag{16}$$

** - Case $\Delta = 0$ and $m'^2 - l' \neq 0$: Then $(2mm' - l)^2 = 4m^2(m'^2 - l') \Longrightarrow m'^2 - l' = \alpha^2$,$\alpha \in \mathbb{N}$. In this case the equation (14 has a double root $T_1 = T_2 = \dfrac{l - 2mm'}{2m^2} = R^2 \Longrightarrow l - 2mm' = 2m^2R^2 > 0$. But $(l - 2mm')^2 = 4m^4R^4 = 4m^2(m'^2 - l') \Longrightarrow m'^2 = m^2R^4 + l' > R^4 \Longrightarrow m' > R^2$. Then the contradiction as $m' < R^2$. The case $\Delta = 0$ and $m'^2 - l' \neq 0$ is impossible.

** - Case $\varDelta = 0$ and $m'^2 - l' = 0$: In this case, $2mm' - l = 0 \Longrightarrow R^2 = 0$. Then the contradiction as $R > 0$. The case $\varDelta = 0$ and $m'^2 - l' = 0$ is impossible.

** - Case $\varDelta > 0$ and $m'^2 - l' = 0$: The equation (14) becomes:

$$F(T) = m^2T^2 + (2mm' - l)T = 0 \Longrightarrow \begin{cases} T_1 = 0 \\ T_2 = \dfrac{l - 2mm'}{m^2} = R^2 \end{cases} \tag{17}$$

Then, we have:

$$l - 2mm' = m^2R^2 \Longrightarrow l = 2mm' + m^2R^2$$

As $m' < R^2 \Longrightarrow l - m^2R^2 < 2mR^2 \Longrightarrow l < 2mR^2 + m^2R^2$, we obtain $lR^2 < m(2 + m)R^4$. We deduce that $c^2 = lR^2 + l' < m(2 + m)R^4 + R^2$. We know that $c < (m + 1)R^2 \Longrightarrow c^2 < (m + 1)^2R^4$. We verify easily that $m(2 + m)R^4 + R^2 < (m + 1)^2R^4$, then the contradiction with $mR^2 < c < (m + 1)R^2$. Hence, the case $\varDelta > 0$ and $m'^2 - l' = 0$ is impossible.

** - Case $\varDelta > 0$ and $m'^2 - l' > 0$: We have: $\varDelta = (2mm' - l)^2 - 4m^2(m'^2 - l') = t^2 \Longrightarrow t^2 < (2mm' - l)^2$. Let the case $|2mm' - l| = 2mm' - l \Longrightarrow t < 2mm' - l$. The expression of the two roots are:

$$\begin{cases} T_1 = \dfrac{l - 2mm' + t}{2m^2b} < 0 \\ \\ T_2 = \dfrac{l - 2mm' - t}{2m^2} < 0 \end{cases} \tag{18}$$

As $R^2 > 0$ is a root of $F(T) = 0$, then the contradiction. Hence, the case $\varDelta > 0$ and $m'^2 - l' > 0$ is impossible.

** - Case $\varDelta > 0$ and $m'^2 - l' < 0$: From $m'^2 < l' \Longrightarrow (c - mR^2)^2 < c^2 - lR^2$, it gives $m^2R^2 + l - 2mc < 0 \Longrightarrow m^2R^2 + l < 2mc < 2m(m + 1)R^2$. Then we obtain $l < m^2R^2 + 2mR^2 \Longrightarrow lR^2 < m(m + 2)R^4 \Longrightarrow c^2 = lR^2 + l' < m(m + 2)R^4 + R^2$. We know that $c < (m + 1)R^2 \Longrightarrow c^2 < (m + 1)^2R^4$. We verify easily that $m(2 + m)R^4 + R^2 < (m + 1)^2R^4$, then the contradiction with $mR^2 < c < (m + 1)R^2$. Hence, the case $\varDelta > 0$ and $m'^2 - l' < 0$ is impossible.

All the cases for the resolution of the equation (14) have given contradictions with the hypothesis $c > mR^2, m > 0$. Then we obtain that $m = 0$ and $0 < c < R^2$. Hence the condition (4) is verified.

We announce the theorem:

Theorem 1 *Let a, b, c positive integers relatively prime with $c = a + b, a > b$, then $c < rad^2(abc)$.*

3 The Proof of the *abc* conjecture

3.1 Case : $\epsilon \geq 1$

Using the result that $c < R^2$, we have $\forall \epsilon \geq 1$:

$$c < R^2 \leq R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad with \;\; K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, \;\; \epsilon \geq 1 \tag{19}$$

We verify easily that $K(\epsilon) > 1$ for $\epsilon \geq 1$. Then the *abc* conjecture is true.

3.2 Case: $\epsilon < 1$

3.2.1 Case: $c < R$

In this case, we can write :

$$c < R < R^{1+\epsilon} < K(\epsilon).R^{1+\epsilon}, \quad with \;\; K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, \;\; \epsilon < 1 \tag{20}$$

here also $K(\epsilon) > 1$ for $\epsilon < 1$ and the *abc* conjecture is true.

3.2.2 Case: $c > R$

In this case, we confirm that :

$$c < K(\epsilon).R^{1+\epsilon}, \quad with \;\; K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)}, 0 < \epsilon < 1 \tag{21}$$

If not, then $\exists \epsilon_0 \in]0,1[$, so that the triple (a,b,c) checking $c > R$ and:

$$c \geq R^{1+\epsilon_0}.K(\epsilon_0) \tag{22}$$

are in finite number. We have:

$$c \geq R^{1+\epsilon_0}.K(\epsilon_0) \Longrightarrow R^{1-\epsilon_0}.c \geq R^{1-\epsilon_0}.R^{1+\epsilon_0}.K(\epsilon_0) \Longrightarrow$$
$$R^{1-\epsilon_0}.c \geq R^2.K(\epsilon_0) > cK(\epsilon_0) \Longrightarrow R^{1-\epsilon_0} > K(\epsilon_0) \tag{23}$$

As $c > R$, we obtain:

$$c^{1-\epsilon_0} > R^{1-\epsilon_0} > K(\epsilon_0) \Longrightarrow$$

$$c^{1-\epsilon_0} > K(\epsilon_0) \Longrightarrow c > (K(\epsilon_0))^{\left(\frac{1}{1-\epsilon_0}\right)} \tag{24}$$

We deduce that it exists an infinity of triples (a,b,c) verifying (22), hence the contradiction. Then the proof of the *abc* conjecture is finished. We obtain that $\forall \epsilon > 0$, $c = a + b$ with a, b, c relatively coprime:

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \quad with \quad K(\epsilon) = e^{\left(\frac{1}{\epsilon^2}\right)} \quad \epsilon > 0 \tag{25}$$

Q.E.D

In the two following sections, we are going to verify some numerical examples.

4 Examples : Case $c = a + 1$

4.1 Example 1

The example is given by:

$$1 + 5 \times 127 \times (2 \times 3 \times 7)^3 = 19^6 \tag{26}$$

$a = 5 \times 127 \times (2 \times 3 \times 7)^3 = 47\,045\,880 \Rightarrow \mu_a = 2 \times 3 \times 7 = 42$ and $rad(a) = 2 \times 3 \times 5 \times 7 \times 127$, in this example, $\mu_a < rad(a)$.
$c = 19^6 = 47\,045\,880 \Rightarrow rad(c) = 19$. Then $rad(ac) = rad(ac) = 2 \times 3 \times 5 \times 7 \times 19 \times 127 = 506\,730$.
We have $c > rad(ac)$ but $rad^2(ac) = 506\,730^2 = 256\,775\,292\,900 > c = 47\,045\,880$.

4.1.1 Case $\epsilon = 0.01$

$c < K(\epsilon).rad(ac)^{1+\epsilon} \Longrightarrow 47\,045\,880 \overset{?}{<} e^{10000}.506\,730^{1.01}$. The expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = e^{\frac{1}{0.0001}} = e^{10000} = 8.7477777149120053120152473488653e{+}4342 \tag{27}$$

We deduce that $c \ll K(0.01).506\,730^{1.01}$ and the equation (25) is verified.

4.1.2 Case $\epsilon = 0.1$

$K(0.1) = e^{\frac{1}{0.01}} = e^{100} = 2.6879363309671754205917012128876e + 43 \Longrightarrow c < K(0.1) \times 506\,730^{1.01}$, and the equation (25) is verified.

4.1.3 Case $\epsilon = 1$

$K(1) = e \Longrightarrow c = 47\,045\,880 < e.rad^2(ac) = 697\,987\,143\,184,212$ and the equation (25) is verified.

4.1.4 Case $\epsilon = 100$

$$K(100) = e^{0.0001} \Longrightarrow c = 47\,045\,880 \overset{?}{<} e^{0.0001}.506\,730^{101} = \\ 1.5222350248607608781853142687284e + 576$$

and the equation (25) is verified.

4.2 Example 2

We give here the example 2 from $https://nitaj.users.lmno.cnrs.fr$:

$$3^7 \times 7^5 \times 13^5 \times 17 \times 1831 + 1 = 2^{30} \times 5^2 \times 127 \times 353 \tag{28}$$

$a = 3^7 \times 7^5 \times 13^5 \times 17 \times 1831 = 424\,808\,316\,456\,140\,799 \Rightarrow rad(a) = 3 \times 7 \times 13 \times 17 \times 1831 = 8497671 \Longrightarrow \mu_a > rad(a)$,
$b = 1$, $rad(c) = 2 \times 5 \times 127 \times 353$ Then $rad(ac) = 849767 \times 448310 = 3\,809\,590\,886\,010 < c$. $rad^2(ac) = 14\,512\,982\,718\,770\,456\,813\,720\,100 > c$, then $c \leq 2rad^2(ac)$. For example, we take $\epsilon = 0.5$, the expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = e^{1/0.25} = e^4 = 54.59800313096579789056 \tag{29}$$

Let us verify (25):

$$c \stackrel{?}{<} K(\epsilon).rad(ac)^{1+\epsilon} \Longrightarrow c = 424808316456140800 \stackrel{?}{<} K(0.5) \times (3\,809\,590\,886\,010)^{1.5} \Longrightarrow \\ 424808316456140800 < 405970304762905691174.98260818045 \tag{30}$$

Hence (25) is verified.

5 Examples : Case $c = a + b$

5.1 Example 1

We give here the example of Eric Reyssat [1], it is given by:

$$3^{10} \times 109 + 2 = 23^5 = 6436343 \tag{31}$$

$a = 3^{10}.109 \Rightarrow \mu_a = 3^9 = 19683$ and $rad(a) = 3 \times 109$,
$b = 2 \Rightarrow \mu_b = 1$ and $rad(b) = 2$,
$c = 23^5 = 6436343 \Rightarrow rad(c) = 23$. Then $rad(abc) = 2 \times 3 \times 109 \times 23 = 15042$.
For example, we take $\epsilon = 0.01$, the expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = e^{9999.99} = 8.7477777149120053120152473488653e + 4342 \tag{32}$$

Let us verify (25):

$$c \stackrel{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow c = 6436343 \stackrel{?}{<} K(0.01) \times (3 \times 109 \times 2 \times 23)^{1.01} \Longrightarrow \\ 6436343 \ll K(0.01) \times 15042^{1.01} \tag{33}$$

Hence (25) is verified.

5.2 Example 2

The example of Nitaj about the ABC conjecture [1] is:

$$a = 11^{16}.13^2.79 = 613\,474\,843\,408\,551\,921\,511 \Rightarrow rad(a) = 11.13.79 \quad (34)$$
$$b = 7^2.41^2.311^3 = 2\,477\,678\,547\,239 \Rightarrow rad(b) = 7.41.311 \quad (35)$$
$$c = 2.3^3.5^{23}.953 = 613\,474\,845\,886\,230\,468\,750 \Rightarrow rad(c) = 2.3.5.953 \quad (36)$$
$$rad(abc) = 2.3.5.7.11.13.41.79.311.953 = 28\,828\,335\,646\,110 \quad (37)$$

5.2.1 Case 1

we take $\epsilon = 100$ we have:

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 \overset{?}{<} e^{0.0001}.(2.3.5.7.11.13.41.79.311.953)^{101} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 < 2.7657949971494838920022381186039e+1359$$

then (25) is verified.

5.2.2 Case 2

We take $\epsilon = 0.5$, then:

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow \quad (38)$$
$$613\,474\,845\,886\,230\,468\,750 \overset{?}{<} e^4.(2.3.5.7.11.13.41.79.311.953)^{1.5} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 < 8\,450\,961\,319\,227\,998\,887\,403,9993 \quad (39)$$

We obtain that (25) is verified.

5.2.3 Case 3

We take $\epsilon = 1$, then

$$c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 \overset{?}{<} e.(2.3.5.7.11.13.41.79.311.953)^2 \Longrightarrow$$
$$613\,474\,845\,886\,230\,468\,750 < 831\,072\,936\,124\,776\,471\,158\,132\,100 \times e \quad (40)$$

We obtain that (25) is verified.

5.3 Example 3

It is of Ralf Bonse about the ABC conjecture [3] :

$$2543^4.182587.2802983.85813163 + 2^{15}.3^{77}.11.173 = 5^{56}.245983 \tag{41}$$
$$a = 2543^4.182587.2802983.85813163$$
$$b = 2^{15}.3^{77}.11.173$$
$$c = 5^{56}.245983$$
$$rad(abc) = 2.3.5.11.173.2543.182587.245983.2802983.85813163$$
$$rad(abc) = 1.5683959920004546031461002610848e + 33 \tag{42}$$

5.3.1 Case 1

For example, we take $\epsilon = 10$, the expression of $K(\epsilon)$ becomes:

$$K(\epsilon) = e^{0.01} = 1.0078157404282956743204617416\,77$$

Let us verify (25):

$$\begin{gathered} c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Rightarrow c = 5^{56}.245983 \overset{?}{<} \\ e^{0.01}.(2.3.5.11.173.2543.182587.245983.2802983.85813163)^{11} \\ \Longrightarrow 3.4136998783296235160378273576498e + 44 < \\ 1.4236200596494908176008120925\,72e + 365 \end{gathered} \tag{43}$$

The equation (25) is verified.

5.3.2 Case 2

We take $\epsilon = 0.4 \Longrightarrow K(\epsilon) = 12.182473474251512159126256696\,08$, then: The

$$\begin{gathered} c \overset{?}{<} K(\epsilon).rad(abc)^{1+\epsilon} \Rightarrow c = 5^{56}.245983 \overset{?}{<} \\ e^{6.25}.(2.3.5.11.173.2543.182587.245983.2802983.85813163)^{1.4} \\ \Longrightarrow 3.4136998783296235160378273576498e + 44 < \\ 3.6255465680011453642792720569685e + 47 \end{gathered} \tag{44}$$

And the equation (25) is verified.

Ouf, end of the mystery!

6 Conclusion

We have given an elementary proof of the *abc* conjecture, confirmed by some numerical examples. We can announce the important theorem:

Theorem 2 *(David Masser, Joseph Œsterlé & Abdelmajid Ben Hadj Salem; 2019) Let* a, b, c *positive integers relatively prime with* $c = a + b$*, then for each* $\epsilon > 0$*, there exists* $K(\epsilon)$ *such that :*

$$c < K(\epsilon).rad(abc)^{1+\epsilon} \tag{45}$$

where $K(\epsilon)$ *is a constant depending of* ϵ *proposed as :*

$$K(\epsilon) = e^{\left(\dfrac{1}{\epsilon^2}\right)} \quad , \epsilon > 0$$

Acknowledgements The author is very grateful to Professors Mihăilescu Preda and Gérald Tenenbaum for their comments about errors found in previous manuscripts concerning proofs proposed of the *abc* conjecture.

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