The Complexity of NonSwapClique

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ABSTRACT

Problem NonSwapClique: Given an undirected graph $G = (V, E)$, does it contain a clique $S \subseteq V$ of size k, such that you cannot obtain another clique of the same size by swapping a pair of vertices? In this note, I settle the complexity of this problem as NP-complete, by a reduction from problem 1-in-3-SAT.

KEYWORDS

NonSwapClique; Complexity

1 RESULT

Definition 1.1. Decision problem NonSwapClique, given an unoriented graph $G = (V, E)$ and an integer k, asks whether there exists a clique $S \subseteq V$ of size k, such that there is no pair of vertices $v \in S$ and $v' \in V \setminus S$ such that $S \setminus \{v\} \cup \{v'\}$ is also a clique of size k. Such a clique is called a non-swap clique. Removing a vertex from S to add a new one is called swapping.

Definition 1.2. Decision problem 1-in-3-SAT, given a 3CNF formula $F = C_1 \wedge \ldots \wedge C_m$ on binary variables $X = \{x_1, \ldots, x_n\}$, asks whether there exists an instantiation $\tau : X \rightarrow \{0, 1\}$ such that in every clause $C_i = \ell_{i,1} \vee \ell_{i,2} \vee \ell_{i,3}$, exactly one literal is true and two are false.

Theorem 1.3. *NonSwapClique is NP-complete.*

Proof. An instance of NonSwapClique, if a non-swap clique $S \subseteq V$ of size k is given, can be verified true in time $O(|V|^2|E|)$. Therefore, problem NonSwapClique is in class NP.

We show NP-hardness by a many-one polynomial-time reduction from problem 1-in-3-SAT. Let 3CNF formula $F = C_1 \wedge \ldots \wedge C_m$ and binary variables $X = \{x_1, \ldots, x_n\}$ be an instance of 1-in-3-SAT, that we reduce to the following NonSwapClique instances. For every clause $C_i \in F$, we introduce a subset V_i of three disconnected vertices $V_i = \{v_{i,1}, v_{i,2}, v_{i,3}\}$ that represent the literals of the clause. For every binary variable $x_j \in X$, we introduce a subset V_{m+j} of two disconnected vertices $V_{m+j} = \{v_{m+j,0}, v_{m+j,1}\}$ that represent the two possible literals on variable x_j, hence its two possible instantiations. The set of $3m + 2n$ vertices is:

$$W = V_1 \cup \ldots \cup V_m \quad \cup \quad V_{m+1} \cup \ldots \cup V_{m+n}.$$

Edges only exist between two different subsets. Given any two different subsets V and V', there exists an edge between nodes $v \in V$ and $v' \in V'$ if and only if the corresponding literals are compatible. In other words, an edge is missing between v and v' if and only if the corresponding literals negate each other. We ask whether a non-swap clique of size $k = m + n$ exists in this graph. Since there are no edges inside subsets V, it amounts to ask whether there exists a clique $S \subseteq W$ with exactly one vertex v in each subset V, such that swapping to an other vertex $v' \in V \setminus \{v\}$ will induce some missing edges between v' and some vertex $u \in S \cap V'$ in some other subset V'.

(yes⇒yes) Assume there exists an instantiation $\tau : X \rightarrow \{0, 1\}$ that one-in-three satisfies formula $C_1 \wedge \ldots \wedge C_m$. Then we have the following non-swap clique $S \subseteq W$ of size $m + n$: in every subset V, take the vertex which corresponding literal is set true by the instantiation. Since an instantiation is a function and does not contradict itself, S is clearly a clique of size $n + m$. Also, in sets $V_{m+1}, \ldots, V_{m+n}$, it contains vertices that fully encode instantiation τ. In any subset from $V_1, \ldots, V_m$, swapping from a vertex v to a vertex v', which corresponding literal on variable x_j was set to false by 1-in-3 satisfying instantiation τ, would contradict the instantiation; hence, $S \setminus \{v\} \cup \{v'\}$ would miss an edge between v' and V_{m+j}. Similarly, every variable appears at least once in formula $C_1 \wedge \ldots \wedge C_m$, e.g. in corresponding vertex $v'' \in V_i$. Therefore, in any subset V_{m+j}, swapping from a vertex v to a vertex v', which corresponds to swapping the instantiation of variable x_j, would contradict v''; hence, $S \setminus \{v\} \cup \{v'\}$ would miss an edge between v' and $v'' \in V_i$.

(yes⇐yes) Assume there exists a non-swap clique $S \subseteq W$ of size $n + m$. It fully defines an instantiation τ_S, since the clique is also defined on $V_{m+1} \ldots V_{m+n}$. The vertices of the clique correspond to the literals set to true in the formula. Then, in any subset V, swapping from $v \in S \cap V$ to $v' \in V \setminus \{v\}$ has some missing edge in $S \setminus \{v\} \cup \{v'\}$. It means that v' contradicts a literal set to true (a vertex in set $S \setminus \{v\}$). Therefore, the literal corresponding to v' must be set to an opposite value in τ_S or F. Hence, τ_S 1-in-3 satisfies the formula. □