If Riemann's Zeta Function is True, it Contradicts Zeta's Dirichlet Series, Causing "Explosion". If it is False, it Causes Unsoundness.

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Abstract

Riemann's "analytic continuation" produces a second definition of the Zeta function, that Riemann claimed is convergent throughout half-plane $s \in \mathbb{C}$, $\text{Re}(s) \leq 1$, (except at $s = 1$). This contradicts the original definition of the Zeta function (the Dirichlet series), which is proven divergent there. Moreover, a function cannot be both convergent and divergent at any domain value. In physics and in other mathematics conjectures and assumed-proven theorems, the Riemann Zeta function (or the class of L-functions that generalizes it) is assumed to be true. Here the author shows that the two contradictory definitions of Zeta violate Aristotle's Laws of Identity, Non-Contradiction, and Excluded Middle. Non-Contradiction is an axiom of classical and intuitionistic logics, and an inherent axiom of Zermelo-Fraenkel set theory (which was designed to avoid paradoxes). If Riemann's definition of Zeta is true, then the Zeta function is a contradiction that causes deductive "explosion", and the foundation logic of mathematics must be replaced with one that is paradox-tolerant. If Riemann's Zeta is false, it renders unsound all theorems and conjectures that falsely assume that it is true. Riemann's Zeta function appears to be false, because its derivation uses the Hankel contour, which violates the preconditions of Cauchy's integral theorem.

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Contents

1 Significance Statement

Riemann's "analytic continuation" of the Zeta function is either true or false. If it is true, then it contradicts the Dirichlet series definition of Zeta in half-plane $s \in \mathbb{C}$, $\text{Re}(s) \leq 1$. The former is convergent, and the latter divergent, throughout said half-plane. This contradiction renders as paradoxes all conjectures and theorems that use the Riemann Zeta function (or the class of L-functions), causing "explosion" and necessitating a new paradox-tolerant foundation of mathematics. However, Riemann's Zeta appears to be false, because its derivation uses the Hankel contour, which violates the preconditions of Cauchy's integral theorem. If Riemann's Zeta function is false, it renders unsound all conjectures and theorems that falsely assume that it (or the class of L-functions) is true.

2 Introduction: Two Contradictory Definitions of the Zeta Function

Bernhard Riemann begins his famous paper *On the Number of Primes Less Than a Given Magnitude* (See [96], p.1) by stating that the Dirichlet series definition of the Zeta function is proven to be divergent for all values of complex variable $s \in \mathbb{C}$ in half-plane $\text{Re}(s) \leq 1$. (See [57], pp.3-5, citing [69], [22], and [19]; [64], pp.117-118, Thm 4.6; [26], p.11, Thm 11).

While not mentioned in Riemann's paper, the "Integral Test for convergence" (a.k.a. the Maclaurin–Cauchy test for convergence) proves that the Dirichlet series of the Zeta function is divergent for all values of s on the Real half-axis $(\text{Re}(s) \leq 1, \text{Im}(s) = 0)$, which is a sub-set of the half-plane $\text{Re}(s) \leq 1$. The "Integral Test for convergence" is commonly taught in introductory calculus textbooks, to prove that the famous "harmonic series" is divergent (See [52], Thm 13.3.4).

In addition, the Dirichlet series of the Zeta function is also proven to be divergent for all values of s on the misleadingly-named "line of convergence", $\text{Re}(s) = 1$, which is a sub-set of the half-plane of divergence, and which is the border line between the half-plane of divergence and the half-plane of convergence. (See [57], p.5, Example (iii), citing [19]). At the point $s = 1$, the Dirichlet series of the Zeta function is the "harmonic series", which is proven divergent by the "Integral test for divergence". At all other values of s on the "line of convergence", the Dirichlet series of the Zeta function is a bounded oscillating function, which by definition is divergent (See [57], p.5, Example (iii), citing Bromwich [19]).

Later in Riemann's paper, his so-called "analytic continuation" [1] of the Zeta function results in an alleged second definition of the Zeta function, one that he claimed "always remains valid" (except at $s = 1$). (See [96], p.1). In other words, a second definition of the Zeta function that is convergent for all values of s in half-plane $\text{Re}(s) \leq 1$ (except at $s = 1$). If true, Riemann's claim would mean that all of these propositions are true:

> The Zeta function is divergent for all s in half-plane $\text{Re}(s) \leq 1$.
>
> The Zeta function is convergent for all s in half-plane $\text{Re}(s) \leq 1$, (except at $s = 1$).

> The Zeta function is divergent for all s in the Real half-axis, $s < 1$.
>
> The Zeta function is convergent for all s in the Real half-axis, $s < 1$.

[1] Riemann himself does not use this name. His method is very different from Weierstrass's "unit disk" method. (See [104]).

The Zeta function is divergent for all s in the "line of convergence" $\mathrm{Re}(s) = 1$.

The Zeta function is convergent for all s in the "line of convergence" $\mathrm{Re}(s) = 1$, (except at $s = 1$).

In summary, Riemann misunderstood the logical concept of "validity", and the contradiction inherent in the definitions of the mathematical concepts of "convergence" and "divergence". The proof that the Dirichlet series of the Zeta function is divergent throughout half-plane $\mathrm{Re}(s) \leq 1$, is a logically *valid* proof. The fact that the Zeta function is divergent throughout the half-plane $\mathrm{Re}(s) \leq 1$ does not render the proof invalid, or does it render the function false (or in Riemann's terminology, render the function "not valid"). In fact, it is Riemann's second definition of the Zeta function that creates an issue of logical validity, by creating two contradictory definitions of Zeta in the above=cited half-plane.

In Riemann's defense, his paper (1859) predates Frege's *Begriffsschrift* (1879) by two decades, and predates the subsequent developments in logic and the foundations of mathematics by at least a half-century. Brouwer's *The Untrustworthiness of the Principles of Logic* (1908), Whitehead and Russell's *Principia Mathematica* (See [126], 1910), Łukasiewicz's *On Three-Valued Logic* (1920), Zermelo–Fraenkel set theory (1920's) were all published long after Riemann's death (1866). The only relevant publication in the field of logic that was contemporaneous with Riemann's was Boole's *The Laws of Thought* (See [14], 1854), of which Riemann was likely unaware. That said, "proof by contradiction" has been a method of proof in mathematics since Euclid's *Elements*.

In number theory, algebraic geometry, and quantum physics, Riemann's second definition of the Zeta function is widely assumed to be true, as is the class of Dirichlet L-functions which generalizes Riemann's Zeta function (See [7], Ch.13, p.200. See also [70], pp.3-4, 12; [46], pp.60,61 and 65).

In number theory and algebraic geometry, Riemann's Zeta function or Dirichlet L-functions are assumed to be true in many important conjectures and presumed-proven theorems. For example: the Riemann hypothesis (See [13], p.1), its analogues (See [78], p.3; [68], pp.4-5; [78], p.49), and its generalizations (See [23], p.4; [67]; [100]), the Birch and Swinnerton-Dyer conjecture (See [27]), the Bloch-Kato conjecture (See [18], p.cxvii; [10], pp.38, 44, 50), the Modularity theorem (See [107], p.13, Thm 25.33; [124]; [43], pp.17-22, Conj 2,3, Thm 5.1, 5.3), the Hasse-Weil theorem (See [128], p.2, citing [129], [112], and [24]; [107], p.14, §25.9), and Fermat's last theorem (See [129], para.1).

In quantum physics, the Riemann's Zeta function is assumed to be true in many theories, for example: the Casimir effect (See [35], pp.30-34; [114], pp.38-40), Quantum

Electrodynamics (See [35], p.34), Quantum Chromodynamics (See [36]; [35], p.34), Yang-Mills theory (See [130]; [2]), Supersymmetry (See [40]), Quantum Field Theory (See [84], pp.656,678), Bosonic String Theory (See [63]; [121]; [42]; [115]; [82]; pp.17-18; [114], pp.39-40), and in "Zeta function regularization" (See [59], p.133, §1).

3 In a Nutshell: Propositional Logic and Foundations of Mathematics

The following discussion is the bare minimum of propositional logic and set theory necessary for this paper. Logic is the study of the methods and principles used to distinguish between valid and invalid arguments (See [74], p.2; [25]). A sound argument is a valid argument, whose premises are *all* true (See [74], p.19). An otherwise valid argument is rendered unsound if one (or more) of its premises are false.

The earliest study of logic in the Western European tradition is traditionally attributed to Aristotle. (Note that Aristotle's life either barely preceded, or partially overlapped, Euclid's). As stated by Russell, Aristotle's "Laws of Thought" are: (See [99], ch.VII. But see [92], p.139. See also [65]; [48]; [51]; [103], §11. See also [14], pp.8, 34-36, 48-49, 99-100)

> (1) The Law of Identity (LOI): 'Whatever is, is.' Its sequent is: $\vdash P \equiv P$.
>
> (2) The Law of Non-Contradiction (LNC): 'Nothing can both be and not be.' Its sequent is: $\vdash \neg(P \wedge \neg P)$.
>
> (3) The Law of the Excluded Middle (LEM): 'Everything must either be or not be.' Its sequent is: $\vdash (P \vee \neg P)$.

These three axioms of logic date back at least to Aristotle's *Organon* (See [6], *De interpretatione* §9) and *Metaphysics* (See [5], Book IV, Part 3). The LNC pre-dates Aristotle (See [87], pp. 264-265, §8; [28], p.328). Aristotle considered the LNC to be the most important axiom of philosophy (See [5], Book IV, Part 3; [32], p.33, citing [14], p.49; [29], §4). Other phrasings of the LNC include: “opposite assertions cannot both be true simultaneously”, and "an unambiguous statement cannot be both true and false" (See [85] p.22, para.4).

An important theorem to add to these axioms is the "Principle of Explosion", or the principle of Pseudo-Scotus, or *Ex Contradictione (Sequitur) Quodlibet* (ECQ), which is attributed to 12th century French philosopher William of Soissons (See [93], p.25, citing [91], vol.6, ch.4). This theorem holds that from a contradiction, any statement can be proven to be true. Its sequent is: $(P, \neg P \vdash Q)$. The result is "explosion": all statements become "trivially true". This is why in logics that have both the LNC and ECQ, a single contradiction in a proof is catastrophic. Both classical and intuitionistic logics have this theorem (See [71], p.101).

The logics of the early 20th century (e.g. classical logic, intuitionistic logic, and three-valued logics (3VLs)) differ as to which of these axioms (and theorem) they include. The so-called "classical logic" of Whitehead and Russell's *Principia Mathematica* (See [126]) includes all of the LOI, LNC, LEM, and ECQ (See [44], Ch 2.6). Brouwer's intuitionistic logic has the LOI, LNC, and ECQ, but rejects the LEM, in the specific case of mathematical propositions lacking both proof and disproof. Heyting's formalized intuitionistic logic rejects the LEM completely (See [80]).

In contrast, three-valued logics (3VLs) have a third truth-value, so for propositions that have this third truth-value, the LOI, LEM, and LNC (and thus also ECQ) all fail (See [54], p.5).

In regards to mathematics, as in classical logic, the LOI and LNC are *inherent* axioms of mathematics, as is ECQ. The creation of Zermelo–Fraenkel set theory (the consensus foundation of mathematics) constitutes evidence that LOI, LNC, and ECQ are axioms of mathematics. While the LOI, LNC, and LEM are not expressly stated axioms of Zermelo–Fraenkel set theory, it was formulated in order to avoid the paradoxes of naive set theory: Russell's paradox, the Burali-Forti paradox, and Cantor's paradox. (But see [79]). It also avoids the Banach-Tarski paradox, if the Axiom of Choice is rejected. (See [122], pp.217-8, citing [8]). In other words, it was created in order to avoid violating the LOI and LNC.

Additional evidence that LOI and LNC are axioms of mathematics is "proof by contradiction", a technique of proof which has been used in mathematics since at least Euclid's second theorem in *Elements* (See [30]). Proof by contradiction establishes the truth or validity of a proposition by first assuming the proposition to be false, and then proving that this false assumption leads to a contradiction with another proposition proven to be true.

Note however that "intuitionists think of logic as secondary to mathematics", instead of as a foundation of mathematics. (See [53], pp.216-7; [80], 2nd para.). This is opposite to the views of Aristotle, Whitehead and Russell (See [126]), who assumed that logic applies to all propositions, and it is also opposite to the views of the proponents of Zermelo–Fraenkel set theory, who assumed that set theory is the basis of all mathematics. Moreover, the intuitionists also object to the LEM, and thus object to the use of "proof by contradiction" (because it depends on the LEM).

4 If Riemann's Zeta is True, it Contradicts Zeta's Dirichlet Series and Causes "Explosion"

Given the *proven* divergence of the Dirichlet series definition of the Zeta function throughout the half-plane $\text{Re}(s) \leq 1$, if Riemann's Zeta function is true, the Zeta function would have both a true *convergent* definition and a true *divergent* definition throughout half-plane $\text{Re}(s) \leq 1$ (except at $s = 1$).

Moreover, if Riemann's Zeta function were true, and thus convergent throughout half-plane $\text{Re}(s) \leq 1$, then it would be *convergent* throughout the Real half-axis $\{\text{Re}(s) < 1, \text{Im}(s) = 0\}$, which is a sub-set of the half-plane $\text{Re}(s) \leq 1$. (Riemann's functional equation of the Zeta function even claims to have "trivial zeros" on this Real half-axis.) This result of "convergence" directly contradicts the results of "divergence" produced by the Integral test for convergence (a.k.a. the Maclaurin-Cauchy test for convergence) when applied to the Dirichlet series definition of the Zeta function, for all values of s on this Real half-axis.

Also, if Riemann's Zeta function were true, and thus convergent throughout half-plane $\text{Re}(s) \leq 1$, then it would be *convergent* at all points on the misleadingly-named "line of convergence" at $\text{Re}(s) = 1$ (except at $s = 1$). This directly contradicts the divergence of the Dirichlet series definition of the Zeta function along this line. (See [57], p.5, Example (iii), citing [19])

Taken individually. each of these results would be sufficient to render the Zeta function a paradox (in the half-plane $\text{Re}(s) \leq 1$). This paradox would cause "deductive explosion" for any other deductive argument that assumes that either definition of the Zeta function is true in that half-plane.

However, according to the mathematical definitions of "convergence" and "divergence", a function cannot be both convergent and divergent at any value in its domain (See [56], p.1). Moreover, the two different definitions of the Zeta function in the half-plane $\text{Re}(s) \leq 1$ violate the definition of a "function" in set theory, due to the one-to-two mapping from domain to range (See [105]). Perhaps most alarmingly, if the two contradictory definitions of the Zeta function were both true in the half-plane $\text{Re}(s) \leq 1$, it would mean that mathematics is inconsistent, resulting in "deductive explosion".

In addition, the two contradictory definitions of the Zeta function would, if both were true, violate all three of Aristotle's three "Laws of Thought". The two definitions of Zeta violate Aristotle's Law of Identity (LOI), according to which each thing is identical with itself. But Zeta has two *different* (non-equivalent) values throughout the cited half-plane. The LOI is inherently an axiom of mathematics, otherwise propositions such as "$-1/2 = \infty$", and "$5 \neq 5$" (wherein both fives are in base ten), would be valid in mathematics.

The two definitions of the Zeta function also violate the Law of Non-Contradiction (LNC). This is because its contradiction with the Dirichlet series definition results in the Zeta function being both divergent and convergent at values of s in the cited half-plane. Such a contradiction violates the LNC. So Riemann's claim is not valid in logics that have the LNC as an axiom, or in the foundations of mathematics (i.e. Zermelo-Fraenkel set theory), which inherently has the LNC as an axiom.

Riemann's claim also violates the Law of the Excluded Middle (LEM), because it asserts that at certain values of s, the Zeta function is simultaneously *both* divergent *and* convergent. But according to the LEM, the Zeta function must be one or the other - it cannot be both simultaneously. In summary, if both the Dirichlet series definition and Riemann's definition of Zeta are true, this result violates all of the LOI, LEM, and LNC. The violation of LNC causes ECQ ("Explosion").

If Riemann's Zeta function were true, its violation of the LNC would mean that the foundation logic of mathematics would have to be a paradox-tolerant "paraconsistent" logic, such as a three-valued logic (e.g. Bochvar's 3VL ([12]), or Priest's *LP* ([89], [90], [62])), or a logic that has the LNC but not ECQ ([92], [94], [91]).

5 If Riemann's Zeta is False, it Renders Unsound All Arguments that Assume it is True

Fortunately (for the consistency of mathematics), there appears to be an error in Riemann's derivation of his definition of the Zeta function, due to Hankel's contour violating the preconditions of Cauchy's integral theorem. This is discussed in detail in section 8 of this paper.

Yet even this result is problematic, because if Riemann's Zeta function is *false* at all values of s in half-plane $\text{Re}(s) \leq 1$ (except at $s = 1$), then all mathematics conjectures and theorems, and physics theories, that falsely assume that Riemann's Zeta function is true are rendered *unsound* (and invalid) in Aristotelian, classical, and intuitionistic logics (and even the paradox-tolerant 3VLs and paraconsistent logics).

For example, the "Zeta Function Regularization" used in physics is rendered invalid, because it equates a true definition of the Zeta function to a false definition. Moreover, because Riemann's Zeta function is one example of the Dirichlet L-functions, the falsity of Riemann's Zeta function is the counter-example that falsifies the assumption that all L-functions are true. More specifically, the false assumption that the L-functions are true includes the false assumption that Riemann's version of "analytic continuation" (See [21], p.4) is valid. In turn, the false assumption that all L-functions are true renders unsound several presumed-proven mathematical theorems (e.g. the Modularity theorem, Fermat's last theorem).

The falsity of Riemann's Zeta function also confirms that $\zeta(1) \neq 0$. This resolves the Birch and Swinnerton-Dyer (BSD) Conjecture in favor of finiteness (See [27]). The Dirichlet series exclusively defines the Zeta function, so at $s = 1$, it is the "harmonic series", which is proven to be divergent by the Integral test for convergence (See [52], Thm 13.3.4). Moreover, the Landau-Siegel zero (See [102]; [31], p.351) is non-existent, due to the invalidity of the "analytic continuation" of Riemann's Zeta function specifically, and thus of L-functions in general

The falsity of Riemann's Zeta function, such that the Zeta function is exclusively defined by Dirichlet series, resolves the BSD conjecture and thereby triggers a "domino effect" of unsoundness (invalidity) through a chain of equivalent conjectures. For example, the BSD conjecture "for elliptic curves over global fields of positive characteristic" is equivalent to the Tate conjecture "for elliptic surfaces over finite fields", (See [116], citing [120], pp.6,31-32; [117], p.578; [77], p.3, Thm 1.4). The Tate and Hodge conjectures are equivalent "for abelian varieties of CM-type" (See [47], p.364, §11.2, citing [88], [86], [15], [16]; [33], p.43, Cor 6.2. See also [101], p.60, citing [88], §2, [81], [73], [95], [60]. See also [9], pp.12-14, Cor 5.5, citing [75], [111], [108]). Therefore, the Tate and Hodge conjectures, which falsely assume that L-functions are true, are rendered unsound by the falsity of Riemann's Zeta function in half-plane $\text{Re}(s) \leq 1$, via the BSD conjecture.

There exist other conjectures rendered unsound by the falsity of Riemann's Zeta function in half-plane $\text{Re}(s) \leq 1$, due to their relationship to the BSD conjecture. These include the finiteness of the Tate–Shafarevich group, and the finiteness of the Brauer group. (See [117], p.579; [128], p.2, citing: [110], p.416,426; [76], Cor 9.7).

Regarding Hadamard and de la Vallée Poussin's respective proofs of the prime number theorem, contrary to the statement in Borwein ([17]), they do not "follow from the truth of the Riemann hypothesis". (See [17], pp.9,61, §7.1, §12.4; [39], pp.68-69). Instead, they are true because the Zeta function is exclusively defined by the Dirichlet series (which has no zeros). Therefore, the resulting Zeta function has no zeros on the misleadingly-named "line of convergence", $\text{Re}(s) = 1$.

6 In Physics, Theories that Use Riemann's Zeta Function Either Cause "Explosion" or are Unsound

Currently, physicists assume that Riemann's Zeta function is true. For example, in "Zeta Function Regularization", they expressly equate Riemann's definition and the Dirichlet series definition of Zeta function where the two contradict, under the false assumption that are both are true definitions of the same function, and that it is logically valid to equate two contradictory statements (See [96], p.1; [59], p.133, §1; [35], p.34. But see [34]; [11], p.4).

So if both of the contradictory expressions of Zeta are true, the resulting contradiction (and paradox) would cause "explosion", due to the Law of Non-Contradiction and the Principle of Explosion. It also means that the traditional foundations of mathematics and physics (e.g. classical logic, Zermelo-Fraenkel set theory) are wrong, because by assuming the Law of Non-Contradiction and the Principle of Explosion, they are paradox-*intolerant*. If Riemann's Zeta function is indeed true, then a foundation logic of mathematics that is paradox-*tolerant* is needed, such as a three-valued logic (e.g. Bochvar's 3VL, Priest's LP).

But if Riemann's Zeta function is false, then all of the mathematics conjectures and presumed-proven theorems, and physics theories, that falsely assume that it is true are rendered unsound.

7 A Third Version of Zeta that is Conditionally Convergent (and Thus a Paradox) in the Critical Strip

Ash ([7]) derives a third version of the Zeta function from the original Dirichlet series version, that contradicts both Dirichlet's and Riemann's versions of the Zeta function. (See [7], pp.169-171). Ash derives this version of Zeta by multiplying the Dirichlet series of $\zeta(s)$ by the term $1/2^s$:

$$\frac{1}{2^s} \cdot \zeta(s) = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \cdots . \tag{7.1}$$

This product is then twice subtracted from the original Dirichlet series, resulting in:

$$(1 - \frac{1}{2^s} - \frac{1}{2^s}) \cdot \zeta(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} - \frac{1}{6^s} + \cdots \tag{7.2}$$

Note that the right side of the above equation is a Dirichlet series $\sum a_n n^{-s}$, such that the coefficients are $1, -1, 1, -1, 1, \ldots$, so $|a_1 + \cdots + a_n| < 2$ for all n. The above equation can be rewritten as:

$$\zeta(s) = \left(1 - \frac{1}{2^{s-1}}\right)^{-1} \cdot \left(1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} - \frac{1}{6^s} + \cdots\right) \tag{7.3}$$

Ash (See [7], p. 169, Thm 11.7) cites the following theorem to prove that this third version of the Zeta function is convergent throughout half-plane $\mathrm{Re}(s) > 0$, and divergent throughout half-plane $\mathrm{Re}(s) \leq 0$:

> THEOREM 11.7: Suppose that there is some constant K so that $|a_1 + \cdots + a_n| < K$ for all n. Then the Dirichlet series $\sum a_n n^{-s}$ converges if $\sigma > 0$.

Therefore, this third definition of the Zeta function contradicts the Dirichlet series version of the Zeta function throughout the "critical strip", $0 < \mathrm{Re}(s) \leq 1$, where the third version is convergent and the Dirichlet series is divergent. It contradicts Riemann's version of the Zeta function throughout half-plane $\mathrm{Re}(s) \leq 0$, where the third version is divergent and Riemann's version is convergent.

Clearly, in a logic that has the LNC as an axiom, at most one of these three versions of the Zeta function can be true. It is impossible for all three versions of the Zeta function to be true, or even for two of the three to be true.

By definition,

> $\sum a_n$ is absolutely convergent if $\sum a_n$ converges and $\sum |a_n|$ converges.
> $\sum a_n$ is conditionally convergent if $\sum a_n$ converges but $\sum |a_n|$ diverges.

So the third definition of the Zeta function, as defined in Eq. 7.3 is absolutely convergent only for values of s in half-plane $\mathrm{Re}(s) > 1$. For values of s in the "critical strip" ($0 < \mathrm{Re}(s) \leq 1$), the third definition of the Zeta function is only *conditionally* convergent.

The "Riemann series theorem" provides an explanation as to why Ash's third version of the Zeta function is a paradox in regards to convergence in the "critical strip". According to the Riemann series theorem: "By a suitable rearrangement of terms, a conditionally convergent series may be made to converge to any desired value, or to diverge" (See [123], citing: [20], p.74; [45], p.171; and [58], p.102).

So the third version of the Zeta function is created by transforming the Dirichlet series of the Zeta function, which is an *unconditionally* convergent series throughout half-plane $\mathrm{Re}(s) > 1$, to a series which is that too, *and also conditionally* convergent throughout the "critical strip", $0 < \mathrm{Re}(s) \leq 1$.

It is this *conditional* convergence in the "critical strip" that gives the third definition of the Zeta function the paradoxical result of "convergence" there. Why is it paradoxical? Because Riemann's series theorem shows that the terms of a conditionally convergent series can be rearranged to result in different values of convergence, and even in divergence.

Such a paradoxical result - a series that is both convergent and divergent at the same domain value - violates both the LOI and the LNC. It also violates the definition of a "function" (due to the one-to-many mapping from domain to range), and violates the associative and commutative properties of addition of real and complex numbers. According to certain three valued logics (e.g. Bochvar's 3VL), a series that is both convergent and divergent at the same domain value should have the third truth-value.

8 Riemann Zeta Function is Invalidly Derived, due to Hankel's Contour and Cauchy's Integral Theorem

In the derivation of the Riemann Zeta function, Riemann uses the following equation (See [96], p.1):

$$\int_0^\infty e^{-nx} x^{s-1}\, dx = \frac{\prod(s-1)}{n^s} \tag{8.1}$$

On the left side of the equation, Riemann uses (See [39], p.9, fn 1) the equation $\sum_{n=1}^\infty r^{-n} = (r-1)^{-1}$ to replace the term e^{-nx} in the integral with the term $(e^x - 1)^{-1}$. On the right side of the equation, Riemann introduces a summation (from $n = 1$ to ∞) for the term $1/n^s$, thereby obtaining:

$$\int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx = \prod(s-1) \cdot \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{8.2}$$

The Dirichlet series definition of the Zeta function defines $\zeta(s) = \sum n^{-s}$, so the above equation is rewritten as:

$$\int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx = \prod(s-1) \cdot \zeta(s) \tag{8.3}$$

Next, Riemann considers the following integral:

$$\int_{+\infty}^{+\infty} \frac{(-x)^s}{(e^x - 1)} \cdot \frac{dx}{x} \tag{8.4}$$

Edwards states (See [39], p.10):

> The limits of integration are intended to indicate a path of integration which begins at $+\infty$, moves to the left down the positive Real axis, circles the origin once once in the positive (counterclockwise) direction, and returns up the positive Real axis to $+\infty$. The definition of $(-x)^s$ is $(-x)^s = \exp[s \cdot \log(-x)]$, where the definition of $\log(-s)$ conforms to the usual definition of $\log(z)$ for z not on the negative Real axis as the branch which is Real for positive Real z; thus $(-x)^s$ is not defined on the positive Real axis and, strictly speaking, the path of integration must be taken to be slightly above the Real axis as it descends from $+\infty$ to 0 and slightly below the Real axis as it goes from 0 back to $+\infty$.

This is the Hankel contour (See [39], pp.10-11; See also [127], pp.85-87, 244-45 and 266). The first use of this contour integral path was by Hankel, in his investigations of the Gamma function. (See [125], citing [72], §13.2.4, p.159; and [55]).

When the Hankel contour is split into three terms, it is written mathematically as follows (See [39], p.10). The first term is "slightly above" the Real axis as it descends from $+\infty$ to δ, the middle term represents the circle with radius δ around the origin, and the third term is "slightly below" the Real axis as it goes from δ back to $+\infty$:

$$\int_{+\infty}^{\delta} \frac{(-x)^s}{(e^x - 1)} \cdot \frac{dx}{x} + \int_{|z|=\delta} \frac{(-x)^s}{(e^x - 1)} \cdot \frac{dx}{x} + \int_{\delta}^{+\infty} \frac{(-x)^s}{(e^x - 1)} \cdot \frac{dx}{x} \tag{8.5}$$

In regards to the middle term (the circle term), Edwards states (See [39], p.10):

> [T]he middle term is $2\pi i$ times the average value of $(-x)^s \cdot (e^x - 1)^{-1}$ on the circle $|x| = \delta$ [because on this circle $i \cdot d\theta = (dx/x)$]. Thus the middle term approaches zero as $\delta \to 0$ provided $s > 1$ [because $x(e^x - 1)^{-1}$ is nonsingular near $x = 0$]. The other two terms can then be combined to give[:]

$$\int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1} \cdot \frac{dx}{x} = \lim_{\delta \to 0} \Big[\int_{+\infty}^{\delta} \frac{\exp[s(\log x - i\pi)]}{(e^x - 1)} \cdot \frac{dx}{x} + \int_{\delta}^{+\infty} \frac{\exp[s(\log x + i\pi)]}{(e^x - 1)} \cdot \frac{dx}{x} \Big] \tag{8.6}$$

resulting in

$$\int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1} \cdot \frac{dx}{x} = (e^{i\pi s} - e^{-i\pi s}) \cdot \int_0^{\infty} \frac{x^{s-1}\, dx}{e^x - 1} \tag{8.7}$$

Given that $(e^{i\pi s} - e^{-i\pi s}) = 2i\sin(\pi s)$, this can be rewritten as:

$$\int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1} \cdot \frac{dx}{x} = 2i\sin(\pi s) \cdot \int_0^\infty \frac{x^{s-1}\,dx}{e^x - 1} \tag{8.8}$$

Rearranging the terms results in:

$$\int_0^\infty \frac{x^{s-1}\,dx}{e^x - 1} = \frac{1}{2i\sin(\pi s)} \cdot \int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1} \cdot \frac{dx}{x} \tag{8.9}$$

The left sides of Equations 8.3 and 8.9 are identical, so Riemann equates the right sides of Equations 8.3 and 8.9, resulting in Equation 8.10:

$$\int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1} \cdot \frac{dx}{x} = 2i\sin(\pi s) \cdot \prod(s-1) \cdot \zeta(s) \tag{8.10}$$

Then, Riemann multiplies both sides of the equation by $\prod(-s) \cdot s/2\pi i s$, resulting in

$$\frac{\prod(-s) \cdot s}{2\pi i s} \cdot \int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1} \cdot \frac{dx}{x} = \frac{\prod(-s) \cdot s}{2\pi i s} \cdot 2i\sin(\pi s) \cdot \prod(s-1) \cdot \zeta(s) \tag{8.11}$$

The s terms on the left side cancel out, as do the $2i$ terms on the right side, so

$$\frac{\prod(-s)}{2\pi i} \cdot \int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1} \cdot \frac{dx}{x} = \frac{\prod(-s) \cdot \prod(s-1) \cdot s}{\pi s} \cdot \sin(\pi s) \cdot \zeta(s) \tag{8.12}$$

Next, the identity (See [39], p.8, Eq.5; and pp.421-425) of $\prod(s) = s \cdot \prod(s-1)$ is substituted into Eq. 8.12, resulting in

$$\frac{\prod(-s)}{2\pi i} \cdot \int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1} \cdot \frac{dx}{x} = \frac{\prod(-s) \cdot \prod(s)}{\pi s} \cdot \sin(\pi s) \cdot \zeta(s) \tag{8.13}$$

Finally, the identity (See [39], p.8, Eq. 6) of $\sin(\pi s) = \pi s \cdot \left[\prod(-s)\prod(s)\right]^{-1}$ is substituted into the right side of Eq. 8.13, resulting in

$$\zeta(s) = \frac{\prod(-s)}{2\pi i} \cdot \int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1} \cdot \frac{dx}{x} \tag{8.14}$$

This is the Riemann Zeta Function. (See [39], pp.10-11. Eq.3). However, as a reminder, in regards to the three terms of the Hankel contour (See [39], pp.10-11; [127], p.244-6, §12.22, citing [55], p.7) shown in Equation 8.5:

$$\int_{+\infty}^{+\infty} \frac{(-x)^s}{(e^x-1)} \cdot \frac{dx}{x} = \int_{+\infty}^{\delta} \frac{(-x)^s}{(e^x-1)} \cdot \frac{dx}{x} \quad + \int_{|z|=\delta} \frac{(-x)^s}{(e^x-1)} \cdot \frac{dx}{x} + \int_{\delta}^{+\infty} \frac{(-x)^s}{(e^x-1)} \cdot \frac{dx}{x} \tag{8.15}$$

Edwards states (See [39], p.10):

> [T]hus $(-x)^s$ is not defined on the positive Real axis and, strictly speaking, the path of integration must be taken to be slightly above the Real axis as it descends from $+\infty$ to 0 and slightly below the Real axis as it goes from 0 back to $+\infty$.

Riemann copied this solution directly from Hankel's derivation of the Gamma function $\Gamma(s)$. (See [127], pp.244-5,266). Riemann uses the Hankel contour in the derivation of the Riemann Zeta function, but neither Hankel nor Riemann (nor anyone else for that matter) resolve the follow-up question to Edwards's comment: what is the mathematical basis for the "trick" of equating the branch cut of $f(x) = \log(-x)$ to the limit of the Hankel Contour (as the Hankel contour approaches the branch cut)?

As every high school algebra student should know, the logarithm of a non-positive Real number is undefined. So, by definition, all points on the branch cut have no defined value. Equating the branch cut to the limit of the Hankel contour is a *de facto* assignment of values to points that, by definition, must have *no* value. Remember that in for $x \in \mathbb{R}$, there are *no values* of x that result in the exponential function $f(x) = \exp x$ being a non-positive number.

Hankel (See [55]), Riemann (See [96]), and Edwards [39] all fail to provide any mathematically valid reason for equating the "strictly speaking" interpretation of the "first contour" on the left side of Eq. 8.15 to the "non-strictly speaking" interpretation of the "first contour" on the right side of Eq. 8.15. Again, the points on the contour represented by the left side of the equation have no defined value, and thus are also non-holomorphic, but the points on the Hankel contour represented by the right side of the equation, ("slightly above the Real axis as it descends from $+\infty$ to 0 and slightly below the Real axis as it goes from 0 back to $+\infty$"), do have defined values

Fortunately, in contrast to Riemann [96] and Edwards [39], Whittaker et al. [127] *does* provide a reason: *the path equivalence corollary of Cauchy's integral theorem* is given as the mathematical basis for equating the Hankel contour to the branch cut. (See [127], pp.85-7, 244, §5.2, Cor 1). However, the basis provided is *neither mathematically nor logically valid.* The Hankel contour and the branch cut contradict the prerequisites of the Cauchy integral theorem (See [127], p.85), and of its corollary (See [127], p.87).

These contradictions invalidate the derivation of Riemann's version of the Zeta function (in logics with LNC, and in mathematics).

Here are the reasons why the Hankel contour contradicts the prerequisites of Cauchy's integral theorem. Cauchy's integral theorem states that if function $f(z)$ of complex variable z is "holomorphic" (complex differentiable) at all points *on* a simple closed curve ("contour") C, and if $f(z)$ is also holomorphic at all points *inside* the contour, then the contour integral of $f(z)$ is equal to zero (See [127], p.85):

$$\int_{(C)} f(z) \cdot dz = 0 \tag{8.16}$$

The path equivalence corollary of Cauchy's integral theorem states the following: (See [127], p.87, Cor 1)

(1) If there exist four distinct points (z_0, Z, A, and B) on the Cartesian plane (that represents the complex domain), and the two points z_0 and Z are connected by two distinct paths z_0AZ and z_0BZ (one path going through A, the other path going through B), and

(2) if function $f(z)$ of complex variable z is holomorphic at all points on these two distinct paths z_0AZ and z_0BZ, and $f(z)$ is holomorphic at all points enclosed by these two paths,

(3) then any line integral connecting the two points z_0 and Z inside this region (bounded by z_0AZ and z_0BZ) has the same value, regardless of whether the path of integration is z_0AZ, or z_0BZ, or any other path disposed between z_0AZ and z_0BZ.

Riemann invalidly used Cauchy's integral theorem to assign, to the branch cut, the value of the limit of the Hankel contour (as the Hankel contour approaches the branch cut of $f(x) = \log(-x)$ at $x \in \mathbb{C}$).

But by definition, $\log(-x)$ has no value (and thus is non-holomorphic) at all points on half-axis $x \in \mathbb{R}, x \geq 0$. The geometric proof that $\log(-s)$ is non-holomorphic at all points on half-axis $s \geq 0$ is as follows: In the Cartesian plane, the first derivative of $f(x) = \log(-x)$, for $x \in \mathbb{R}$ at a value of x, is represented by the slope of the line tangent to $f(x)$ at x. However, $f(x)$ has no values at $x \geq 0$, so its first derivative cannot have any values at $x \geq 0$.

(Note however, that for $s \in \mathbb{C}$, there exists a definition for the branch cut of $f(s) = \log(-s)$ that assigns to it the values of $f(s) = \log(|s|)$ (and remains undefined at $s = 0$). This definition contradicts the definition of logarithms of Real numbers. (See [3])).

Moreover, the Hankel contour is either open, or closed, at $x = +\infty$ (the latter enclosing non-holomorphic points). In both cases, the Hankel contour violates prerequisites of Cauchy's integral theorem.

If the Hankel contour is open, the Cauchy integral theorem cannot be used, because it only applies to closed contours. In the alternative, if the Hankel contour is indeed closed at $+\infty$ on the branch cut, as assumed by Riemann, (See [127], p.245), then the Hankel contour still contradicts the requirements of the Cauchy integral theorem. This is because the closed Hankel contour encloses the entire branch cut of $f(z)$, and the branch cut consists entirely of non-holomorphic points. Also, there would be a non-holomorphic point on the Hankel contour itself, at the point where it intersects the branch cut at $+\infty$ on the Real axis. These reasons disqualify the use of the Cauchy integral theorem with the Hankel contour.

For these reasons it is not valid to use the Cauchy integral theorem's path equivalence corollary to find the limit of the Hankel contour, as the Hankel contour approaches the branch cut of $f(x) = \log(-x)$ at $x \in \mathbb{C}$. So the derivation of Riemann's Zeta function violates the LNC.

9 The "Calculated Zeros" are not of Riemann's Zeta Function

The so-called "calculated zeros" of Riemann's Zeta function are actually zeros of other functions (which are approximations of Riemann's Zeta function), that assume that Riemann's Zeta function is true. For example, before attempting to calculate the "zeros" of Riemann's Zeta function. Odlyzko (See [83], citing [39], [66], [113]) assumes that "by analytic continuation [Zeta] can be extended to an analytic function of s for all $s \neq 1$".

However, as discussed above, assuming that the "analytic continuation" of the Zeta function is true creates a contradiction in the half-plane where the Dirichlet series in divergent, thus generating a paradox. In classical and intuitionistic logics, this violation of the LNC triggers ECQ, and thus renders "trivially true" (and *de facto* invalidates) everything that is built on the assumption (that uses the Euler-Maclaurin Formula).

Odlyzko (See [83], p.798) then discloses the following in regards to calculating the "zeros" of Riemann's Zeta function, not by use of Riemann's Zeta function, but by use of the Euler-Maclaurin formula: "The [Dirichlet series of the Zeta function] suggests the idea of using the Euler-Maclaurin summation formula [citing Abramowitz et al.'s [1] Equation 23.1.30] to evaluate [Riemann's Zeta function]", and "All calculations of zeros of the zeta function that were published before 1930 relied on this method. Its advantages include the ease of estimating the error term."

However, the use of the Euler-Maclaurin summation formula fails at Odlyzko et al.'s [83] first sentence: "[The Dirichlet series definition of the Zeta function] suggests the idea of using the Euler-Maclaurin summation formula ... to evaluate [Riemann's Zeta function]".

Apostol (See [4], p.409) discloses why the Euler-Maclaurin summation formula cannot be used to calculate "zeros" of the Dirichlet series of the Zeta function in half-plane $\mathrm{Re}(s) \leq 1$. As Apostol [4] indicates: "[t]he integral test for convergence of infinite series compares a finite sum $\sum_{k=1}^{n} f(k)$ and an integral $\int_1^n f(x)\,dx$ where f is positive and strictly decreasing", and "[t]he difference between a sum and an integral can be represented geometrically". As discussed in the present paper, at all values of s in half-plane $\mathrm{Re}(s) \leq 1$, the Dirichlet series of the Zeta function *fails* the integral test for convergence of infinite series. This is sufficient reason to disqualify the use of the Euler-Maclaurin summation formula to calculate "zeros" of the Zeta function in half-plane $\mathrm{Re}(s) \leq 1$.

Also Odlyzko (See [83], p.798) discusses the use of the Riemann-Siegel formula for calculating the "zeros" of Riemann's Zeta function: "The Riemann-Siegel formula is the fastest method for computing the zeta function to moderate accuracy that is currently known, and has been used for all large scale computations since the 1930s."

However, the Riemann–Siegel formula is an approximation of the zeta function by a sum of two finite Dirichlet series. But summing two finite series results in a finite value, and cannot be a valid "approximation" of any infinite series (e.g. Dirichlet series of the Zeta function). So in logics with LNC, the Riemann–Siegel formula is an invalid approximation of the Zeta function.

Moreover, Edwards (See [39], pp.137-8, §7.2, citing pp.12-15, §1.5) confirms that the "functional equation" of the Zeta function is used in the derivation of the Riemann-Siegel formula (and the "functional equation" of the Zeta function assumes that the "analytic continuation" of the Dirichlet series of the Zeta function is true).

Unfortunately, the "functional equation" of Riemann's Zeta function is *not* valid in logics with LNC, because the "analytic continuation" of the Dirichlet series of the Zeta function is not valid in those logics. The Dirichlet series of the Zeta function is proven to be divergent throughout half-plane $\mathrm{Re}(s) \leq 1$, so the "analytic continuation" of the Dirichlet series of the Zeta function violates the LNC.

Odlyzko (See [83], p.800, Eq.1.7; pp.803-804, §3) also discloses other methods for calculating the "zeros" of Riemann's Zeta function, including a method by Turing [118], a method using Fast Fourier Transforms, etc. See also Gourdon (See [49]).

However, these other methods share the same problems as the Euler-Maclaurin and

Riemann-Siegel formulas. All of these formulas are approximations of Riemann's Zeta function, which is *invalid* in half-plane $\text{Re}(s) \leq 1$ in logics with LNC. Therefore, the "functional equation" of Riemann's Zeta function must also be invalid in logics with LNC. All zeros calculated by these "approximations" falsely assume that "analytic continuation" of the Dirichlet series of the Zeta function is true.

10 The Riemann Hypothesis is Either Trivially True or a Paradox

The Riemann Hypothesis states that "all non-trivial zeros of Riemann's Zeta function are on the critical line $\text{Re}(s) = 0.5$". Turing (See [119], p.165) argued that the Riemann Hypothesis is a "number-theoretic" problem. This classification is incorrect, and may be a contributing factor as to why the problem has remained unsolved for so long. In fact, the Riemann hypothesis is a logic problem.

As discussed in the previous sections of this paper, the Dirichlet series of the Zeta function is proven to be divergent throughout half-plane $\text{Re}(s) \leq 1$. Therefore, the Riemann Zeta function's claim to be convergent in that same half-plane violates the LNC (if it is true). Then, due to ECQ, the Riemann hypothesis is "trivially true".

However, if there is an error in the derivation of Riemann's Zeta function, then it is false. And consequently, the Zeta function is exclusively defined by the Dirichlet series, which has *no zeros* and *no poles.* None of the zeros assumed by the Riemann hypothesis would exist. Also, Riemann's functional equation of the Zeta function (See [46], p.60) would be invalidated by the proven divergence of the Dirichlet series throughout half-plane $\text{Re}(s) \leq 1$.

The non-existent zeros of the Riemann hypothesis constitute "vacuous subjects" of a proposition, just like Russell's (See [98]) famous example of "the present King of France is bald", and other examples discussed by Frege (See [41]) and Strawson (See [106]). Another term for this is "reference failure" (See [54], pp.14-15).

In both classical and intuitionistic logics, "material implication" is the conditional "if p then q". It defines "if p then q", (whose sequent is $p \to q$) as being logically equivalent to the sequent $\neg(p \wedge \neg q)$. So in classical and intuitionistic logics, the material implication $(p \to q)$ is always "true" when p is "false" ([109], pp.25-26; [50], p.329, citing [97]). The same result is obtained from ECQ: a false statement leads to "deductive explosion". (Note that by De Morgan's Laws, which are accepted in classical logic but not in intuitionistic logic, $\neg(p \wedge \neg q)$ is further equivalent to $(\neg p \vee q)$.)

When the material conditional is applied to the Riemann hypothesis, it holds that the Riemann hypothesis is true, because the Riemann hypothesis states:

If $\zeta(s) = 0$, then $\text{Re}(s) = 0.5$.

and because the Zeta function, as defined by Dirichlet series, has no zeros. So $\zeta(s) = 0$ is false, and therefore the Riemann hypothesis is true.

However, according to material implication, the following "anti-Riemann hypothesis" is true too. It states:

If $\zeta(s) = 0$, then $\text{Re}(s) \neq 0.5$.

If the Zeta function has no zeros, then the "anti-Riemann hypothesis" is true. Yet it is paradoxical for the Riemann hypothesis and the "anti-Riemann hypothesis" to both be true.

The end result from the Riemann hypothesis being a paradox, due to LNC and ECQ, is that it causes all conjectures that assume it is true to be "trivially true". In certain 3VLs, the Riemann hypothesis is a paradox that has the third truth-value.

These results (except for the result of "trivial truth") are inconsistent with the presumed-proven analogues of the Riemann hypothesis. See e.g. (1) Hasse's proof of the Riemann hypothesis for elliptic curves of genus 1 (See [78], p.3), (2) Deligne's proof of the Weil conjecture III (See [78], p.49), and (3) Weil's proof of the Riemann hypothesis for elliptic curves of arbitrary genus g (See [68], pp.4-5).

All of these proofs include a violation of the LNC, either because: (1) Riemann's Zeta function is true, and contradicts the true Dirichlet series of Zeta, or (2) the Riemann's Zeta function is false, so the Zeta function has no poles or zeros, no "functional equation", etc.

11 Further Research: Paradox-Tolerant Foundations of Mathematics and Physics?

In regards to a 3VL or 4VL as a possible foundation logic of mathematics, instead of classical logic, Hazen and Dunn state that if one tries to formulate a second-order logic of a 3VL or 4VL, the resultant system collapses to its classical counterpart. (See [61], p.507, citing [38], p.261, citing [37]). Moreover, another Hazen article states that: "it will be extremely difficult to appeal to [Priest's] second-order LP for the purposes that its proponents advocate, until some deep, intricate, and hitherto unarticulated metaphysical advances are made." (See [62]).

As an aside, the author notes that in the twenty years since the initial announcement of the Millenium Problems, *none* of the official descriptions of the problems have ever listed "paradox" as a possible answer, nor has the mathematical community argued that it should be listed as a possible answer. The fact that a central problem of mathematics (the Riemann hypothesis) is either a paradox, or "trivially true" due to paradox, demonstrates that the mathematical community has still not internalized the results of Gödel's famous work.

Finally, in regards to the quantum physicists, they *know* their theories contain contradictions (See [84], §26.9, p.678), but these theories are built on mathematics that has the LNC as an axiom. Thus, these models are logically and mathematically invalid, and must be replaced with valid models.

For example, as is well known, the "Schrödinger cat" thought experiment is a paradox according to LNC. Just as we do not currently know whether or not there will be a sea battle tomorrow (but the admiral issuing the orders does), we also do not know whether the cat is alive or dead, until the box is opened. In certain three-valued logics (e.g. Łukaszewicz's 3VL, Bochvar's 3VL, Priest's *LP*), this scenario corresponds to a third truth-value. Other logical foundations for the "Schrödinger cat" problem can be found in the solutions to Aristotle's "future contingents" problem, one of which is Łukaszewicz's 3VL (See [6], *De interpretatione* §9; and [131], §6). Yet another possible solution is Bayesian probability.

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