From Kirchhoff to Faraday

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Kirchhoff's law of voltages is a standard in the education of students learning circuits for the first time. Even when this is done in physics classes, there is no attention paid to the origins of this law. This is a disservice to the students, because this robs them of an opportunity to see a real world application of conservation laws that they don't need to take on faith. Furthermore, if we introduce a magnetic field to the systems, things get more interesting. This allows the student to see a firsthand application and emergences of a fundamental property of the universe and its effects on everyday things, Faraday's law. In this work we will be exploring these two phenomena from first principles, and showing how Kirchhoff is a limiting case of Faraday's Law.

Kirchhoff from Conservation of Energy

The forces that a charged particles experiences in an electric and a magnetic field are given by the Lorentz Force[3]:

$$\vec{F} = q\vec{E} + q(\vec{v} \times \vec{B}) \quad (1)$$

The Work done on this particle to move it from a to b is

$$W = \int_a^b \vec{F} \cdot d\vec{l} \quad (2)$$

Applying Eq. 1 to Eq.2 gives

$$W = -\int_a^b [q\vec{E} + q(\vec{v} \times \vec{B})] \cdot d\vec{l} \quad (3)$$

When looking at the effects separately we recall that since magnetic fields do not do work on charged particles. From this we see that the integral simplifies to:

$$W = -\int_a^b q\vec{E} \cdot d\vec{l} \quad (4)$$

To keep things clear we will look at the one dimensional case. Just know that these results are true in general. So, recall that the electric field is defined in terms of a changing potential

$$\vec{E} = -\frac{\partial V}{\partial x}\hat{x} \quad (5)$$

Applying Eq.5 to Eq.4 and noting $d\vec{l} = \hat{x}dx$ we get:

$$W = \int_a^b q \frac{\partial V}{\partial x} \hat{x} \cdot \hat{x} dx = q \int_a^b \frac{\partial V}{\partial x} dx = q[V(b) - V(a)] \quad (6)$$

For a shorthand we will use $V_{ab} = V(b)-V(a)$. Thus we have

$$V_{ab} = \frac{W}{q} = -\int_a^b \vec{E} \cdot d\vec{l} \qquad (7)$$

There is nothing new here. So, let's take this a bit further. Now we are going to divide the interval [a,b] into the partition

$$[a, b] = \{x_i\}_{i=0}^{n} \qquad (8)$$

With $a = x_0$ and $b = x_n$. This leads to the expansion

$$V_{ab} = -\sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} \vec{E} \cdot d\vec{l} \qquad (9)$$

To simplify the notation, look at the jth term

$$V_j = -\int_{x_{j-1}}^{x_j} \vec{E} \cdot d\vec{l} \qquad (10)$$

Leading to the simpler form

$$V_{ab} = \sum_{j=1}^{n} V_j \qquad (11)$$

So far this solution has assumed an open path. What happens if we use this same process for a closed current path? Let's set a = b. Now we have no separation between the start and end points. This gives the result that

$$V_{ab} = \sum_{j=1}^{n} V_j = 0 \qquad (12)$$

This gives us Kirchhoff's law as a consequence of conservation of energy.

The above derivation was done with the assumption that because the magnetic field does not do work on charged particles we should ignore them all together. However, we know that we can induce an electric field with a changing magnetic flux. So, the natural question should then be, what affects would that have on Kirchhoff? More generally how would this affect electric and electronic circuits? For the purpose of this analysis we are going to look at an electronic circuit as a single closed current loop.

We know that the magnetic flux of a system is given by

$$\Phi_B = \int \vec{B} \cdot d\vec{a} \qquad (13)$$

We know from Faraday that if we take this flux a change it over time we get an induced electromotive force (emf) in a direction to oppose the change in the magnetic field. Summarized as follows

$$\varepsilon = -\frac{d\Phi_B}{dt} \qquad (14)$$

Where epsilon is the emf discussed. We know that the emf is just the potential difference between two points, giving:

$$\varepsilon = -\int_a^b \vec{E} \cdot d\vec{l} \qquad (15)$$

Combining Eq.15 with Eq.9-11 we have

$$\varepsilon = \sum_{j=1}^{n} V_j \qquad (16)$$

So then combining Eq.16 with Eq.14 we arrive at the final result

$$\sum_{j=1}^{n} V_j = -\frac{d\Phi_B}{dt} \qquad (17)$$

This show that if a circuit is in the presents of a changing magnetic flux then the total potential difference of the circuit no longer vanishes. Instead we get a measurable potential difference. There is a beautiful video of Walter Lewin demonstrating this phenomenon.[1]

The most interesting implications of this result are a) Kirchhoff is a limiting case of Faraday when there is either a constant or non-existent magnetic flux, and b) We have demonstrated earlier that Kirchhoff is the result of conservation of energy. Since Kirchhoff is the limiting case of Faraday we can conclude that Faraday is the result of non-conservative energy. So this means that there are non-conservative forces. Other than just measuring the voltage of the components of a circuit, can we measure this phenomenon in any other way? Yes.

We ultimately know that for the universe as a whole energy must be conserved. So, what is the source of the balancing energy for this system? It turns out that there is energy radiated away in the form of electromagnetic waves. These can be measured experimentally.[2]

References:

1. https://www.youtube.com/watch?v=nGQbA2jwkWI&list=PLyQSN7X0ro2314mKyUiOILaOC2hk6Pc3j&index=17
2. Wood, Rottman, Barrera, Am. J. Phys., Vol. 72, No. 3, March 2004
3. Griffiths, Introduction to Electrodynamics, 3rd Ed., PHI, 1999