Calculating the Hawking temperature in a variant of the near horizon metric

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A modified version of the near horizon metric is introduced, that puts the near horizon metric in the same form as one of the most commonly-used metric variants of Rindler space. The metric is then used to calculate the Hawking temperature, using the WKB tunneling approximation.

INTRODUCTION

The emission of thermal radiation by black holes is a combined prediction of quantum field theory and general relativity. The idea that black holes emit photons, and in the process, slowly evaporate, eventually out of existence, was first introduced to the physics community by Stephen Hawking in 1975 [1]. The temperature of the black hole's radiation, which by definition is measured by an observer located infinitely far away from the black hole, has been termed the Hawking temperature T_H.

The Hawking temperature of an astronomical black hole is very cold.[1] This is because the radiation has to undergo an enormous redshift on its climb out of the black hole's gravitational potential well. More massive black holes have stronger gravitational fields, which causes the emitted radiation to lose more energy on its way to infinity, where the observer measures a correspondingly lower temperature.[2]

Soon after its introduction, the phenomenon of Hawking radiation was extended to related concepts, such as Unruh radiation in 1976 [2], seen by a uniformly accelerating observer in Rindler space; and the radiation emitted by an observer's cosmological horizon, in 1977 [3].

Hawking's original approach has also been extended to different methods of calculation [4–10], and to different black hole geometries, such as for charged and rotating black holes [11, 12]. Of great relevance to this paper, a tunneling model of Hawking radiation, introduced by Parikh and Wilczek in 1999, replaces the original calculation's use of quantum field theory with the WKB approximation of particle tunneling in quantum mechanics [4]. This method can be used to give a very quick and straight-forward calculation of the Hawking temperature.

The Hawking temperature of a Schwarzschild black hole has been calculated in many different metrics of the Schwarzschild geometry, including the near horizon metric in [9]. In this paper, we derive a modified version of the near horizon metric, that puts it in a form that is similar to a specific Rindler metric variant. The new metric is then used to calculate the Hawking temperature. The primary focus of this paper can be divided into two parts. Part 1 presents the derivation of a metric variant of the near horizon approximation. In Part 2, the metric is used to calculate the Hawking temperature, to show that it works.

This paper uses $c = G = \hbar = k_B = 1$.

PART 1: METRIC VARIANT DERIVATION

Motivating the metric variant's derivation

The local geometry of a Schwarzaschild black hole looks a lot like Rindler space. One way to see this is by comparing some of their metrics. Two commonly appearing forms of the Rindler metric are

$$ds^2 = -\left(ax_R\right)^2 dt_R^2 + dx_R^2 + dy_R^2 + dz_R^2 \tag{1a}$$
$$ds^2 = -\left(1 + ax_R\right)^2 dt_R^2 + dx_R^2 + dy_R^2 + dz_R^2 \tag{1b}$$

where a represents the Rindler observer's coordinate acceleration.[3] Subscript R is used to distinguish the

[1] $T_H \approx 6.1 \times 10^{-8}$ K for a hypothetical one solar mass black hole.

[2] Even though it's typical to talk about black holes being very cold, if you were to get really close to a black hole's event horizon, it would feel very hot, since the photons have not yet traveled through an extremely strong gravitational potential well. A better discussion on this can be found in [9].

[3] The coordinate acceleration, in contrast to proper acceleration, is the Rindler observer's acceleration as it is measured by an observer in the coordinates of the underlying Minkowski space.

coordinates of the Rindler observer's tetrad from coordinates of the underlying Minkowski space. The near horizon metric from [9], looks a lot like the first of the two Rindler metrics (1a)

$$ds^2 = -\left(\kappa r_{NH}\right)^2 dt^2 + dr_{NH}^2 + dY^2 + dZ^2 \tag{2}$$

where κ is the black hole's surface gravity, r_{NH} represents proper distance of a spacetime event above the event horizon, and Y, Z are the local coordinates of a Cartesian plane, tangent to the event horizon.

The fact that (2) is symmetric to (1a) motivates the question as to whether or not there is a coordinate transformation that can turn (2) into an equation that looks like (1b). As it turns out, this can be done, but is not as straight-forward as the Rindler space transformation that is used to turn (1a) into (1b). The reason for this has to do with the fact that the underlying geometry in Rindler space is flat Minkowski space, instead of the curved Schwawrzschild spacetime. A Rindler observer accelerates through flat spacetime, while a near horizon observer accelerates through curved spacetime.

Derivation of the original near horizon approximation

It makes sense to start with a derivation of the original near horizon metric, from the Schwarzschild metric, because it will be useful to refer back to some of the equations that come up later on. This derivation is based primarily on [9]. An alternative approach can be found in [13].

The near horizon metric's derivation starts from the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega^2 \tag{3}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$,

The near horizon radial coordinate r_{NH} represents proper distance of a spacetime event, above the event horizon, with $r_{NH} = 0$ corresponding to $r = 2M$, and is calculated as $dr_{NH} = \sqrt{g_{rr}}dr$

$$dr_{NH} \equiv \frac{dr}{\sqrt{1 - \frac{2M}{r}}} \tag{4}$$

Using this to replace the Schwarzschild metric's radial component

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + dr_{NH}{}^2 + r^2 d\Omega^2 \tag{5}$$

Next, dr_{NH} needs to be turned into a finite distance before it can be used to replace r in the metric's time component. Integrating (4) to a finite distance

$$\begin{aligned} r_{NH}(r) &= \int_0^r \frac{dr'}{\sqrt{1 - \frac{2M}{r'}}} \\ &= 2M \sinh^{-1}\left(\sqrt{\tfrac{r}{2M} - 1}\right) + \sqrt{r(r - 2M)} \end{aligned} \tag{6}$$

This integral can be calculated using a change of variables $r' = 2M \sec^2 t$, and is shown in appendix Section A 1.

The inverse of (6) is needed to rewrite the time component explicitly in terms of r_{NH}. This is done using a separate approximation for each term on the right hand side of (6).

First, the argument of $\sinh^{-1}\left(\sqrt{r/2M - 1}\right)$ goes to zero in the limit $r \to 2M$, so it can be approximated with a first order Taylor expansion of $\sinh^{-1}\left(\sqrt{r/2M - 1}\right)$ at $r = 2M$. Using

$$\sinh^{-1} x = x - O\left(x^3\right) \tag{7}$$

to approximate the first term on the right hand side of (6) at $r = 2M$

$$2M \sinh^{-1}\left(\sqrt{\tfrac{r}{2M} - 1}\right) \approx \sqrt{2M(r - 2M)} \tag{8}$$

Second, Taylor expanding the argument under the square root, in the second term on the right hand side of (6), at $r = 2M$ gives

$$\sqrt{r(r - 2M)} \approx \sqrt{2M(r - 2M)} \tag{9}$$

Adding (8) and (9)

$$r_{NH}(r) = 2\sqrt{2M(r - 2M)} \tag{10}$$

whose inverse is given by

$$r(r_{NH}) = \tfrac{r_{NH}{}^2}{8M} + 2M \tag{11}$$

This can be used to rewrite the metric's time-component explicitly in terms of r_{NH}

$$-\left(1-\frac{2M}{r\left(r_{NH}\right)}\right)dt^2=-\frac{1}{(4M)^2}r_{NH}^2\left(\frac{1}{\left(\frac{r_{NH}}{4M}\right)^2+1}\right)dt^2$$
$$\approx-\frac{1}{(4M)^2}r_{NH}^2dt^2 \qquad (12)$$

Using this to replace the metric's time component in (5)

$$ds^2=-\frac{1}{(4M)^2}{r_{NH}}^2dt^2+d{r_{NH}}^2+r^2d\Omega^2 \qquad (13)$$

The near horizon time coordinate is usually defined by rescaling Schwarzschild time $t\to t/4M$. However, it will make sense here to leave the Schwarzschild time coordinate unchanged, because it makes the symmetry between the near horizon and Rindler metrics easier to see.

Normally the angular component is approximated as a disc situated on top of the event horizon, by setting $r=2M$ and $\sin\theta\approx\theta$ for small deviations from $\theta=0$

$$r^2d\Omega^2\approx(2M)^2\left(d\theta^2+\theta^2d\phi^2\right)$$
$$=dx^2+dy^2 \qquad (14)$$

In this limit, angular components can also be expressed as a Cartesian plane, tangent to the event horizon. That being said, we will not need the angular component for the remainder of this paper. Taking just the near horizon metric's time-radial component

$$ds^2=-\frac{1}{(4M)^2}{r_{NH}}^2dt^2+d{r_{NH}}^2 \qquad (15)$$

THE METRIC VARIANT

The surface gravity of a Schwarzschild black hole is $\kappa=1/4M$, and can intuitively be thought of as a redshifted-to-infinity measure of the proper acceleration that would be required for an observer to remain radially at rest at the event horizon. A derivation of κ that demonstrates this interpretation is given in the appendix B 2. Replacing $1/4M$ in (15) with κ puts the metric in a form that is symmetric to the Rindler metric's relevant $(1+1)$ component in (1a)

$$ds^2=-\kappa^2{r_{NH}}^2dt^2+d{r_{NH}}^2 \qquad (16)$$

a^{-1} equals proper distance to the accelerating observer's Rindler horizon

The coordinates of the Rindler metric in (1a) are related to the coordinates of an underlying Minkowski space by

$$t\left(t_R,x_R\right)=x_R\sinh at_R \qquad (17a)$$
$$x\left(t_R,x_R\right)=x_R\cosh at_R \qquad (17b)$$

Showing that a^{-1} equals the proper distance to an accelerating observer's Rindler horizon, is done here to set up a situation where a similar insight can be deduced for our alteration of the original near horizon metric, and can be shown in two steps.

First, the accelerating observer's Rindler horizon is located at $x_R=0$. Physically, (17a) tells us that time freezes for events located at $x_R=0$, and only at $x_R=0$, indicating $x_R=0$ as the Rindler horizon. Mathematically, the metric is guaranteed to be singular for arbitrary values of t_R, only when $x_R=0$. Since $x_R=0$ is the only spatial location where the metric is guaranteed to be singular, it must be the location of the accelerating observer's Rindler horizon. A similar argument is made in [10].

Second, it can be shown using a very long derivation, which starts from first principles, that the parameterized worldline of the Rindler observer with acceleration a, is given by [14]

$$t\left(t_R\right)=a^{-1}\sinh at_R \qquad (18a)$$
$$x\left(t_R\right)=a^{-1}\cosh at_R \qquad (18b)$$

Comparing this to (17) we can see that $x_R=a^{-1}$ is constant for a Rindler observer traveling along their worldline. Since we found that the accelerating observer's horizon is located at $x_R=0$ in the first step, and found that the Rindler observer is located at $x_R=a^{-1}$ as they travel along their worldline in the second step, it follows that the Rindler observer is located a proper distance $x_R=a^{-1}$ away from their Rindler horizon.

Proper acceleration equals inverse proper distance to the event horizon of a Schwarzschild black hole, in the limit $r\to 2M$

The near horizon metric in (16) has the same kind of relation to Minkowski coordinates as is shown for the Rindler coordinates in (17)

$$T\left(t, r_{NH}\right) = r_{NH} \sinh \kappa t \tag{19a}$$

$$X\left(t, r_{NH}\right) = r_{NH} \cosh \kappa t \tag{19b}$$

which can be shown by putting T and X into a Minkowski space line element

$$-dT^2 + dX^2 = -\kappa^2 {r_{NH}}^2 dt^2 + d{r_{NH}}^2 \tag{20}$$

The origin of this local Minkowski space is centered on the event horizon, since $r_{NH} = 0$ corresponds to $r = 2M$.

This also, should be expected, given what was just shown for the case of an accelerating observer in Rindler space.

Given these facts, along with the symmetry between (16) and (1a), it's tempting to think that r_{NH} should be equal to κ^{-1} in the limit $r \to 2M$ (since the near horizon approximation becomes exact in that limit). However, it turns out that $r_{NH} = a^{-1}$, not κ^{-1}, in the limit $r \to 2M$.

It's shown in the appendix Section B 1, that in order for an observer to remain radially at rest with respect to the global Schwarzschild geometry, the observer must maintain a constant proper acceleration

$$a\left(r\right) = \frac{M}{r^2\sqrt{1-\frac{2M}{r}}} \tag{21}$$

which exactly offsets the black hole's downward gravitational pull.[4] This can be inverted and rearranged, to write the inverse proper acceleration

$$a^{-1}\left(r\right) = \tfrac{r}{M}\sqrt{r\left(r-2M\right)} \tag{22}$$

Taylor expanding the argument under the square root to first order at $r = 2M$

$$a^{-1}\left(r\right) \approx \tfrac{r}{M}\sqrt{2M\left(r-2M\right)} \tag{23}$$

In the limit $r \to 2M$, the outside factor of $\frac{r}{M}$ goes to 2 without any problems, so we can replace $\frac{r}{M} \to 2$, which gives

$$\begin{aligned}\lim_{r\to 2M} a^{-1}\left(r\right) &= 2\sqrt{2M\left(r-2M\right)} \\ &= r_{NH}\end{aligned} \tag{24}$$

with r_{NH} as it is given in (10).

We can use this result to replace $r_{NH} \to a^{-1}$ in (19a) and (19b)

$$T\left(t, r\right) = a^{-1}\left(r\right) \sinh \kappa t \tag{25a}$$

$$X\left(t, r\right) = a^{-1}\left(r\right) \cosh \kappa t \tag{25b}$$

However, an additional step is needed, beyond what would be done for a Rindler space version of this calculation, since (25) is not symmetric to (18). Specifically, we need to rescale the proper length in the local tangent space at $r = 2M$, to proper length at infinity, which will make (25) symmetric to (18), by rescaling $a^{-1}\left(r\right) \to \kappa^{-1}$.

Rescaling the local coordinates to measure proper distance at infinity

The interpretation of surface gravity mentioned earlier, suggests a rescalling of (19) by a redshift factor

$$f(r) \equiv \frac{ds}{dr} = \frac{1}{\sqrt{1-\frac{2M}{r}}} \tag{26}$$

where r represents the location from where the radiation is emitted, which in this case is $r = 2M$. In the limit $r \to 2M$ we find that $f\left(r\right) \to \infty$ and $r_{NH}\left(r\right) \to 0$, but their product remains finite

$$f(r) r_{NH}(r) = 2\sqrt{2Mr} \tag{27}$$

which at $r = 2M$ becomes

$$f(2M) r_{NH}(2M) = \kappa^{-1} \tag{28}$$

Both T and X in (19) are proportional to r_{NH}, so we can easily rescale them by multiplying (19) by the redshift factor $f\left(r\right)$, and then evaluate the new coordinates at $r = 2M$ to get

$$T_{\infty}\left(t\right) = \kappa^{-1} \sinh \kappa t \tag{29a}$$

$$X_{\infty}\left(t\right) = \kappa^{-1} \cosh \kappa t \tag{29b}$$

which is symmetric to (18).

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[4] We've run into an overuse of notation with $a, a(r)$ being used to represent two different accelerations. From here on out, a refers to the coordinate acceleration of a Rindler observer, and $a\left(r\right)$ indicates the proper acceleration of a near horizon observer.

T_∞, X_∞ have a flat metric that measures proper distance at infinity

To show that T_∞ and X_∞ have a flat metric that measures proper distance at infinity, we must first replace κ^{-1} in (29), by the unevaluated form of (27)

$$T_\infty = 2\sqrt{2Mr}\sinh\kappa t\big|_{r=2M} \tag{30a}$$
$$X_\infty = 2\sqrt{2Mr}\cosh\kappa t\big|_{r=2M} \tag{30b}$$

Taking coordinate differentials, evaluating them at $r = 2M$, and then using $\kappa = 1/4M$ to make simplifying cancellations

$$dT_\infty = \sinh(\kappa t)\, dr + \cosh(\kappa t)\, dt \tag{31a}$$
$$dX_\infty = \cosh(\kappa t)\, dr + \sinh(\kappa t)\, dt \tag{31b}$$

Putting this into a Minkowski line element shows that they measure proper distance at infinity

$$\begin{aligned} -dT_\infty{}^2 + dX_\infty{}^2 &= -dt^2 + dr^2 \\ &= ds_\infty{}^2 \end{aligned} \tag{32}$$

Transforming T_∞ and X_∞ through an arbitrary spatial translation ρ, to give them spatial dependence in the rescaled local tangent space at $r = 2M$

The metric's flatness is important for this section, because it means that a calculation of the Hawking temperature (and any other scalar quantity) that is started from these coordinates, is invariant under Poincaré transformation. This means we can add spatial-dependence to (29), by Poincaré transforming T_∞, X_∞ through an arbitrary spatial translation ρ.

Starting from a spacelike translation in Minkowski space

The spatial translation on T_∞, X_∞ can be set up by starting from the more simple case of a Poincaré transformation in Minkowski space. This can then be extended to the current situation through some minor adjustments. A Lorentz transformation through one space and one time dimension is given by

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} \tag{33}$$

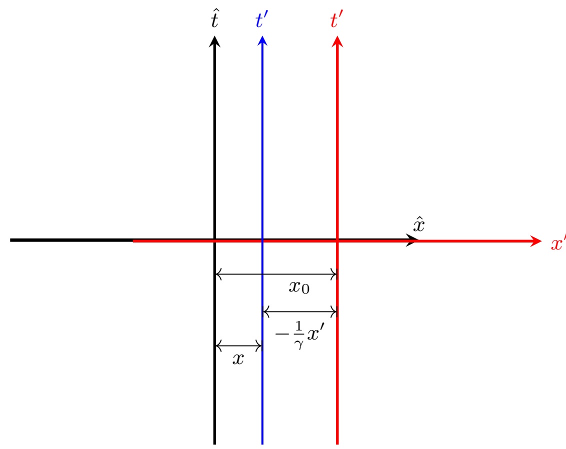

FIG. 1: A Poincaré transformation, with a spatial translation being made in the negative x'-direction ($(1/\gamma)\, x' \to -(1/\gamma)\, x'$ in (34)). The coordinate axes of the rest frame S are labeled with hatted S-frame coordinates $\hat{t}, \hat{x}$, since the un-hatted x is being used to represent the distance of the event in S' from the origin of S.

We can rewrite the spatial component as an expression for $x(t)$

$$x(t) = x_0(t) + \frac{1}{\gamma}x' \tag{34}$$

where $x_0(t){=}vt$ represents the not-Lorentz-contracted distance between the origins of the two inertial frames, and is the one-dimensional version of what we already have in (29).

$(1/\gamma)\, x'$ in (34) represents the distance of an event, with respect to the origin of the moving frame S', as seen by an observer in the coordinates of S. Although (29) lacks a corresponding term, we know that it is possible to add one, since spatial translations are a symmetry of special relativity. Doing this for the current situation is more complicated than for the one-dimensional situation shown in Figure 1, because it requires us to keep track of components for two dimensions. In the current situation, the factor of $(1/\gamma)\, x'$ will be replaced by an inverse Lorentz transformation.

The situation described by (34) is depicted in Figure 1, except with the figure being shown for a translation in the negative x'-direction, $(1/\gamma)\, x' \to -(1/\gamma)\, x'$ in (34), since that will be the direction to the event horizon in the current situation.

The analogous situation that takes place in the presence of a Schwarzschild black hole

In the current situation, S represents an inertial frame whose axes instantaneously coincide with the axes of the near horizon observer's tetrad at $r = 2M$. S is situated on the event horizon, but since S is inertial, at the next instant in time, it will have fallen into the black hole, and its axes will no longer coincide with the axes of a fixed local tangent space at $r = 2M$.

The accelerating frame S'' is noninertial, and is located a very small distance above the event horizon. Locally, from the perspective of the inertial observer in S, the noninertial frame S'' appears to be moving through space, as it accelerates away from the observer in S, while in fact S'', is actually at rest relative to the global Schwarzschild geometry. The proper acceleration of S'' exactly offsets the acceleration due to gravity at its location.

S' is an inertial infalling frame, like S, whose axes instantaneously coincide with the axes of S''. From the perspective of an observer in S, the other inertial frame S' appears to be instantaneously comoving with S'', along the parameterized worldline that is defined by (29).

Assuming instantaneously comoving frames in place of a noninertial frame

The Poincaré transformation for the current situation takes the same basic form as (34), but is made more complicated by the fact that the accelerating frame S'' is noninertial. This is a problem because Lorentz transformations are technically only defined for coordinate transformations between inertial frames, neither frame can be noninertial. This means that there technically is no Lorentz transformation that takes events in S'', into the coordinates of S for all instances in time.

The way around this problem is to assume that we are planning to define a completely new inverse Lorentz transformation for each new instant in time. Practically-speaking, this just means that our inverse Lorentz transformation now has time-dependence $\Lambda^{\nu}{}_{\alpha'} = \Lambda^{\nu}{}_{\alpha'}(t)$. This time-dependence is due to the acceleration of the noninertial frame causing the velocity parameter in (33) to become time-dependent.

We can calculate the velocity $v(t) = (dX_{\infty}/dT_{\infty})(t)$ by taking coordinate differentials of (29), which gives

$$v(t) = \tanh \kappa t \tag{35}$$

And $\gamma = \left(1 - v^2\right)^{-1/2}$ gives

$$\gamma(t) = \cosh \kappa t \tag{36}$$

which comes from using the trig identity $\operatorname{sech}^2 \kappa t = 1 - \tanh^2 \kappa t$. An inverse Lorentz transformation $\Lambda^{\nu}{}_{\alpha'}$ is defined by replacing $v \to -v$ in (33), to represent the fact that we are measuring the velocity of S with respect to S'. After making this replacement, we can insert (35) and (36) to get

$$\Lambda^{\nu}{}_{\mu'}(t) = \begin{pmatrix} \cosh \kappa t & \sinh \kappa t \\ \sinh \kappa t & \cosh \kappa t \end{pmatrix} \tag{37}$$

Spatial translation in the rescaled local tangent space at the event horizon

The acceleration of S'' further complicates things in another way. It is easy to see in Figure 2, that as time increases, the acceleration of S'' causes its worldline to increase in distance from the origin of S, along *both* axes. This is in contrast to what is seen in Figure 1, where the distance between S and S' changes only along one dimension (the time-axes of S and S' remain parallel in Figure 1, but do not remain parallel in Figure 2). The fact that the current situation requires us to keep track of the increase in separation between S and S' along two dimensions instead of one, means that (34) must be re-expressed as a position 4-vector equation in the rescaled local tangent space at $r = 2M$

$$\boldsymbol{x}(t, \rho) = \boldsymbol{x}_0(t) + \boldsymbol{x}'(t, \rho) \tag{38}$$

where ρ is a spacelike coordinate of S' that measures distance along the direction of acceleration of S''.

The objective is to find the 4-vector components of this equation in the basis vectors $\boldsymbol{e}_{\nu}$ of frame S. The first term on the right hand side of (38) is a time-dependent position vector, representing the worldline of S' as seen from the origin of S. Since the basis vectors $\boldsymbol{e}_{\nu}$ of S are at rest in their own frame, the time-dependence of $\boldsymbol{x}_0(t)$ is entirely attributed to the time dependence of its coordinates $x_0{}^{\nu} = x_0{}^{\nu}(t)$ when expressed in the $\boldsymbol{e}_{\nu}$ basis

$$\boldsymbol{x}_0(t) = x_0{}^{\nu}(t)\, \boldsymbol{e}_{\nu} \tag{39}$$

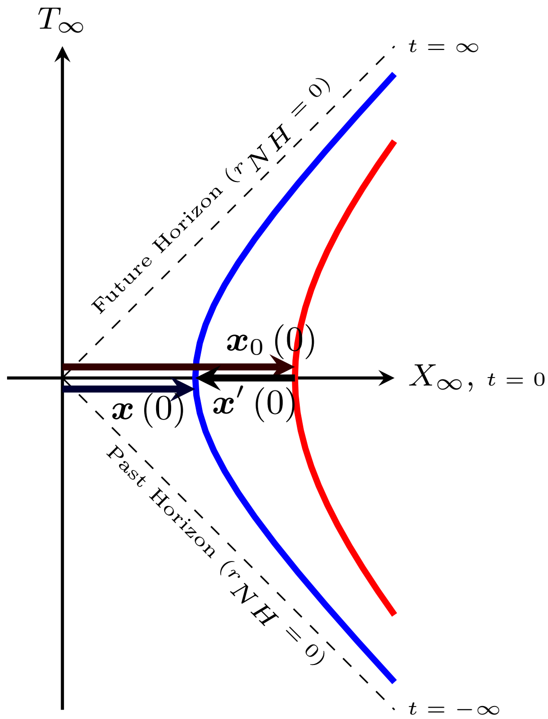

FIG. 2: The red, and blue hyperbolas, represent the worldlines of the accelerating frame S'' (and comoving frame S'), and the event in S' that is a spacelike translation $\boldsymbol{x}'$ (depicted in the negative direction, since that is the direction of the event horizon) with respect to the origin of the comoving frame.

whose first and second components are $T_\infty\,(t)$, $X_\infty\,(t)$. From (29), we can write

$$x_0{}^{\nu}\,(t) = \left(\kappa^{-1}\sinh\kappa t,\;\; \kappa^{-1}\cosh\kappa t\right) \tag{40}$$

which correspond to the position vector that traces out the worldline that is depicted as a red hyperbola in Figure 2.

The second term $\boldsymbol{x}'\,(t,\rho)$ on the right hand side of (38), represents a separation vector in that local tangent space, extending from the origin of S', to an arbitrary event in that local tangent space. The event differs from the origin of S' by a spatial translation ρ, when expressed in the coordinates $x'^{\mu'}$ of S'. So

$$x'^{\mu'} = \left(0,\;\; \rho\right) \tag{41}$$

where ρ is a spatial coordinate defined on the interval $[-\kappa^{-1}, \infty)$, and can be viewed as a gravitationally-redshifted-to-infinity rescaling of r_{NH}, and therefore reflects measurements on r_{NH}, made by an observer at infinity.

The condition that $\boldsymbol{x}'\,(t,\rho)$ correspond to a purely spacelike translation when expressed in the coordinates of S', means that the time-dependence of $\boldsymbol{x}'\,(t,\rho)$ in (38) is entirely attributed to its basis vectors

$$\boldsymbol{x}'\,(t,\rho) = x'^{\mu'}\,(\rho)\,\boldsymbol{e}_{\mu'}\,(t) \tag{42}$$

The basis vectors of S' are related to the basis vectors of S by the inverse Lorentz transformation in (37)

$$\boldsymbol{e}_{\mu'}\,(t) = \Lambda^{\nu}{}_{\mu'}\,(t)\,\boldsymbol{e}_{\nu} \tag{43}$$

Putting this into (42) allows us to rewrite (38) entirely in the basis vectors $\boldsymbol{e}_\nu$ of S, which has components

$$x^{\nu}\,(t,\rho) = x_0{}^{\nu}\,(t) + x'^{\mu'}\,(\rho)\,\Lambda^{\nu}{}_{\mu'}\,(t) \tag{44}$$

Calculating the second term on the right hand side of (44) from (41) and (37), and then adding it to (40), which contributes the components for the first term on the right-hand side of (44), gives the new coordinate relations

$$T_\infty(t,\rho) = \left(\kappa^{-1} + \rho\right)\sinh\kappa t \tag{45a}$$
$$X_\infty\,(t,\rho) = \left(\kappa^{-1} + \rho\right)\cosh\kappa t \tag{45b}$$

As a consistency check, it's easy to see that we get (29) back when $\rho = 0$.

Taking coordinate differentials of (45a) and (45b) gives us a new variant of the near horizon metric

$$ds^2_\infty = -\left(1+\kappa\rho\right)^2 dt^2 + d\rho^2 \tag{46}$$

where the infinity subscript is used to emphasize that the metric measures proper distance at infinity, which follows from (32). This is in contrast to the original near horizon metric in (2), which measures distance on the lengthscale of proper distance just above the event horizon.

Conclusion for Part I

A variant of the near horizon metric is shown in (46). It takes the same form as the Rindler metric variant in (1b). In some situations, the fact that all of its parameters are defined in terms of their measurements at infinity, may make it slightly more useful than the original near horizon metric in (2). However, the fact that the spatial origin of (45) corresponds to the spatial origin of a worldline above the event horizon, instead of to the event horizon itself (as it does for the original near horizon metric), will probably make (46) slightly less convenient for most calculations than the original near horizon metric. Nevertheless, taking the time to go through its derivation provides a very good drill on some of the basics concepts in general relativity.

PART II: CALCULATING THE HAWKING TEMPERATURE

In Part II, a calculation of the Hawking temperature is made in the metric variant (46) that was derived in Part I. The calculation is made using the tunneling model of Hawking radiation that was mentioned earlier. In this approach, the WKB approximation treats that the tunneling event is an instantaneous process (the positive-energy outgoing virtual photon just happened to be above the event horizon when the measurement was made), so the action's time-contribution is equal to zero

$$S=\int p_\mu dx^\mu=\int p_i dx^i \tag{47}$$

The outgoing positive-energy virtual photon, or ingoing negative-energy virtual photon, follows a lightlike radial geodesic, and is modeled as an s-wave of energy $\pm\omega$,[5] tunneling across the spherical surface at $\rho=-\kappa^{-1}$, which corresponds to the event horizon in (46). This process is be described by (47) as

$$\begin{aligned} S_\pm &= \lim_{\epsilon\to 0}\int_{-\kappa^{-1}}^{-\kappa^{-1}\pm\epsilon} p_\rho d\rho \\ &= \lim_{\epsilon\to 0}\int_{-\kappa^{-1}}^{-\kappa^{-1}\pm\epsilon}\int_0^{p_\rho} dp_\rho d\rho' \end{aligned} \tag{48}$$

where the subscript $\pm$ distinguishes the cases of positive-energy outgoing and negative-energy ingoing virtual particles.

[5] $+\omega$ for outgoing positive-energy virtual photons, and $-\omega$ for ingoing negative energy virtual photons.

The Hamiltonian is defined as the total energy contained in the spacetime, so it has a scalar value that is equal to the black hole's initial rest mass $H=M$. Each tunneling event is preceded by pair creation

$$M\to -\omega+\omega+M \tag{49}$$

During each tunneling event, a positive/negative-energy virtual photon tunnels out-of/into the black hole. Once the virtual photons are separated across the event horizon they materialize into real photons. The positive-energy photon materializes outside of the black hole, and travels off to infinity. The negative-energy photon materializes on a negative-energy geodesic inside of the black hole, and manifests as a decrease in the black hole's mass.

The classical Hamilton for this process takes the form $H(p_\rho,\rho)=T(p_\rho)+U(\rho)$. Hamilton's equation for the particle's velocity $\dot\rho=\partial H/\partial p_\rho$ is equal to

$$\dot\rho=\frac{dT}{dp_\rho} \tag{50}$$

where dT is a transfer of kinetic energy. In both outgoing and ingoing cases, this energy transfer decreases the black hole's mass, and transfers the rest mass lost by the black hole, into the kinetic energy of the outgoing photon $dT=+d\omega$. So we can rewrite (50) as

$$dp_\rho=\frac{d\omega}{\dot\rho} \tag{51}$$

The photons travel along lightlike geodesics, which are characterized by $ds^2=0$. Imposing this condition on (46), and then solving for $\dot\rho$ gives

$$\dot\rho=\pm(1+\kappa\rho) \tag{52}$$

where $+/-$ corresponds to the geodesics of outgoing/ingoing virtual photons. Since negative-energy photons travel in the negative-Schwarzschild-time-direction inside of the black hole, we need to account for this by replacing $dt\to\pm dt$, which eliminates the $\pm$ sign in (52), giving

$$\dot\rho=1+\kappa\rho \tag{53}$$

This can be used together with (51), to rewrite (48) as

$$S_\pm = \lim_{\epsilon\to 0}\int_{-\kappa^{-1}}^{-\kappa^{-1}\pm\epsilon}\int_0^{\pm\omega}\frac{\kappa^{-1}d\omega'}{\kappa^{-1}+\rho}d\rho \tag{54}$$

Setting $\varepsilon e^{i\phi} = 4M + \rho$, which gives $i\varepsilon e^{i\phi}d\phi = d\rho$. It is explained in Figure 3, that ϕ is integrated counterclockwise/clockwise through an angle of $\frac{\pi}{2}$ for outgoing/ingoing virtual photons.

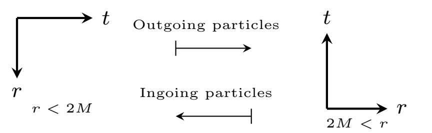

FIG. 3: The axes of the rt-plane undergo a counterclockwise rotation through $\pi/2$ for outgoing particles, which corresponds to integration from 0 to $\pi/2$. For ingoing particles, the axes undergo a clockwise rotation through $\pi/2$, which corresponds to integration from 0 to $-\pi/2$.

Making this change of variables in (54) gives

$$\begin{aligned} S_\pm &= \int_0^{\pm\pi/2}\int_0^{\pm\omega} i4M d\omega' d\phi \\ &= 2\pi i\omega M \end{aligned} \tag{55}$$

The total action $S = S_- + S_+$ is the sum of the contributions to the action from ingoing negative-energy and outgoing positive-energy photons

$$S = i4\pi\omega M \tag{56}$$

The tunneling rate for the quantum mechanical WKB approximation, is given by [16]

$$\Gamma \sim e^{-2\mathrm{Im}\ S_{QM}} \tag{57}$$

In the tunneling model of Hawking radiation, the Hawking temperature is found by assuming that the black hole radiates a Boltzmann distribution of Hawking radiation, which is imposed by setting the tunneling rate equal to a Boltzmann factor for the particle's energy level

$$\Gamma \sim e^{-\omega/T_H} \tag{58}$$

Setting the exponents of (57) and (58) equal to each other gives the Hawking temperature as

$$T_H = \frac{\omega}{2\mathrm{Im}\ S} \tag{59}$$

Putting (56) into this equation, and using $\kappa = 1/4M$ gives the Hawking temperature for a Schwarzschild black hole

$$T_H = \frac{\kappa}{2\pi} \tag{60}$$

Conclusion for Part II

In Part II, the metric variant (46) that was derived in Part I was used to calculate the Hawking temperature using the tunneling model of Hawking radiation. The calculation gave us the well-known equation for the Hawking temperature of a Schwarzschild black hole, shown in (60), demonstrating that (46) does what it is supposed to. Calculating the Hawking temperature in the metric variant is actually slightly more complicated than the same calculation when the original near horizon metric is used, instead of its variant that was derived here. That being said, the calculation of T_H made in Part II was just intended to demonstrate that (46) does what it is supposed to.

Appendix A: Near horizon approximation calculations

1. Proper distance from $r = 2M$ to $r > 2M$

For integration over radial proper distance $ds = \sqrt{g_{rr}}dr$, we can use the Schwarzschild metric to write

$$\begin{aligned} r_{NH}(r) &= \int_0^{r_{NH}} ds \\ &= \int_{2M}^{r}\frac{\sqrt{r'}dr'}{\sqrt{r'-2M}} \end{aligned} \tag{A1}$$

Using $r' = 2M\sec^2 t$, this can be written as

$$r_{NH} = 4M\int \sec^3 t dt \tag{A2}$$

Applying the identity $\sec^2 t = 1 + \tan^2 t$, this becomes

$$\int \sec^3 t dt = \int \sec t dt + \int \sec t \tan^2 t dt \tag{A3}$$

The first integral can be solved using $l = \sec t + \tan t$ and $dl = l \sec t dt$, to give

$$\int \sec t dt = \ln|\sec t + \tan t| + C \tag{A4}$$

The second integral on the right hand side of (A3) can be solved using $\sec t \tan t = d \sec t/dt$, followed by integration by parts $f'g = (fg)' - fg'$

$$\int \sec t \tan^2 t dt = \tan t \sec t \Big|_{t_1}^{t_2} - \int \sec^3 t dt \tag{A5}$$

Putting equations (A4) and (A5) into (A3)

$$\int \sec^3 t dt = \frac{1}{2}\left(\ln|\sec t + \tan t| + \tan t \sec t \Big|_{t_1}^{t_2}\right) \tag{A6}$$

Putting this into (A2), and then using $\sec t = \sqrt{r'/2M}$ and $\tan t = \sqrt{(r' - 2M)/2M}$

$$r_{NH}(r) = 2M\left[\ln\left(\sqrt{\tfrac{r}{2M}} + \sqrt{\tfrac{r}{2M} - 1}\right) + \sqrt{\tfrac{r}{2M} - 1}\sqrt{\tfrac{r}{2M}}\right] \tag{A7}$$

The first term on the right hand side can be rewritten as

$$\ln\left(\sqrt{\tfrac{r}{2M}} + \sqrt{\tfrac{r}{2M} - 1}\right) = \ln\left(\sqrt{1 + \left(\sqrt{\tfrac{r}{2M} - 1}\right)^2} + \sqrt{\tfrac{r}{2M} - 1}\right) \tag{A8}$$

Putting $x = \sqrt{r/2M - 1}$ into $\sinh^{-1} x = \ln\left(x + \sqrt{1 + x^2}\right)$, this becomes

$$\ln\left(\sqrt{\tfrac{r}{2M}} + \sqrt{\tfrac{r}{2M} - 1}\right) = \sinh^{-1}\left(\sqrt{\tfrac{r}{2M} - 1}\right) \tag{A9}$$

Using this to replace the first term on the right hand side of (A7) gives

$$r_{NH}(r) = 2M \sinh^{-1}\left(\sqrt{\tfrac{r}{2M} - 1}\right) + \sqrt{r(r - 2M)} \tag{A10}$$

Appendix B: Proper acceleration and surface gravity

1. Proper radial acceleration of an observer who is radially at rest in the presence of a Schwarzschild black hole

The 4-acceleration of an object is defined by

$$\boldsymbol{a} \equiv \frac{d\boldsymbol{u}}{d\tau} = \left[\partial_\mu u^\nu + \Gamma^\nu_{\mu\alpha} u^\alpha\right] u^\mu \boldsymbol{e}_\nu \tag{B1}$$

Assuming the object to be at rest with respect to the global Schwarzschild geometry

$$\begin{aligned} u^\mu &= \left(\frac{dt}{d\tau}, 0, 0, 0\right) \\ &= \left(\left(1 - \tfrac{2M}{r}\right)^{-1/2}, 0, 0, 0\right) \end{aligned} \tag{B2}$$

which comes from $u^t = 1/\sqrt{-g_{tt}}$ since the observer is at rest, and $g_{\mu\nu}$ is the Schwarzschild metric.

$\partial_t u^\nu = 0$ for all ν in (B1) since u^μ does not depend on t. This, together with the fact that the 4-velocity is nonzero only in its time-component simplifies (B1) to

$$\boldsymbol{a} = \Gamma^\nu_{tt}\left(u^t\right)^2 \boldsymbol{e}_\nu \tag{B3}$$

The Christoffel symbols can be calculated from the metric, using

$$\Gamma^\nu_{\mu\alpha} = \frac{1}{2} g^{\nu\beta}\left[\partial_\mu g_{\alpha\beta} + \partial_\alpha g_{\beta\mu} - \partial_\beta g_{\mu\alpha}\right] \tag{B4}$$

Setting $\mu = \alpha = t$, using the fact that the metric is diagonal to eliminate off-diagonal terms, and using the fact that the metric is time-independent to eliminate the time derivatives gives

$$\Gamma^\nu_{tt} = -\frac{1}{2} g^{\nu\nu} \partial_\nu g_{tt} \tag{B5}$$

The Schwarzschild metric's time-component depends only on r, so we only need to compute one Christoffel symbol

$$\Gamma^r_{tt} = \frac{M}{r^2}\left(1 - \frac{2M}{r}\right) \tag{B6}$$

Putting this and (B2) into (B3) gives

$$\boldsymbol{a} = \frac{M}{r^2}\boldsymbol{e}_r \tag{B7}$$

whose scalar quantity is defined by $a = \sqrt{\boldsymbol{a}\cdot\boldsymbol{a}} = \sqrt{g_{rr}}a^r$. Taking g_{rr} from the Schwarzschild metric, and a^r from (B7), the proper acceleration of an observer who is at rest in the Schwarzschild coordinates is

$$a(r) = \frac{M}{r^2\sqrt{1-\frac{2M}{r}}} \tag{B8}$$

2. Surface gravity κ

Surface gravity κ can be thought of as a redshifted-to-infinity measure of the proper acceleration that is required for an observer who is located at the event horizon, to remain radially at rest (which from (B8), requires an infinite proper acceleration).

We can derive the surface gravity $\kappa = 1/4M$ for a Schwarzschild black hole from this interpretation. Consider an observer of mass m, located at some $r > 2M$. To remain radially at rest, the observer must be acted on by a force $F(r)$, directed radially-outward, that exactly offsets the observer's acceleration due to gravity. Over an interval of proper time $\delta\tau_r$, the force does an amount of work

$$\begin{aligned}\delta W_r &= F(r)\delta\tau_r \\ &= ma(r)\delta\tau_r\end{aligned} \tag{B9}$$

to hold the observer in place at constant r. The proper acceleration provided by $F(r)$ is given by (B8), and can be substituted into (B9) to give

$$\delta W_r = \frac{mM}{r^2\sqrt{1-\frac{2M}{r}}}\delta\tau_r \tag{B10}$$

Now suppose that all of this work is transferred into the energy a single photon, with energy

$$\delta\omega_r = \frac{mM}{r^2\sqrt{1-\frac{2M}{r}}}\delta\tau_r \tag{B11}$$

which then travels off to infinity, where it is measured to have energy ω_∞. We can relate ω_r to ω_∞ by first relating the photon's period at the two locations

$$\delta\tau_r = \sqrt{1-\frac{2M}{r}}\delta\tau_\infty \tag{B12}$$

This can be inverted to find the relation between the photon's energy at the two locations

$$\delta\omega_r = \frac{\delta\omega_\infty}{\sqrt{1-\frac{2M}{r}}} \tag{B13}$$

Setting the right hand sides of (B11) and (B13) equal to each other, and then multiplying both sides by $\sqrt{1-2M/r}$ gives

$$\delta\omega_\infty = \frac{mM}{r^2}\delta\tau_r \tag{B14}$$

Since $\delta\tau_r$ is a scalar invariant, we can divide both sides of (B14) by $\delta\tau_r$ to get a redshifted-to-infinity measure of the force that was required for the observer to remain radially at rest

$$F_\infty(r) = \frac{mM}{r^2} \tag{B15}$$

Dividing both sides by m gives a redshifted-to-infinity measure of the proper acceleration that was required to hold the observer at rest in the presence of the black hole

$$\kappa(r) \equiv \frac{M}{r^2} \tag{B16}$$

When evaluated at $r = 2M$, this is equal to the surface gravity of a Schwarzschild black hole

$$\kappa(2M) = \frac{1}{4M} \tag{B17}$$

[1] S. W. Hawking. "Particle creation by black holes," Commun. Math. Phys. **43**. 199-220 (1975).
[2] W. G. Unruh, "Notes on black hole evaporation," Phys. Rev. D **14**, 870-892 (1976).
[3] G. W. Gibbons and S. W. Hawking. "Cosmological event horizons, thermodynamics, and particle creation," Phys. Rev. D **15**, 2738-2751 (1977).
[4] M. K. Parikh and F. Wilczek. "Hawking radiation as tunneling," Phys. Rev. D **85** 5042-5045 (2000).
[5] M. K. Parikh, "A secret tunnel through the horizon," Int. J. Mod. Phys. D **13**, 2351-2354 (2004).
[6] J. F. Donoghue and B. R. Holstein, "Temperature measured by a uniformly accelerated observer," Am. J. Phys. **52**, 730-734 (1984).
[7] S. P. Kim, "Hawking radiation as quantum tunneling in Rindler coordinate," JHEP **11**, 048-1-17 (2007).
[8] P. M. Alsing and P. W. Milonni, "Simplified derivation of the Hawking-Unruh temperature for an accelerated observer in vacuum," Am. J. Phys. **72**, 1524-1529 (2004).
[9] L. Susskind, J. Lindesay, *An Introduction to Black Holes, Information, and the String Theory Revolution: The Holographic Universe*, (World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005).
[10] A. de Gill, D. Singleton, V. Akhmedova, and T. Pilling. "A WKB-like approach to Unruh radiation," Am. J. Phys. **78** 685-691 (2010).
[11] Q. Jiang, S. Wu, and X. Cai, "Hawking radiation as tunneling from the Kerr and Kerr-Newman black holes," Phys. Rev. D **73**, 064003-1-10 (2006).
[12] J. Zhang and Z. Zhao. "Charged particles tunneling from Kerr-Newman black hole," Phys. Lett. B **638**, 110-113 (2006).
[13] L. Susskind and J. Uglum. "Black hole entropy in canonical quantum gravity and superstring theory," Phys. Rev. D **50** 2700-2711 (1994).
[14] T. Dobrian, "Hawking Radiation with the WKB and Graviational WKB approximations" *Undergraduate Honors Theses*. 1496 (2017).
[15] C. W. Misner, K. S. Thorne, and J. A Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
[16] D. J. Griffiths, *Introduction to Quantum Mechanics*, 2nd ed. (Pearson-Prentice-Hall, Upper Saddle Rive, NJ 2005).