# On the role of mean proportionals in the analysis of uniformly accelerated motion

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## Introduction

A simple relation between two quantities a, b is proportionality. Two quantities a and b (say, weight and volume) are said to be directly proportional to each other, if their ratio ($a_i$ / $b_i$) is a constant. Graphical representation of this relation gives a straight line between a and b. These facts are familiar to students even at high school level. However, when it comes to mean proportionals, the familiarity is not high. They are very important in understanding uniformly accelerated motion. We can have more than one mean proportional between two given quantities, a and b. The first mean proportional is the simplest as it is simply the square root of the product of a and b. The more familiar name for the first mean proportional is, 'geometric mean'[1]. The second, third, fourth etc., mean proportionals are not as simple and are not as useful as the first mean proportional. Descartes gives[2] a simple method of obtaining them. He also constructed a simple and elegant instrument that gives those values.

Galileo uses, proportionals and mean proportionals extensively, in his analysis of uniformly accelerated motion[3]. We concern here, mainly with his analysis of motion of two types. One - that on inclined planes[3], and that includes vertical planes (free fall) – and the other - that of projectile motion[4]. In the analysis of both these types of motion, proportionals and mean proportionals play an important role. In particular, mean proportionals are of great significance since they are directly related to relative times of motion. Therefore, it is important to understand clearly, the concept, the definition, their representation and use of mean proportionals to follow Galileo's analysis of uniformly accelerated motion.

We discuss in this article, the above mentioned attributes of mean proportionals through their geometric representation. We also illustrate their use by giving, two examples: one for motion on inclined planes and one for projectile motion. These examples highlight the role played by mean proportionals in appreciating Galileo's conceptual framework of time and motion. We hope this article would be useful to general reader, senior high school, college students, teachers and researchers studying Galileo's works.

## Definition and geometric representation of mean proportionals

If quantities a, b, c are related such that

$$a : b = b : c \tag{1}$$

then, by definition, b is said to be the mean proportional between a and c.
From the above definition, it follows that

$$a \times c = b^2, \qquad and\ b = \sqrt{a \times c} \qquad (2)$$

From equation (2) it is evident that, finding mean proportional involves taking square root of the product of two quantities[1]. It is same as finding the side of a square having equal area as that of a reactance of given sides.

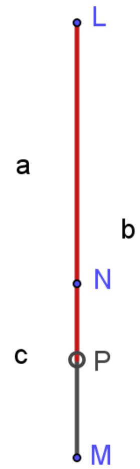


Fig.1. Line segment LM is divided internally in the ratio, a : c = LN : NM by the point N. The red line LP (= b) represents the mean proprtional of a and c.

Finding mean proportionals using geometric method is much simpler than using other methods[1,2]. It also gives a feel for the concept and offers a visual display of the quantity. We give this method below.

Let a line segment LM be divided by a point N lying between L and M (internal division), into two parts in an arbitrarily chosen ratio, a : c (see Fig. 1). Draw a semicircle with LM as diameter.

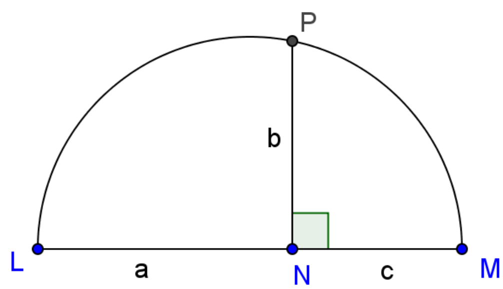


Fig. 2. b gives the mean proportional of the quantities a and c.

Draw a perpendicular NP to LM at N to meet the semicircle at P. NP (= b) gives the square root of the product of a and c. Therefore, b is the mean proportional of a and c.

Proof: Join LP and MP. Right triangles LPN, PMN and LMP are similar. Therefore, their sides are proportional and give,

$$LN : NP = NP : NM, \text{that is}, a : b = b : c \tag{3}$$

$$\therefore a \times c = b^2 \quad and \quad b = \sqrt{a \times c} \tag{4}$$

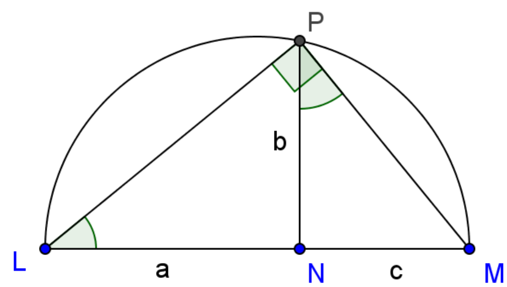


Fig. 3. Triangles LPN, PNM and LPM are similar. The proportionality of their sides leads to the result b = √(a × c).

The corresponding case for external division was recently discussed by the author[7].

## Relative times, distances and mean proportional

To highlight the importance of the mean proportionals we quote from Galileo's 'The two New Sciences'[8], the following.

"… starting from any initial point, if we take any two distances, travelled in any time intervals whatsoever, these time intervals bear to one another the same ratio as one of the distances to the mean proportional of the two distances."

We elucidate the above by using a diagram (Fig. 4). Let LN, LM be two distances measured from the initial point L. Let the mean proportional of LN and LM be LP. Then starting from rest, the time of fall with uniform acceleration through the distance LN is to the time of fall through the distance LM is as LN is to LP. Equally valid is the statement that the time of free fall through the distance LM is to the time of free fall through the distance LN is as LM is to LP.

Since the times of fall and the respective distances are related as $t_{LN} : t_{LP}$ = LN : LM, it becomes very important for us to understand how to locate the position of the point P on LM.

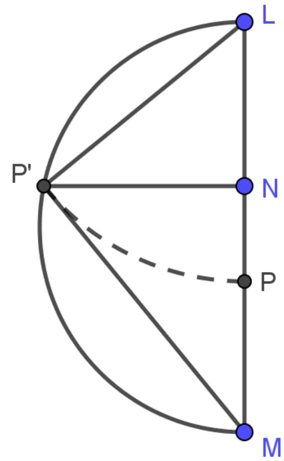


Fig.4. Figure shows the relation between relative distances of free fall and the corresponding relative times.

This task of locating P on LM, that is, obtaining the mean proportional of LN and LM, becomes very easy by the geometrical method given by Galileo[9].

If the body falls through the distance LN in a time $t_{LN}$, then LP gives the time of fall ($t_{LM}$) through the distance LM. That is, the distances of fall and the times of fall are related as $t_{LN} : t_{LM} = LN : LP$. Since this is the crucial concept that is to be understood, we elaborate this further.

If the body falls through the distance LN in unit time, then it falls through the distance LM in LP units of time. On the other hand, if the body falls through the distance LM in unit time, then it falls through the distance LN in LP units of time.

Similarly, if the body falls through the distance MN in unit time, then it falls through the distance ML in MP′ (the mean proportional of MN and ML) units of time. On the other hand, if the body falls through the distance ML in unit time, then it falls through the distance MN in MP′ units of time.

## Applications of mean proportional in Galileo's concept of time and motion

Application 1: Motion on inclined plane. The times of descent along two inclined planes of the same length but different inclinations[5].

From a single point, L, draw inclined planes LK and LJ having the same length but different inclinations (see Fig. 5). Draw the vertical through L and horizontals through K and J. Let the horizontals through K and J meet the vertical at N and M respectively. LN represents the height of the plane LK and LM represents the height of the plane LJ. Let LP be the mean proportional of LN and LM. Then, the ratio of LM to LP is equal to the ratio of LM to LN.

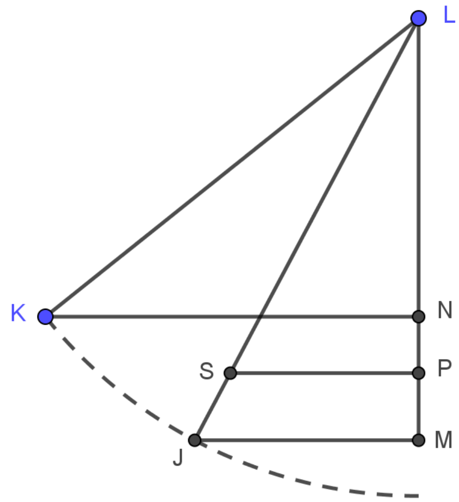


Fig.5. The figure deoicts two inclined plaes, LK and LJ of equal lenghts, their respective heights LN and LM. The times of fall along the planes is equal to the square root of the inverse ratio of their heights.

Now, the proposition[7] says, the ratio of times of descent along LK and LJ is equal to the square root of the inverse ratio of the heights LN to LM, so that the time of descent along LK is related to the height LM of the other plane LJ, as the time of descent along LJ is related to the height LN of the other plane LK. We must prove that the time of descent along LK is to that along LJ as the square root of the inverse ratio of LN to LM.

Proof

Draw PS parallel to MJ. We know that the time of fall along LK is to that along the vertical LN is as the length LK is to the height LN. We also know that time of fall along LN is that along LM is as LN is to LP. Similarly, the time of fall along LM is to that along LJ is as the length LM is to LJ or as LP is to LS. Therefore, it follows that the time along LK is to that along LJ is as LK is to LS or LJ is to LS. However, LJ is to LS is as LM is to LP.

$$t_{LK} : t_{LN} = LK : LN \tag{5}$$

$$t_{LN} : t_{LM} = LN : LP \tag{6}$$

$$\text{from (5) and (6) we get} \quad t_{LK} : t_{LM} = LK : LP \tag{7}$$

$$\text{Similarly,} \quad t_{LM} : t_{LJ} = LM : LJ = LP : LS \tag{8}$$

$$\text{or} \quad LJ : LS = LM : LP \tag{9}$$

From equations (7) and (8) we get

$$t_{LK} : t_{LJ} = LK : LS = LJ : LS \quad (\,since\ LK = LJ) \tag{10}$$

$$from\ (9)\ we\ get \qquad \left(\frac{LJ}{LS}\right) = \left(\frac{LM}{LP}\right) \tag{11}$$

$$But, \qquad \left(\frac{LM}{LP}\right) = \sqrt{\left(\frac{LM}{LN}\right)} \tag{12}$$

$$\therefore \quad From\ (10), (11), (12)\ we\ get \quad t_{LK} : t_{LJ} = \sqrt{LM} : \sqrt{LN} \tag{13}$$

Therefore, it follows that the times of fall along two inclined planes of equal length are proportional to the square root of the inverse ratio of their heights.

Application 2: Projectile motion[6,]

Given a parabola in the vertical plane, find out the point on the axis extended upwards, from which a particle must be dropped, so that when reflected at the apex to move along the horizontal, traces the parabola.

We note that, when the particle on reaching the apex is reflected by an elastic collision with a mirror located at the apex at an angle of $45^0$ to the horizontal. There afterwards, it moves with a constant speed (acquired in falling to the apex) along the horizontal and uniformly accelerated

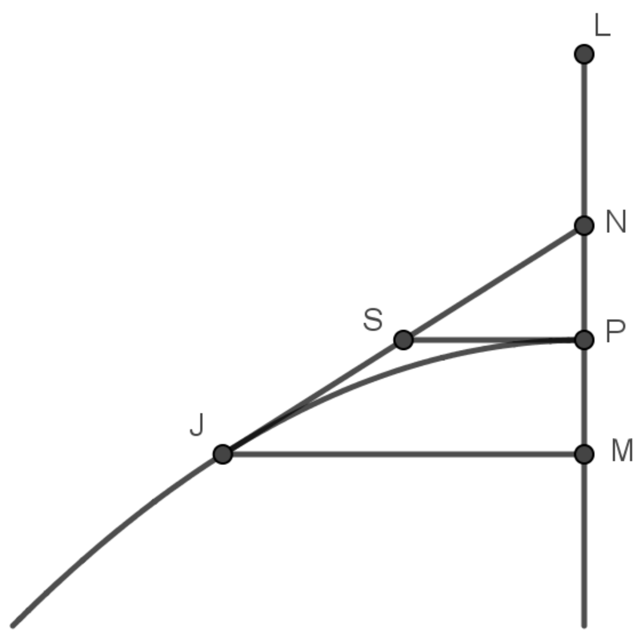


Fig. 6. PJ is the given semi parabola. The vertical through P is its axis. The problem is to locate L, the point from which a particle dropped and reflected at P to move along the horizontal traces the parabola.

along the vertical. The composition of these two independent motions gives the parabolic motion.

Proof

Let PJ be the given semi parabola in the vertical plane (see Fig. 6). Let JM be its amplitude and LM its axis extended. The problem is to find the location of the point L, from which the particle is to be dropped to trace the given parabola.

Draw the horizontal through the apex P. Mark N such that NP is equal to PM. Join NJ. It is also the tangent to the parabola at J. Let it cut the horizontal through P at S. Locate L such that PS is the mean proportional between LN and NM. L gives the point we are seeking.

We give a geometric method that gives the location of L.

## Geometric Method to obtain the point of start from rest

Semi parabola through PJ is given (see Fig. 7). Draw the horizontal PS and the vertical MP (axis) through P. Draw the horizontal JM and, the tangent at J, to the parabola. Let the horizontal through P and the tangent intersect at S. Join SM. Draw perpendicular to SM at S. Let it intersect the axis extended upwards, at L. L is the point that we are seeking. Draw a semi-circle through MSL. We can immediately recognize that PS is the mean proportional between MP and PL by comparing this figure to Fig. 2, confirming the result.

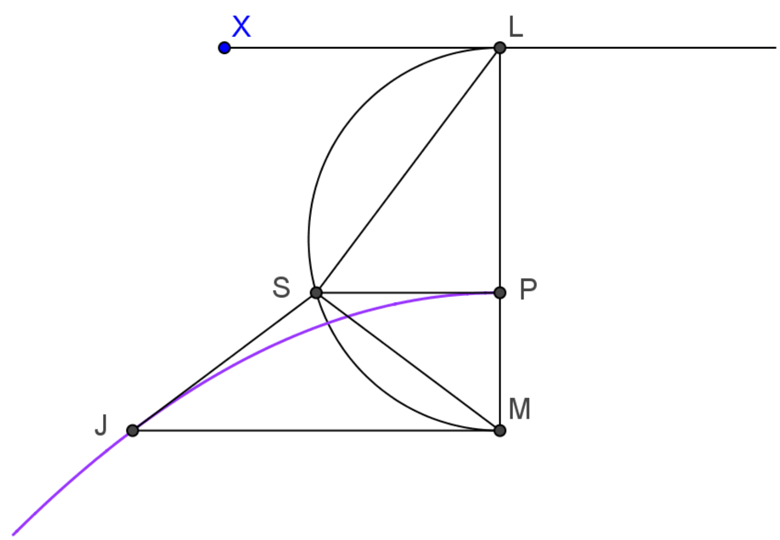


Fig. 7. PJ is the given parabola in the vertical plane. The vertical through P is the axis. The problem is to locate the point L on the axis.

Before ending the discussion, we merely quote, but not discuss (since it does not directly involve time and motion) another very interesting result involving mean proportional[10]. It is this:

The area of the circle is a mean proportional between any two regular and similar polygons of which one circumscribes it and the other is isoperimetric with it (the circle). In addition, the area of the circle is less than that of any circumscribed polygon and greater than that of any isoperimetric polygon. And further, of these circumscribed polygons, the one which has the greater number of sides is smaller than the one which has a less number of sides, but on the other hand, that isoperimetric polygon which has the greater number of sides is the larger.

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## References

[1]. C Stanley Ogilvy, ‘Excursions in Geometry’, Dover Pub., N. Y. (1990) p. 10.
[2]. Rene Descartes, ‘ The Geometry of Rene Descartes’, Tr. by David E Smith and Marcia L. Latham, D, Dover Pub., N. Y. (1954) p. 155.
[3]. Galileo Galili, The Two New Sciences, ‘Dialogues concerning Two New Sciences’, Third Day, Tr. by Henry Crew and Alfonso De Salvio The Macmillan Co., N. Y. (1914).
[4]. Ibid, Fourth Day.
[5]. Ref. 3, page 187.
[6]. Ref. 3, page 272.
[7]. Radhakrishnamurty, P., ‘The relation between mean proportionals of the parts and the whole of a line segment’, submitted to Journal of Geometry for publication.
[8]. Ref. 3, page 180.
[9]. Ref. 3, page
[10]. Ref. 3, First day, page 57.