Polygonal numbers in terms of the Beta function

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Abstract

In this article we study some characteristics of polygonal numbers, which are the positive integers that can be ordered, to form a regular polygon.

The article is closed, showing the relation of the polygonal numbers, with the Beta function when expressing any polygonal number, as a sum of terms of the Beta function.

Keywords: Beta function, polygonal number

Introduction

The beta function is among the transcendent functions, this being its vital importance, since it is related to the Gamma function and the Bessel functions, important for ordinary differential equations and in mathematical analysis.

$$\mathrm{B}_{(\mathrm{x,y})} = \int_0^1 \mathrm{t}^{\mathrm{x}-1}(1-\mathrm{t})^{\mathrm{y}-1}\mathrm{dt} \qquad (\mathrm{x} > 0, \mathrm{y} > 0)$$

The Beta function is the name used by Legendre and Whittaker and Watson for the Beta integral (first class Eulerian integral)

The Beta function lacks primitive, when x, y are natural numbers. The value of the beta function can be obtained through a series of integration by parts. However, there is an expression of the Beta function with natural variables deduced from the properties of the Gamma function:

$$B_{(x,y)} = \frac{(x-1)!\ (y-1)!}{(x+y-1)!}$$

On the other hand, figurative numbers are those whole numbers that can be represented by a set of equidistant points, forming a geometric figure. If the representation is a regular polygon, they are called polygonal numbers.

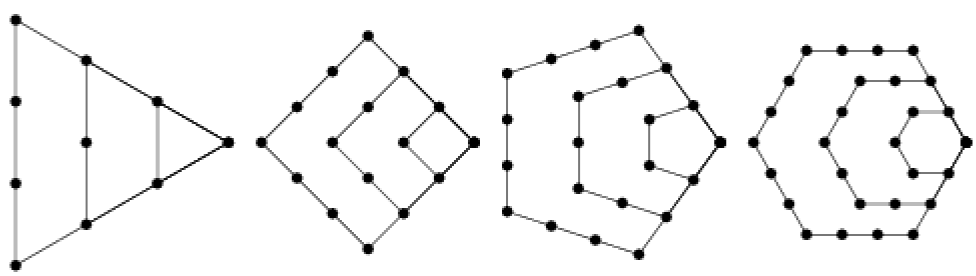

Figure 1 triangular, square, pentagonal and hexagonal numbers.

Polygonal numbers were discovered in ancient times by the Pythagoreans between the 6th and 5th centuries BC. its origin is due to the fact that the Pythagoreans represented the numbers by means of points on a parchment or pebbles in the sand and classified them according to the polygonal figure they represented, that is, they associated the numbers with geometric figures obtained by the arrangement regular of points, whose sum determines the number represented. Thus they obtained the different types of polygonal or figurative numbers:

- Triangular numbers: 1, 3, 6, 10, 15

- Square numbers: 1, 4, 9, 16, 25
- Pentagonal numbers: 1, 5, 12, 22, 35

Polygonal numbers are presented in number theory, since they allow the representation of any positive integer as a sum of polygonal numbers; such that a natural number can be written as the sum of three triangular numbers, the sum of four squares, five pentagonal, etc.

This article will show how to write the polygonal numbers from in terms the Beta function.

Definition 1

If l is the number of sides of a polygon, then the formula for the polygonal n-number of l sides is

$$p(l, n) = \frac{n((l-2)n-(l-4))}{2}$$

Example

Triangular $l = 3$

$$p(3, n) = \frac{n(n+1)}{2}$$

Squares $l = 4$

$$p(4, n) = \frac{n(2n)}{2} = n^2$$

Pentagonal $l = 5$

$$p(5, n) = \frac{n(3n-1)}{2}$$

Relationships of the Beta function with the polygonal numbers.

<u>Theorem 1</u>

We see an expression of polygonal numbers as sum of triangular numbers:

Theorem 1

$$p(l, n) = p(3, n) + (l-3)p(3, n-1)$$

Proof

$$\frac{n((l-2)n-(l-4))}{2}=\frac{n(n+1)}{2}+(l-3)\frac{(n-1)n}{2}$$

$$\frac{n((l-2)n-(l-4))}{2}=\frac{n(n+1)+(l-3)(n-1)n}{2}$$

$$\frac{n((l-2)n-(l-4))}{2}=\frac{n(n+1+nl-l-3n+3)}{2}$$

$$\frac{n((l-2)n-(l-4))}{2}=\frac{n(nl-2n-l+4)}{2}$$

$$\frac{n((l-2)n-(l-4))}{2}=\frac{n((l-2)n-(l-4))}{2}$$

□

In this way it is demonstrated that every polygonal number can be expressed as the sum of triangular numbers.

Theorem 2

We see an expression that relates triangular numbers to the Beta function.

Definition 2

Given two positive integers x, y; the Beta function is defined as

$$B(x,y)=\frac{(x-1)!\,(y-1)!}{(x+y-1)!}$$

Theorem 2

$$\frac{1}{2B(2,n)}=\frac{n(n+1)}{2}$$

Proof

From definition 2 we have

$$\frac{1}{\mathrm{B(x,y)}} = \frac{(\mathrm{x+y-1})!}{(\mathrm{x-1})!\,(\mathrm{y-1})!}$$

Evaluating at x = 2, y = n

$$\frac{1}{\mathrm{B(2,n)}} = \frac{(2+\mathrm{n}-1)!}{(2-1)!\,(\mathrm{n}-1)!}$$

$$\frac{1}{\mathrm{B(2,n)}} = \frac{(\mathrm{n}+1)!}{(\mathrm{n}-1)!}$$

$$\frac{1}{\mathrm{B(2,n)}} = \frac{(\mathrm{n}+1)(\mathrm{n})(\mathrm{n}-1)!}{(\mathrm{n}-1)!}$$

$$\frac{1}{\mathrm{B(2,n)}} = \mathrm{n}(\mathrm{n}+1)$$

Dividing both members by 2.

$$\frac{1}{2B(2,n)} = \frac{n(n+1)}{2}$$

□

In this way it is shown that any triangular number can be expressed in terms of the Beta function.

Theorem 3

We see an expression of the polygonal numbers in terms of the Beta function.

Theorem 3

$$\mathrm{p}(\mathrm{l},\mathrm{n}) = \frac{1}{2\mathrm{B}(2,\mathrm{n})} + \frac{\mathrm{l}-3}{2\mathrm{B}(2,\mathrm{n}-1)}$$

Proof

By theorem 1

$$\mathrm{p}(\mathrm{l},\mathrm{n}) = \mathrm{p}(3,\mathrm{n}) + (\mathrm{l}-3)\mathrm{p}(3,\mathrm{n}-1)$$

By theorem 2

$$\mathrm{p}(\mathrm{l},\mathrm{n}) = \frac{1}{2\mathrm{B}(2,\mathrm{n})} + \frac{\mathrm{l}-3}{2\mathrm{B}(2,\mathrm{n}-1)}$$

□

Conclusions

This article showed the relationship between polygonal numbers, and the Beta function when expressing polygonal numbers as terms of the latter.

Since every positive integer can be written as the sum of three triangular numbers, and by theorem 2 it is that the triangular numbers are related to the Beta function, it is concluded that every natural number can be written in terms of the Beta function.

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