# Integrals and Series

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May 27 , 2019

Abstract
This note present some elementary integrals and series.

Resumen
Esta nota presenta algunas integrales y series elementales

## 1. Introducción

En la tabla Gradshteyn & Ryzhik (seventh edition, p.116,hyperbolic functions,formula 2.423.9), aparece la integral:

$$\int \frac{dx}{\cosh x} = 2\tan^{-1}\left(e^{x}\right) \tag{1}$$

Utilizando (1) podemos obtener interesantes integrales y series.

## 2. Integrales y Series

Entry 1. Si $a \geq 0$ , entonces

$$\int_{0}^{a} \frac{dx}{\cosh x} = 2\tan^{-1}\left(e^{a}\right) - \frac{\pi}{2} = \frac{\pi}{2} - 2\tan^{-1}\left(e^{-a}\right) \tag{2}$$

Entry 2. Si $a \geq 0$ , entonces

$$\frac{\pi}{2} - 2\tan^{-1}\left(e^{-a}\right) = a\,\mathrm{sech}\,a + 1 - \mathrm{sech}\,a\;\ln \mathrm{sech}\,a + \int_{\mathrm{sech}\,a}^{1} \ln\left(1+\sqrt{1-x^{2}}\right)dx \tag{3}$$

Proof. Entry 1. $\Rightarrow$ Entry 2.

Entry 3. Si $0 \leq b \leq 1$ , entonces

$$\frac{\pi}{2} - 2\tan^{-1}\left(\frac{b}{1+\sqrt{1-b^{2}}}\right) = b\ln\left(1+\sqrt{1-b^{2}}\right) + 1 - b + \int_{b}^{1} \ln\left(1+\sqrt{1-x^{2}}\right)dx \tag{4}$$

Proof. Entry 2. $\Rightarrow$ Entry 3. ( $b = \mathrm{sech}\,a$ ).

Entry 4.

$$\int_{\sqrt{3}/2}^{1} \ln\left(1+\sqrt{1-x^{2}}\right)dx = \frac{\pi}{6} - \frac{\sqrt{3}}{2}\ln\frac{3}{2} - 1 + \frac{\sqrt{3}}{2} \tag{5}$$

$$\int_{1/\sqrt{2}}^{1} \ln\left(1+\sqrt{1-x^2}\right)dx = \frac{\pi}{4} - \frac{1}{\sqrt{2}}\ln\left(1+\frac{1}{\sqrt{2}}\right) - 1 + \frac{1}{\sqrt{2}} \tag{6}$$

$$\int_{1/2}^{1} \ln\left(1+\sqrt{1-x^2}\right)dx = \frac{\pi}{3} - \frac{1}{2}\ln\left(1+\frac{\sqrt{3}}{2}\right) - \frac{1}{2} \tag{7}$$

$$\int_{0}^{1} \ln\left(1+\sqrt{1-x^2}\right)dx = \frac{\pi}{2} - 1 \tag{8}$$

Proof. Entry 3. $\Rightarrow$ Entry 4.

Entry 5.

$$\frac{\pi}{2} + 1 = -\int_{0}^{1} \ln\left(1-\sqrt{1-x^2}\right)dx \tag{9}$$

Proof. (8)$\Rightarrow$ (9).

Entry 6.

$$\pi = \int_{0}^{1} \ln\left(\frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}\right)dx \tag{10}$$

Proof. (8) $\wedge$ (9) $\Rightarrow$ (10).

Entry 7.

$$\frac{\pi}{2} - 1 = \int_{0}^{1} \frac{x^2}{1-x^2+\sqrt{1-x^2}}\,dx \tag{11}$$

$$\frac{\pi}{2} + 1 = \int_{0}^{1} \frac{x^2}{\sqrt{1-x^2}+x^2-1}\,dx \tag{12}$$

Proof. (8)$\Rightarrow$ (11),(9)$\Rightarrow$ (12),(integración por partes).

Entry 8.

$$\frac{\pi}{2} - 1 = \int_{0}^{\infty} \left(1-\frac{\sqrt{2x+x^2}}{1+x}\right)dx = \int_{0}^{\infty} \frac{dx}{(1+x)\left(1+x+\sqrt{2x+x^2}\right)} = \int_{1}^{\infty} \frac{dx}{x\left(x+\sqrt{x^2-1}\right)} \tag{13}$$

Proof. Entry (7) $\Rightarrow$ Entry (8) .

La siguiente fórmula se utiliza en Entry (9) y Entry (10):

$$\ln\left(1+\sqrt{1-x^2}\right) = \ln 2 - \sum_{n=1}^{\infty} \binom{2n}{n} \frac{2^{-2n}}{2n} x^{2n} \quad ,|x| \le 1 \tag{14}$$

Observación: ver [2], p.54,fórmula 1.515.1.

Entry 9.

$$\sum_{n=1}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{2n(2n+1)}\left(\frac{3}{4}\right)^{n}=1-\ln 3+2\ln 2-\frac{2\pi}{3\sqrt{3}} \tag{15}$$

$$\sum_{n=1}^{\infty}\binom{2n}{n}\frac{2^{-3n}}{2n(2n+1)}=1+\ln\left(4-2\sqrt{2}\right)-\frac{\pi}{2\sqrt{2}} \tag{16}$$

$$\sum_{n=1}^{\infty}\binom{2n}{n}\frac{2^{-4n}}{2n(2n+1)}=1+\ln\left(8-4\sqrt{3}\right)-\frac{\pi}{3} \tag{17}$$

Proof. Entry (3) , Entry (4) y (14) $\Rightarrow$ Entry (9).

Entry 10.

$$\begin{aligned}&\sum_{n=1}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{2n(2n+1)}\left(\left(\frac{\sqrt{3}}{2}\right)^{2n+1}-\left(\frac{\sqrt{2}}{2}\right)^{2n+1}\right)=\\&=\frac{1}{\sqrt{2}}\ln\left(1+\sqrt{2}\right)-\frac{\sqrt{2}}{12}(6+9\ln 2)+\frac{\sqrt{3}}{12}(6-6\ln 3+12\ln 2)-\frac{\pi}{12}\end{aligned} \tag{18}$$

$$\begin{aligned}&\sum_{n=1}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{2n(2n+1)}\left(\left(\frac{\sqrt{3}}{2}\right)^{2n+1}-\left(\frac{1}{2}\right)^{2n+1}\right)=\\&=\frac{1}{2}\ln\left(2+\sqrt{3}\right)+\frac{\sqrt{3}}{6}(3-3\ln 3+6\ln 2)-\ln 2-\frac{1}{2}-\frac{\pi}{6}\end{aligned} \tag{19}$$

$$\begin{aligned}&\sum_{n=1}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{2n(2n+1)}\left(\left(\frac{\sqrt{2}}{2}\right)^{2n+1}-\left(\frac{1}{2}\right)^{2n+1}\right)=\\&=\frac{1}{2}\ln\left(2+\sqrt{3}\right)-\frac{\sqrt{2}}{2}\ln\left(1+\sqrt{2}\right)+\frac{\sqrt{2}}{12}(6+9\ln 2)-\ln 2-\frac{1}{2}-\frac{\pi}{12}\end{aligned} \tag{20}$$

Proof. Entry (4) y (14) $\Rightarrow$ Entry (10).

Entry 11. Si $n=0,1,2,3,\ldots$ , entonces

$$\frac{\pi}{2}\binom{2n}{n}2^{-2n}-\frac{1}{2n+1}=\int_0^1\ln\left(1+\sqrt{1-x^{2/(2n+1)}}\right)dx \tag{21}$$

Proof. En la referencia (2), p.372, fórmula 3.512.2:

$$\int_0^{\infty}\frac{\sinh^{\mu}x}{\cosh^{\nu}x}dx=\frac{1}{2}B\left(\frac{\mu+1}{2},\frac{\nu-\mu}{2}\right)\quad ,\operatorname{Re}\mu>-1,\operatorname{Re}(\mu-\nu)<0 \tag{22}$$

Con $\mu=0,\nu=2n+1,n=0,1,2,3,\ldots$ , se tiene

$$\int_0^{\infty}\operatorname{sech}^{2n+1}x\,dx=\frac{1}{2}B\left(\frac{1}{2},n+\frac{1}{2}\right)=\frac{\pi}{2}\binom{2n}{n}2^{-2n} \tag{23}$$

(23) $\Rightarrow$ (21).

Observación: $B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ es la función Beta.

Entry 12.

$$\frac{\pi}{4}-\frac{1}{3}=\int_0^1 \ln\left(1+\sqrt{1-\sqrt[3]{x^2}}\right)dx \tag{24}$$

Proof. Entry (11) con $n=1$ .

Entry 13. Si $n=0,1,2,3,\dots$ , entonces

$$\frac{\pi}{2}\binom{2n}{n}2^{-2n}-\frac{1}{2n+1}=\ln 2-(2n+1)\sum_{k=1}^{\infty}\binom{2k}{k}\frac{2^{-2k}}{2k(2k+2n+1)} \tag{25}$$

Proof. (14) y (21) $\Rightarrow$ (25) .

Entry 14. Si $n=0,1,2,3,\dots$ , entonces

$$\pi\binom{2n}{n}2^{-2n-1}(2n+1)-1=\int_0^1 \frac{x^{2/(2n+1)}}{1-x^{2/(2n+1)}+\sqrt{1-x^{2/(2n+1)}}}dx \tag{26}$$

Proof. Entry (11) $\Rightarrow$ Entry (14).

Entry 15. Si $n=0,1,2,3,\dots$ , entonces

$$\pi\binom{2n}{n}2^{-2n-1}-\frac{1}{2n+1}=\int_0^1 \frac{x^{2n+2}}{1-x^2+\sqrt{1-x^2}}dx \tag{27}$$

Proof. Entry (14) $\Rightarrow$ Entry (15).

Entry 16. Si $n=0,1,2,3,\dots$ , entonces

$$\pi\binom{2n}{n}2^{-2n-1}(2n+1)-1=\int_0^{\infty}\left(1-\sqrt{\left(\frac{x(2+x)}{(1+x)^2}\right)^{2n+1}}\right)dx \tag{28}$$

Proof. Entry (14) $\Rightarrow$ Entry (16).

Entry 17.

$$\pi\left(\frac{1}{2\sqrt{2}}-\frac{1}{4}\right)=\int_0^1 \frac{x^2}{(1+x^2)\left(1-x^2+\sqrt{1-x^2}\right)}dx \tag{29}$$

Proof. Entry (15) $\Rightarrow$ Entry (16).

Entry 18.

$$\frac{\pi}{2\sqrt{2}}=\int_0^1 \frac{1}{(1+x^2)\sqrt{1-x^2}}dx \tag{30}$$

Proof. Entry (17) $\Rightarrow$ Entry (18).

Entry 19. Si $n=0,1,2,3,\ldots;0\le a\le 1$ , entonces

$$\pi\binom{2n}{n}2^{-2n-1}-\frac{1}{2n+1}=a\ln 2-(2n+1)a\sum_{k=1}^{\infty}\binom{2k}{k}\frac{2^{-2k}a^{2k/(2n+1)}}{2k(2k+2n+1)}+ \\ +\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\left(1-a^{2/(2n+1)}\right)^{k/2}\left(F\left(-\frac{2n+1}{2},\frac{k}{2};\frac{k}{2}+1;1-a^{2/(2n+1)}\right)-a\right) \tag{31}$$

Proof. Entry (11) $\Rightarrow$ Entry (19).

Observación: $F(a,b;c;x)={}_2F_1(a,b;c;x)$ es la función hipergeométrica de Gauss.

Entry 20. Si $n=0,1,2,3,\ldots;0\le a\le 1$, entonces

$$\pi\binom{2n}{n}2^{-2n-1}-\frac{1}{2n+1}=a\ln 2-a\ln\left(1+\sqrt{1-a^{2/(2n+1)}}\right) \\ -(2n+1)a\sum_{k=1}^{\infty}\binom{2k}{k}\frac{2^{-2k}a^{2k/(2n+1)}}{2k(2k+2n+1)}+ \\ +\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\left(1-a^{2/(2n+1)}\right)^{k/2}F\left(-\frac{2n+1}{2},\frac{k}{2};\frac{k}{2}+1;1-a^{2/(2n+1)}\right) \tag{32}$$

Proof. Entry (19) $\Rightarrow$ Entry (20).

Observación: $\pi=4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}=3.141592\ldots$ .

# Referencias


1. G. Boros and V. Moll : Irresistible Integrals: Symbolics, Analysis, and Experiments in the Evaluation of Integrals, Cambridge University Press, Cambridge, 2004.
2. I.S. Gradshteyn and I.M. Ryzhik : Table of Integrals, Series, and products. Edited by A. Jeffrey and D. Zwillinger. Academic Press, New York, 7th edition, 2007.