A First Look at Feature Optics

Paul Mirsky
paulmirsky633@gmail.com

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Abstract

Feature optics is a new proposal for modeling diffraction and interference in terms of simple elements called *features*. We combine features to form composite systems by taking the outer product of the individual features' state vectors. The Fourier transform takes on a surprisingly simple form. We apply the new method to the beam and the diffraction grating.

1 Introduction

This article introduces *feature optics* (FO), a new framework for analyzing some diffraction and interference phenomena that ordinarily fall within the purview of Fourier optics.

The most elementary system in Fourier optics is the lens along with its two focal planes[1,2], referred to here as the *2f system* because its total length is two times the focal length *f* of the lens. We refer to the front focal plane as the *input plane*, and the rear focal plane as the *output plane*. Any pattern of monochromatic light may be thought of as a complex function vs position in the plane – a magnitude and a phase factor at each point. When such a pattern is placed in the input plane, the 2f system naturally acts to perform the *Fourier transform* (FT) of that complex function and project it onto the output plane. (Note that the output is *not* an image of the input. Imaging occurs between a different pair of planes, and is beyond the scope of this work).

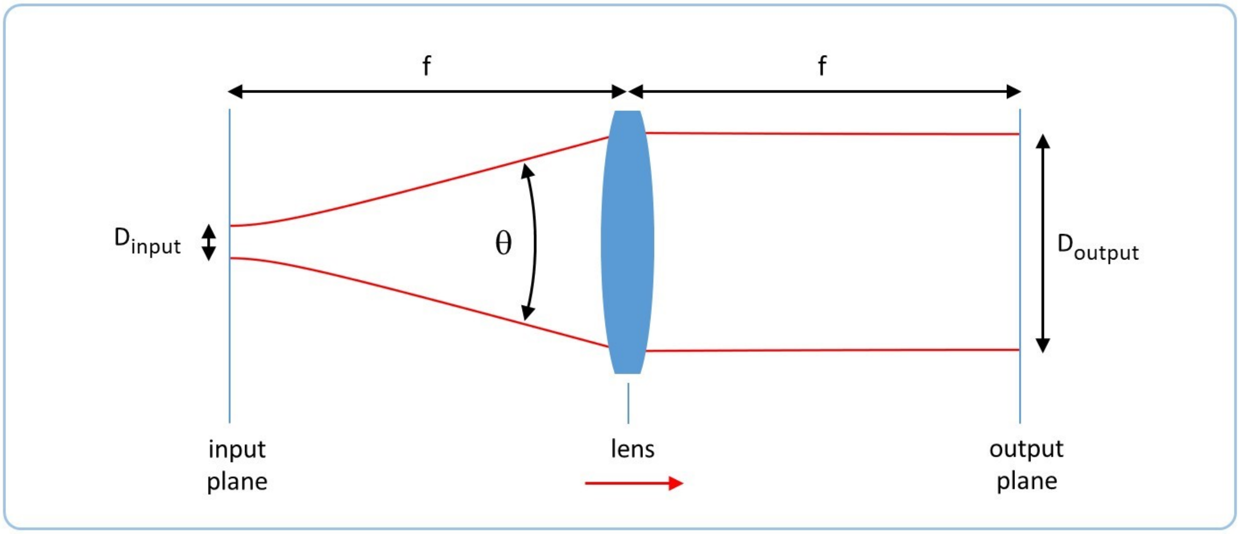
We will consider only two simple model patterns as inputs: the *beam* and the *grating*. The beam describes a single-mode laser, which is used ubiquitously in optics as a coherent light source. The diffraction grating is an opaque screen with a periodic pattern of apertures; when it is illuminated by a wide beam, the transmitted light carries the shape of the apertures. The grating is often used to separate polychromatic light into its component wavelengths, but it is also studied as a paradigm of interference from periodic monochromatic sources. We will consider each of these systems twice – first in terms of established theory to provide a benchmark for comparison, and second in terms of FO.

2 Preview

Before introducing the foundational concepts of FO in the following section, we first orient the reader with a brief preview. The easiest entry point to the subject is to consider the example of the *beam* from two perspectives: first the conventional Gaussian beam model, then the FO model.

The Gaussian beam is shown in figure 2.1. It begins with a small diameter and a flat wavefront in the input plane. As it propagates forward, it grows by diffraction and becomes much wider. The lens intercepts the beam and changes its curvature. After further propagation and a tiny decrease in diameter, a second flat wavefront forms in the output plane.

figure 2.1, the Gaussian beam



The system is parameterized by three independent numbers:

- D_{input}, the diameter of the beam in the input plane, measured at $1/e^2$ of the maximum power
- λ, the wavelength of the light
- f, the focal length of the lens

The remaining parameters are calculated from these. We calculate the far-field divergence angle θ of the beam[3]

$$\theta = \frac{4}{\pi} \cdot \frac{\lambda}{D_{input}}$$

The lens converts the divergence angle into the linear width D_{output}, by multiplying the angle by the lens focal length

$$D_{output} = \theta \cdot f = \frac{4}{\pi} \cdot \frac{\lambda}{D_{input}} \cdot f$$

Up to this point, we have been reviewing a conventional model of the beam. Now we will analyze it in terms of Feature Optics.

We begin by introducing a new set of parameters. All the parameters above have units of *length*. Our new parameters are dimensionless, and specify the *number of wavelengths* that make up various sizes:

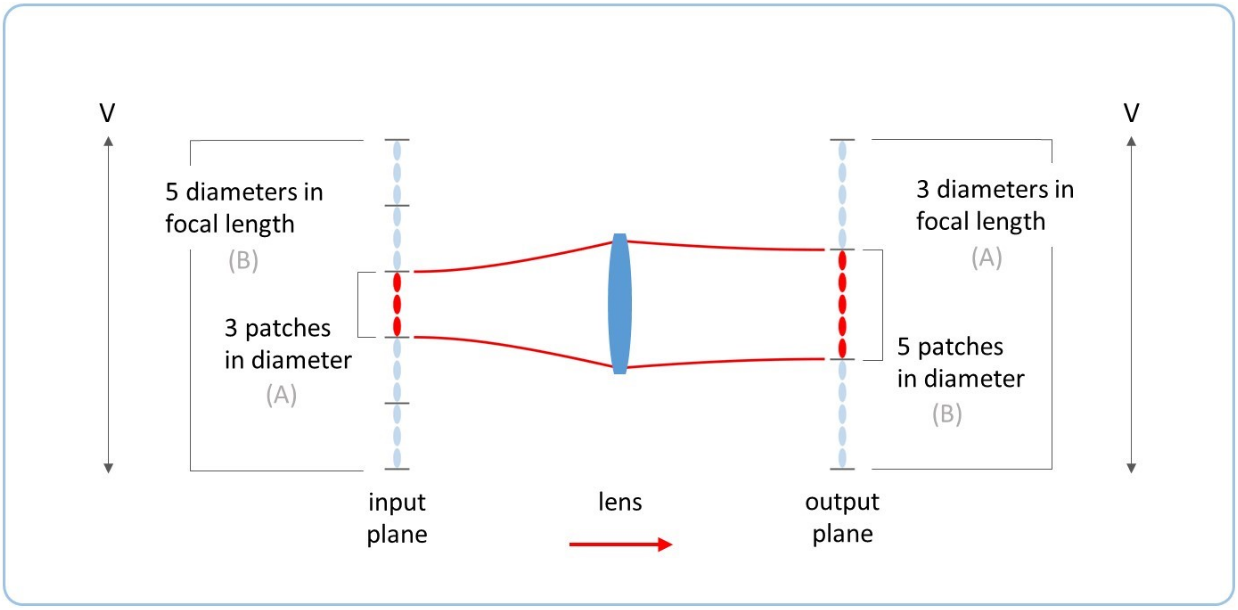
- A, the number of wavelengths in the input diameter
- B, the number of wavelengths in the output diameter
- V, the number of wavelengths in the lens focal length
- τ, a *Gaussian correction factor* of $4/\pi$, or roughly 1.27. This value often appears as a small discrepancy between the FO model and the Gaussian beam. We note that feature optics is only approximately accurate, because it frequently neglects minor inaccuracies such as this for the sake of having a simpler model.

Using these parameters, the previous equation can be re-expressed as

$$A \cdot B = V \cdot \tau$$

This equation is manifested in the system shown in figure 2.2. The 2f system acts to perform the discrete FT on an input vector of length V. Each element of the vector is a 'sample' of the wavefunction taken at each discrete (λ-sized) patch. Note that in addition to the area occupied by the beam, this vector also includes enough *empty* space adjacent to the beam to make up the full space V (we neglect τ). This dark space (magnitude = 0) is typically many times larger than the beam diameter.

figure 2.2, the beam in feature optics



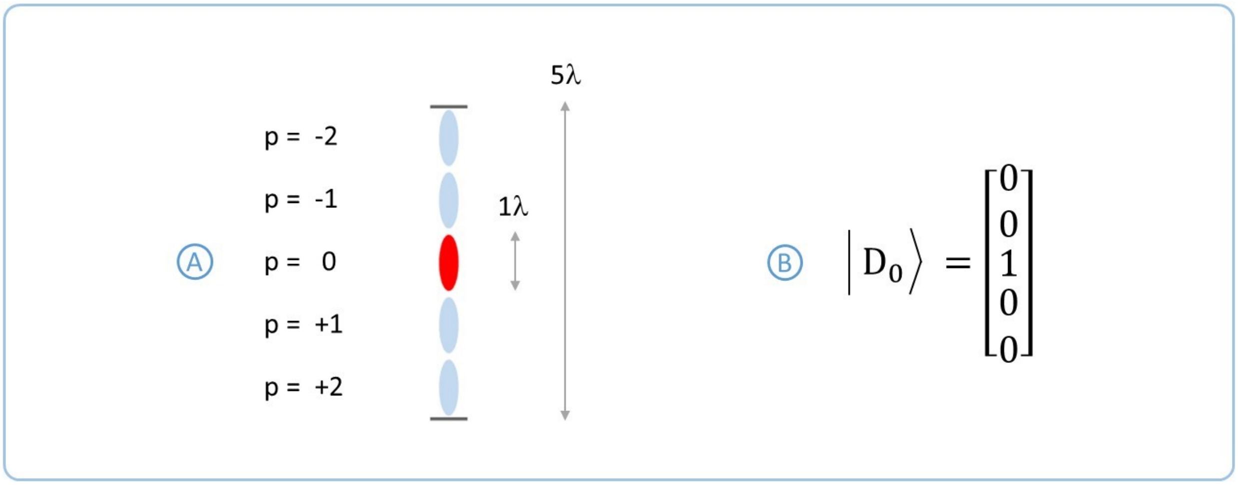
In the example in the figure, A=3 and B=5, meaning that the input beam is 3 wavelengths wide and is embedded in a space 5 times its width. Comparing the output against the input, we see that the FT has *reversed* the roles of A and B; i.e. a beam 5 wavelengths wide is now embedded in a space 3 times its width. In the conventional understanding, this coincidence carries no particular significance. However, in FO this is the very essence of how light transforms by propagation, as the next section will describe.

3 Principles of feature optics

3.1 Dark and bright features

In FO, patterns of light are composed of smaller, simpler patterns. The most elementary patterns are called *features* and exist in two types. The first type, shown in figure 3.1a, is the *dark feature*. It consists of a single bright patch of monochromatic light one wavelength λ wide, embedded in a space of 5 available positions (or in general, any positive integer n); in other words, the single bright patch is surrounded by n-1 dark patches of adjacent empty space. A second parameter p indicates *which* of the available positions the bright patch is located in.

figure 3.1, the dark feature

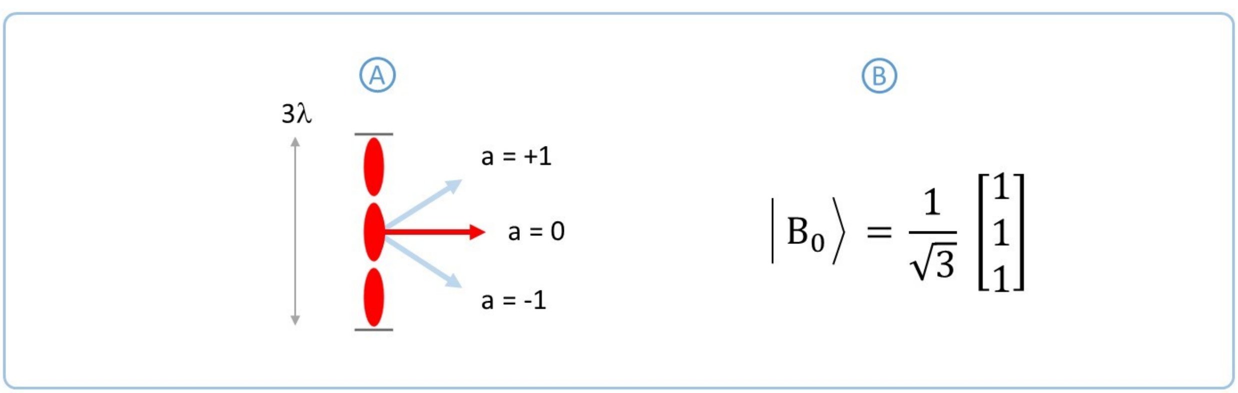


A feature is expressed mathematically by a state vector, as in quantum mechanics, see figure 3.1b. The squared magnitude at each point of the vector represents the probability of detecting a photon there. Alternatively, we may interpret it as the amplitude of the electric field.

The state vector $|D_0\rangle$ indicates that the bright patch is located at the central position, labeled p=0. The zeros in the vector indicate that four other dark states $|D_{-2}\rangle$, $|D_{-1}\rangle$, $|D_{+1}\rangle$, and $|D_{+2}\rangle$ also exist in which the bright patch is located elsewhere within the space, but these other states lie beyond the scope of this work.

The second type of feature is called the *bright feature*; an example is shown in figure 3.2a. It consists of 3 adjacent bright patches of light, or *n* in general. The light fills all of the available positions; there is no empty space. As with the dark feature, there are *n* possible states, because the bright feature can be tilted at any angle *a* out of *n* possible angles; however, the present work will consider only the zero angle. Choosing the zero angle allows us to write the state vector using real numbers; if we choose any other angle, we must use complex numbers.

figure 3.2, the bright feature



The corresponding state vector is equally distributed in amplitude across the n patches, but is normalized to an overall modulus of 1 (see figure 3.2b).

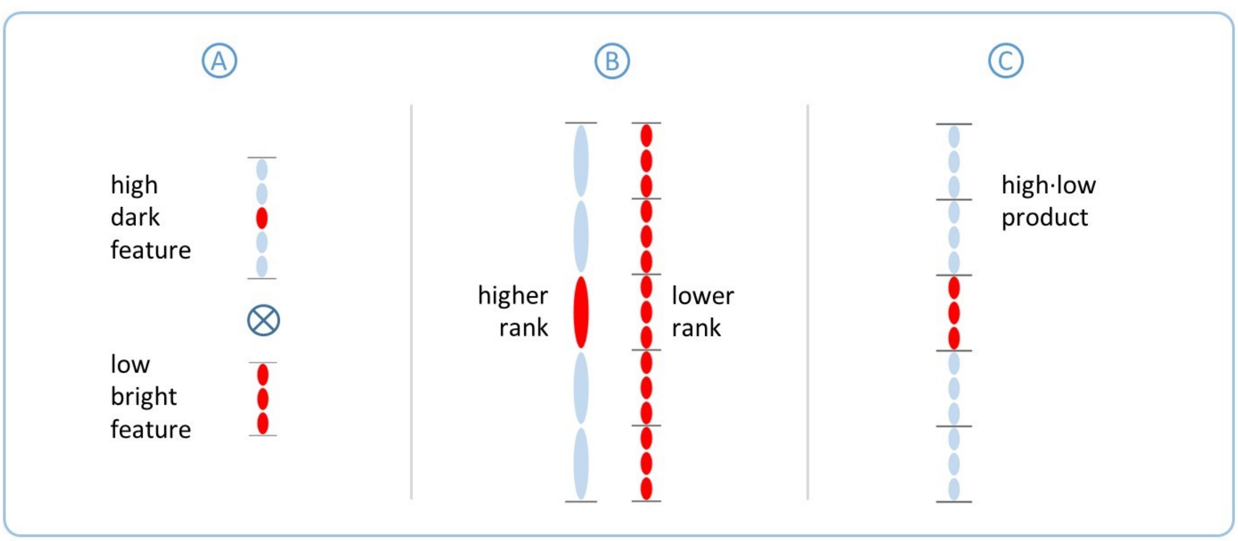
3.2 Combining features

Dark and bright features act as elemental building blocks which can be combined to form composite patterns. Again following the methods of quantum mechanics, we treat the states of the two features as two commuting observables. In other words, it is possible to observe both the position of the dark feature *and* the angle of the bright feature simultaneously, without being limited by uncertainty relations. We therefore represent the state of the combined system by forming the *outer product* of the individual state vectors.

$$|\mathrm{D}_0\mathrm{B}_0\rangle = |\mathrm{D}_0\rangle \otimes |\mathrm{B}_0\rangle = \frac{1}{\sqrt{3}} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A physical interpretation is drawn in figure 3.3. The *feature diagram* in figure 3.3a shows the two separate features described above, written one above the other with the outer-product symbol ⊗ between them. They are also labeled as *high* and *low* features, respectively.

figure 3.3, the physical meaning of the outer product



As figure 3.3b shows, an *entire* copy of the low feature is *nested* inside *each* patch of the high feature, and the high feature 'expands' accordingly. By convention, the feature nested inside is considered to be at *lower rank*. Each of the 15 positions is a unique pairing of one of the 3 positions of the low feature and one of the 5 positions of the high feature. The value at each point is then the product of the two features' values. Where either one is dark, the product is dark; if both are bright, the product is bright. The final result is a shown in figure 3.3c, a *spatial diagram* which shows how the pattern actually looks in physical space.

For some applications beyond the scope of this work, we may also write the state as a single vector which matches the spatial diagram.

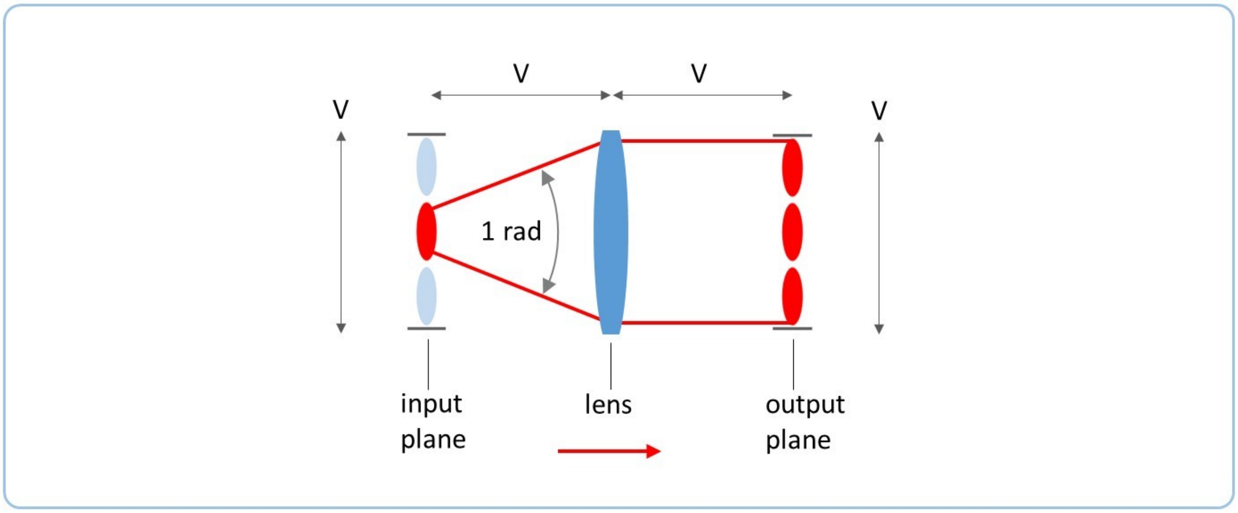
$$|D_0B_0\rangle = \frac{1}{\sqrt{3}} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Note that this is a low-fidelity model, in that it does not match experiment precisely. In FO, the intensity is uniform inside the beam and falls abruptly to zero outside of it. Actual beams typically have a Gaussian (normal distribution) profile, but FO accepts this discrepancy for the sake of simplicity.

3.3 FT from input to output

We next consider the behavior of a single feature in the 2f system, as drawn in figure 3.4. The feature begins at the input plane as a dark feature. It propagates forward in a cone 1 radian in divergence, which is the maximum that may be captured by the lens. The lens approximately collimates the expanded beam, projecting to a bright feature in the output plane.

figure 3.4, the feature in two planes of the 2f system



The overall effect is to perform the discrete FT, which changes the dark feature to a bright feature.

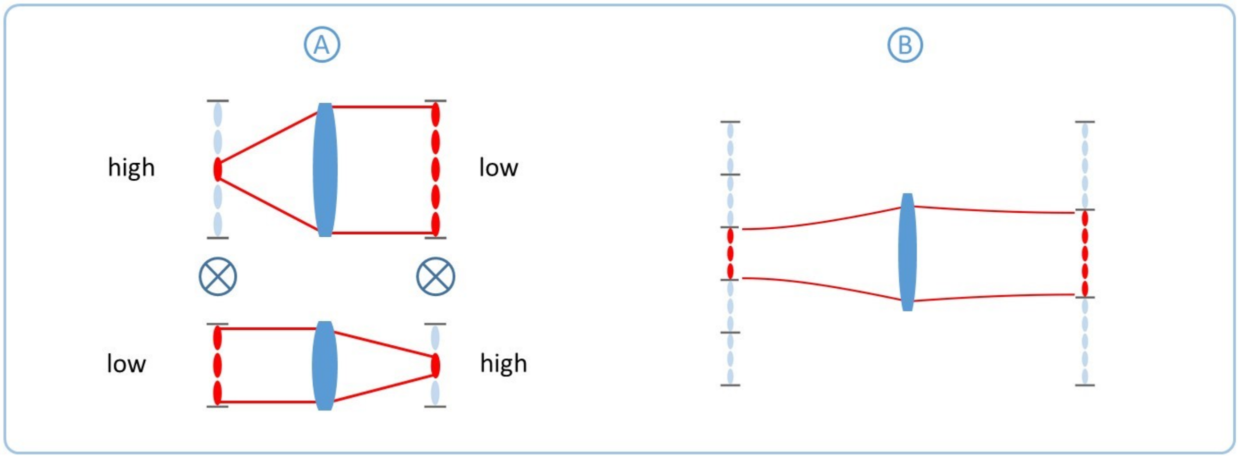
$$\mathrm{FT}\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\frac{1}{\sqrt{3}}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}$$

The system also exists in the mirror-image configuration, i.e. the bright feature is the input and the dark feature is the input. The entire system is simply reversed, including the FT.

$$\mathrm{FT}\left(\frac{1}{\sqrt{3}}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}\right)=\begin{bmatrix}0\\1\\0\end{bmatrix}$$

Returning now to the overall beam in the 2f system (shown once more in figure 3.5b), we see that the overall FT process consists of two parts: Firstly, each individual component feature undergoes the FT as seen in figure 3.5a. Secondly, the nesting rank of the features is reversed – the high feature becomes the low feature, and vice-versa.

figure 3.5, the beam as the product of two features



When representing the beam as an outer product, we actually calculate a 2-d FT which is equivalent to an independent 1-d FT on each of the two features. We express it as

$$\mathrm{FT}\left(\frac{1}{\sqrt{3}}\cdot\begin{bmatrix}0&0&0\\0&0&0\\1&1&1\\0&0&0\\0&0&0\end{bmatrix}\right)=\frac{1}{\sqrt{5}}\cdot\begin{bmatrix}0&1&0\\0&1&0\\0&1&0\\0&1&0\\0&1&0\end{bmatrix}$$

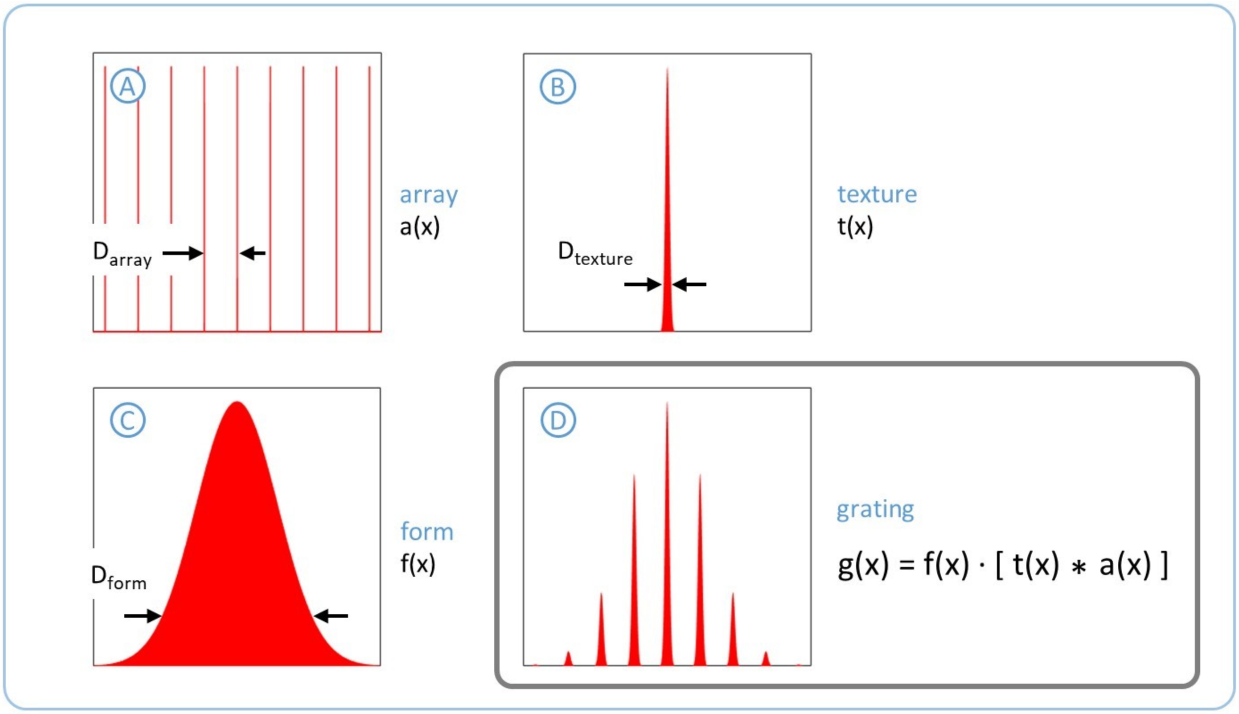
However, this does not account for the reversal of the nesting rank. To complete the FT we must flip our interpretation of the axes of the array, so that the rows refer to the positions of the low feature while the columns refer to the positions of the high feature.

It is puzzling that a 2-d FT takes place on a pattern that is physically arranged along only 1 dimension, by a 2f system which ordinarily performs a simple 1-d FT. After all, the 2f system has no obvious way of 'recognizing' that the state vector is factorable into two components.

4 The Gaussian grating

We now turn from the beam to a second, more complex model system: the Gaussian grating, whose transmitted light has a pattern like the one shown in figure 4.1d. Like the Gaussian beam, this is a conventional model which we will later treat with FO. It is a composite of three separate functions[4,5] which we refer to as the *array*, the *texture*, and the *form*.

figure 4.1, the Gaussian grating as a composite of functions



The Gaussian grating is parameterized by 5 independent numbers:

- $D_{formInput}$, the width of the *form* in the input plane
- $D_{textureInput}$, the width of the *texture* in the input plane
- $D_{arrayInput}$, the period of the *array* in the input plane
- λ, the wavelength of the light
- f, the focal length of the lens (not shown in the figure)

The array a(x) (see figure 4.1a) is the sum of an infinite set of delta functions spaced period D_{array} apart. Each individual delta function is a rectangle of infinitesimally small width and infinitely tall height, which multiply to unit area. The array describes the period and translational symmetry of the pattern, which extends infinitely in either direction.

$$a(x) = \sum_{n=-\infty}^{+\infty} \delta(x - D_{array} \cdot n)$$

The texture t(x) (see figure 4.1b) is a narrow Gaussian beam which is *convolved* with the array, so that a replicate of the texture appears in place of *each* delta function in the array. Physically, it describes the small beams which emerge from the individual slits in a grating screen. It is a single degree of freedom, because all of these small beams are constrained to share a common shape by virtue of the grating's symmetry.

The form f(x) (see figure 4.1c) is a wide Gaussian function which is *multiplied* by the array and texture. The form scales the amplitude and acts like an invisible 'boundary' constraining the periodic function inside it. Without the form to modulate its amplitude, the grating would extend forever along with the infinite array function. In practice, the envelope is often set by the width of the source that illuminates the grating optic.

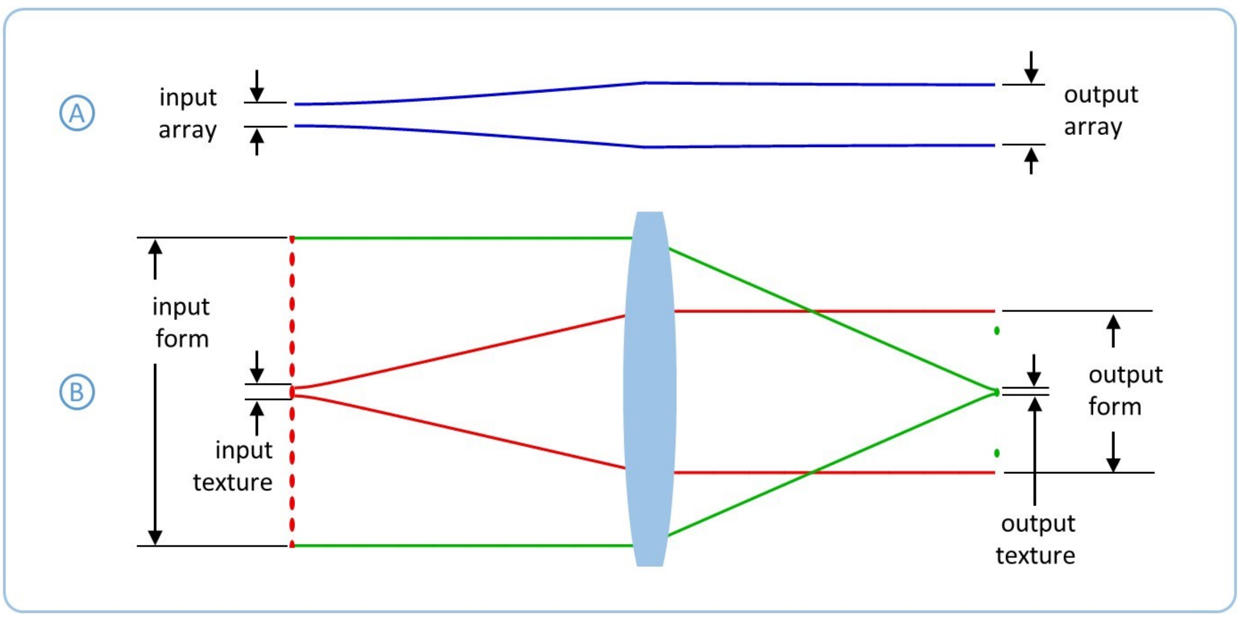
The combination of the three component functions yields the equation of the grating (see figure 4.1d)

$$g(x) = f(x) \cdot [\, t(x) * a(x) \,]$$

When the grating is placed in a 2f system (see figure 4.2b), the output has the same basic array-texture-form structure as the input but with different parameters. A simple set of rules governs the transformation:

- The output form is the FT of the input texture
- The output texture is the FT of the input form
- The output array is the FT of the input array

figure 4.2, the Gaussian grating in two planes of the 2f system



The FTs of the texture and array beams are calculated individually following the formulas used above in the discussion of the Gaussian beam. Note that while the array is not actually a beam, it can be treated like one for the purpose of computing the period in the output plane (see figure 4.2a).

Designating the output grating function as h(x),

$$h(x) = FT(g(x)) = FT(t(x)) \cdot [\, FT(f(x)) * FT(a(x)) \,]$$

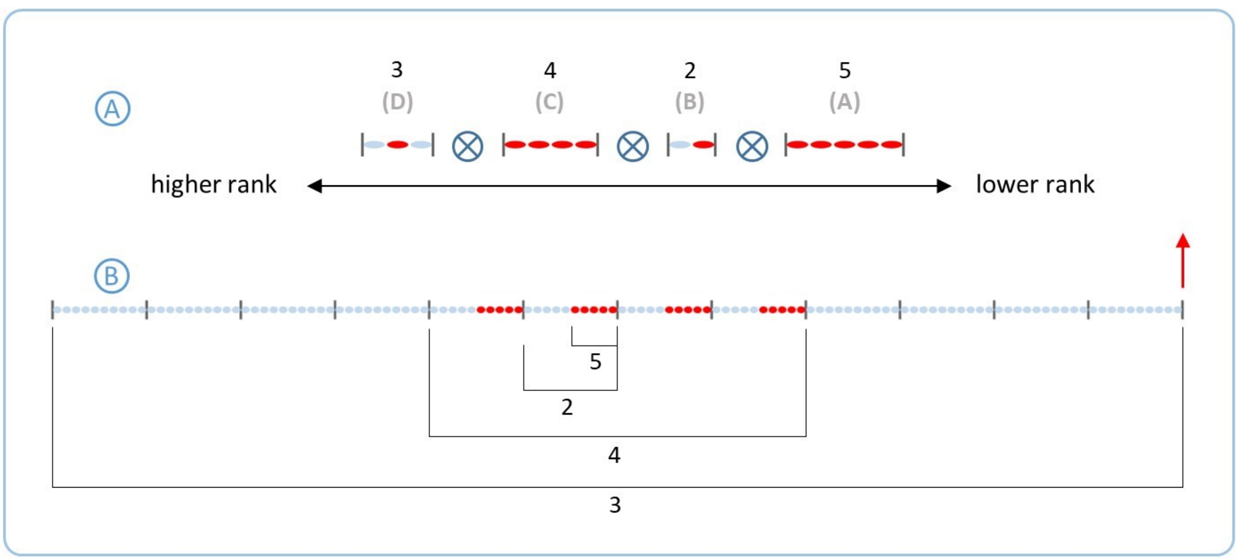
5 The Grating in FO

We describe the grating in FO by extending the principles used earlier for the beam, and re-express its equations in terms of dimensionless parameters. Here, we define all parameters in the input plane:

- A, the number of wavelengths in the input texture
- B, the number of input textures in the input period
- C, the number of input periods in the input form
- D, the number of input forms in the focal length of the lens (take care not to confuse the dimensionless parameter D with the various width parameters such as $D_{formInput}$, which have units of length).
- V, the number of wavelengths in the focal length of the lens

We view the grating pattern as the outer product of 4 different features: a bright feature, nested under a dark feature, nested under a second bright feature, nested under a second dark feature. The feature diagram for such a grating is shown in figure 5.1a, and the corresponding spatial diagram is shown in figure 5.1b (the direction of propagation is upwards).

figure 5.1, the grating in feature optics



When we express the grating transformation rules with the new parameters, they appear as

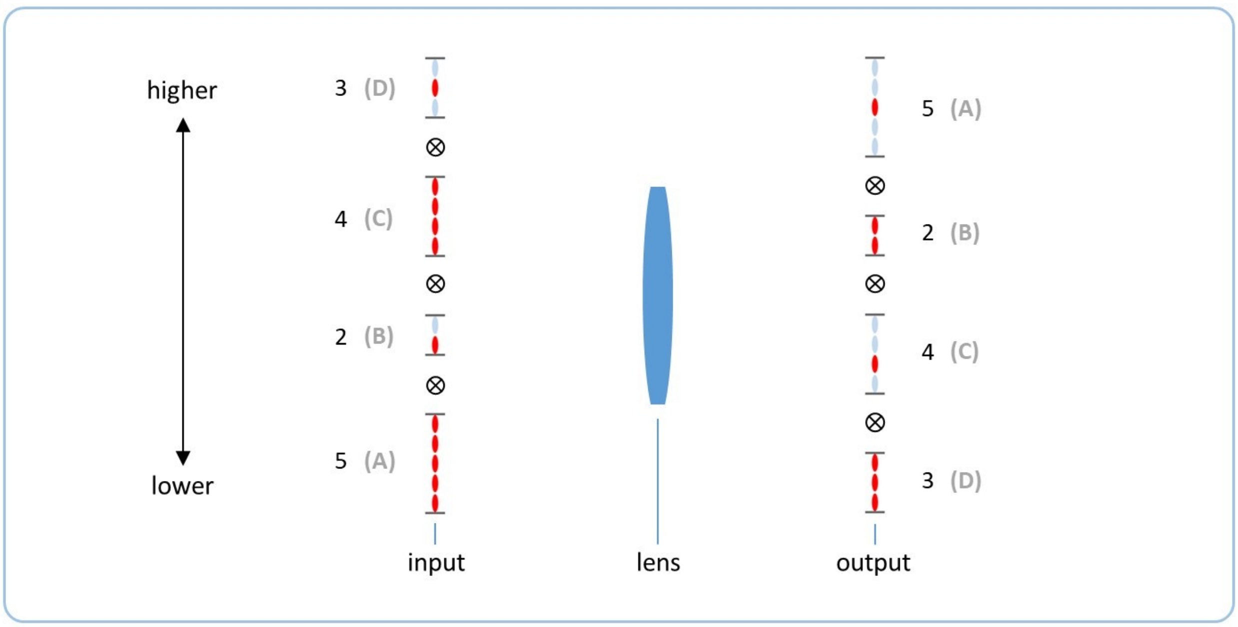
$$A \cdot B \cdot C \cdot (D_{texOut}/\lambda) \; = \; V \cdot \tau \qquad \textit{FT(input form) = output texture}$$

$$A \cdot (D_{formOut}/\lambda) = V \cdot \tau$$ *FT(input texture) = output form*

$$A{\cdot}B \cdot (D_{arrayOut}/\lambda) = V \cdot \tau$$ *FT(input array) = output array*

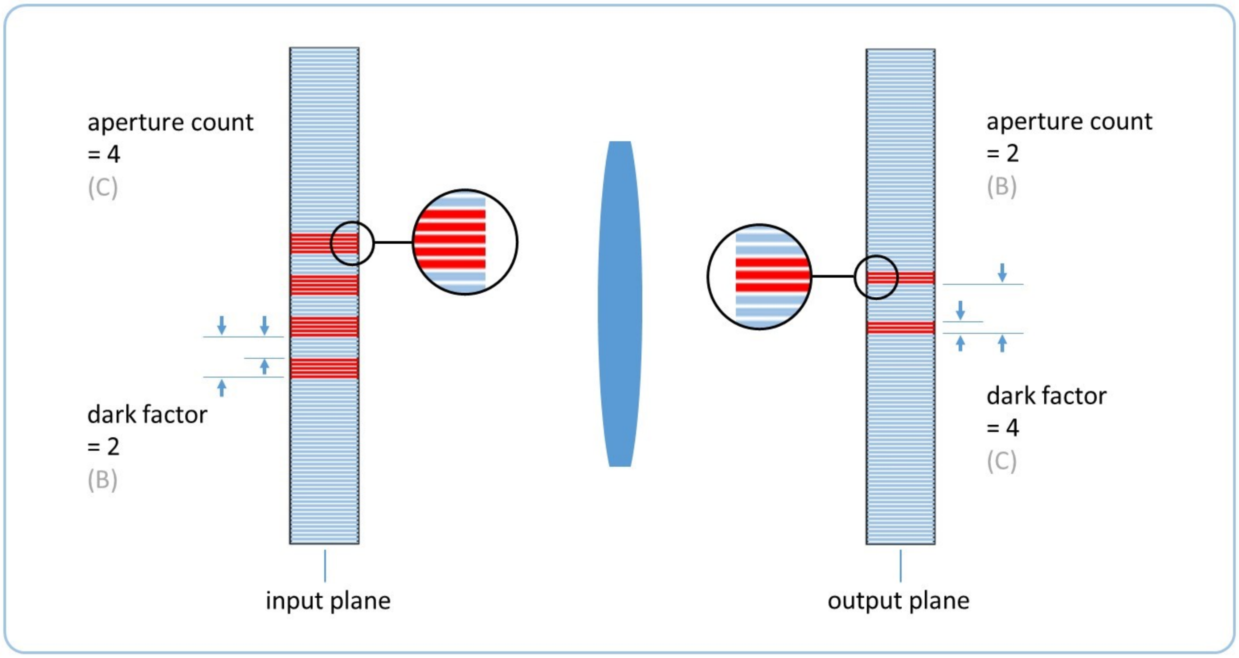
From these we deduce the structure of the output plane, as show in figure 5.2 in terms of a nested chain of features. The grating undergoes the FT following the same principles that apply to the beam: 1. Each individual feature flips from bright to dark or vice-versa, and 2. The nesting rank of the features is reversed. Indeed, these principles can be seen even more clearly in the grating than in the beam, because the grating contains more features and the rules are seen to apply consistently for them all. Again, we approximate by neglecting τ.

figure 5.2, feature diagram of the grating in the 2f system



A spatial diagram of this grating is shown in figure 5.3. We can draw some simple physical interpretations: B is the *dark factor* of the input, i.e. the ratio of total space to bright space within each period. C is the *aperture count* of the input, i.e. the number of times the aperture is repeated in the pattern. We see that under the FT of the grating, these two values exchange physical meaning in the output plane. Note that values A and D can vary arbitrarily with no effect on this rule.

figure 5.3, spatial diagram of the grating in the 2f system



6 Conclusions

FO is a set of practical rules for finding the FT of an input pattern. At present, its scope of applicability is limited to the beam and grating systems which we have analyzed. Curiously, it makes no explicit use of wave optics or Gaussian beam theory, yet it yields the same results.

FO has not been reduced to a set of fundamental axioms; it is a matter for future research to determine whether any such axioms exist. However, the remarkable fact that such simple rules-of-thumb actually work suggests that a deeper explanation may be possible.

7 References

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