MSC 2010: 31A05, 30C15 УДК: 517.574, 517.547

Affine Balayage of Measures in Domains of the Complex Plane with Applications to Holomorphic Functions

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Abstract

Let $u \not\equiv -\infty$ and $M \not\equiv -\infty$ are two subharmonic functions in a domain D in the complex plane $\mathbb{C}$. We investigate two related but different problems. The first is to find the conditions on the Riesz measures υ_u and μ_M of functions u and M respectively under which there exists a subharmonic function $h \not\equiv -\infty$ on D such that $u+h \leq M$. The second is the same question, but for a harmonic function h on D. The answers to these questions are given in terms of the special affine balayage of measures introduced in our recent previous works. Applications of this technique concern the description of distribution of zeros for holomorphic functions f on the domain D satisfying the restriction $|f| \leq \exp M$.

Keywords: subharmonic function, Riesz measure, balayage, holomorphic function, sequence of zeros, uniqueness set, weighted class

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We have are considered in the survey [5] general concepts of *affine balayage.* In this article we deal with a particular case of such balayage with respect to special classes of test subharmonic functions. Here we give some results from [4], [6], [3] as well as their generalizations.

As usual, $\mathbb{N} := \{1, 2, \dots\}$, $\mathbb{R}$ and $\mathbb{C}$ are the sets of all *natural, real* and *complex* numbers, respectively. For the *real line* $\mathbb{R}$ with *Euclidean norm-module* $|\cdot|$,

$$\mathbb{R}_{-\infty} := \{-\infty\} \cup \mathbb{R}, \quad \mathbb{R}_{+\infty} := \mathbb{R} \cup \{+\infty\}, \ |\pm\infty| := +\infty; \quad \mathbb{R}_{\pm\infty} := \mathbb{R}_{-\infty} \cup \mathbb{R}_{+\infty}$$

is *extended real line* in the end topology with two ends $\pm\infty$, with the order relation $\leq$ on $\mathbb{R}$ complemented by the inequalities $-\infty \leq x \leq +\infty$ for $x \in \mathbb{R}_{\pm\infty}$, with the *positive real axis*

$$\begin{gathered}\mathbb{R}^+ := \{x \in \mathbb{R} \colon x \geq 0\}, \quad x^+ := \max\{0, x\}, \quad x^- := (-x)^+, \quad \text{for } x \in \mathbb{R}_{\pm\infty},\\ S^+ := \{x \geq 0 \colon x \in S\}, \quad S_* := S \setminus \{0\} \quad \text{for } S \subset \mathbb{R}_{\pm\infty}, \quad \mathbb{R}^+_* := (\mathbb{R}^+)_*,\\ x \cdot (\pm\infty) := \pm\infty =: (-x) \cdot (\mp\infty) \quad \text{for } x \in \mathbb{R}^+_* \cup (+\infty),\\ \frac{x}{\pm\infty} := 0 \quad \text{for } x \in \mathbb{R}, \quad \text{but } 0 \cdot (\pm\infty) := 0\end{gathered}$$

unless otherwise specified. An open connected (sub-)set of $\mathbb{R}_{\pm\infty}$ is a *(sub-)interval* of $\mathbb{R}_{\pm\infty}$. The *Alexandroff* one-point *compactification* of $\mathbb{C}$ is denoted by $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ [7].

The same symbol 0 is used, depending on the context, to denote the number zero, the origin, zero vector, zero function, zero measure, etc. Given $z \in \mathbb{C}$ and $r \in \mathbb{R}_{+\infty}$, we set

$$\begin{aligned} D(z,r) &:= \{z' \in \mathbb{C} \colon |z'-z| < r\}, \quad \overline{D}(z,r) := \{z' \in \mathbb{C} \colon |z'-z| \leq r\}, \\ D(\infty,r) &:= \{z \in \mathbb{C}_\infty \colon |z| > 1/r\}, \quad \overline{D}(\infty,r) := \{z \in \mathbb{C}_\infty \colon |z| \geq 1/r\}, \\ D(r) &:= D(0,r), \quad \mathbb{D} := D(0,1), \quad \overline{D}(r) := \overline{D}(0,r), \quad \overline{\mathbb{D}} := \overline{D}(0,1). \end{aligned}$$

Thus, the basis of open (respectively closed) neighborhood of the point $z \in \mathbb{C}_\infty$ is *open* (respectively *closed*) *disks* $D(z,r)$ (respectively $\overline{D}(z,r)$) centered at z with radius $r > 0$.

Given a subset S of $\mathbb{C}_\infty$, the *closure* $\operatorname{clos} S$, the *interior* $\operatorname{int} S$ and the *boundary* ∂S will always be taken relative $\mathbb{C}_\infty$. For $S' \subset S \subset \mathbb{C}_\infty$ we write $S' \Subset S$ if $\operatorname{clos} S' \subset \operatorname{int} S$. An open connected (sub-)set of $\mathbb{C}_\infty$ is a *(sub-)domain* of $\mathbb{C}_\infty$. By $\operatorname{dist}(\cdot,\cdot)$ denote the Euclidean distance function in $\mathbb{C}_\infty$. So, $\operatorname{dist}(S,\infty) := +\infty$ for $S \Subset \mathbb{C}$.

For a subset $S \subset \mathbb{C}$, $\operatorname{har}(S)$, $\operatorname{sbh}(S)$, $\operatorname{Hol}(S)$ and $C^k(S)$ with $k \in \mathbb{N} \cup \{\infty\}$ are the restrictions to S of *harmonic, subharmonic,* and *k times continuously differentiable functions* in some (in general, its own for each function) open set $O \subset \mathbb{C}$ containing S, respectively. But $C(S)$ is the class of all continuous functions on S. The class $\operatorname{sbh}(S)$ contains the minus-infinity function $\boldsymbol{-\infty} \colon z \mapsto -\infty$ identically equal to $-\infty$; $\operatorname{sbh}_*(S) := \operatorname{sbh}(S) \setminus \{\boldsymbol{-\infty}\}$, $\operatorname{Hol}_*(S) := \operatorname{Hol}(S) \setminus \{0\}$, $\operatorname{sbh}^+(S) := \{u \in \operatorname{sbh}(S) \colon u \geq 0 \text{ on } S\}$.

Let $\operatorname{Borel}(S)$ be the class of all Borel subsets in $S \in \operatorname{Borel}(\mathbb{C}_\infty)$. We denote by $\operatorname{Meas}(S)$ the class of all Borel signed measures, or, *charges* on $S \in \operatorname{Borel}(\mathbb{C}_\infty)$; $\operatorname{Meas}_{\mathrm{c}}(S)$ is the class of charges $\mu \in \operatorname{Meas}(S)$ with a compact support $\operatorname{supp}\mu \Subset S$; $\operatorname{Meas}^+(S) := \{\mu \in \operatorname{Meas}(S) \colon \mu \geq 0\}$, $\operatorname{Meas}_{\mathrm{c}}^+(S) := \operatorname{Meas}_{\mathrm{c}}(S) \cap \operatorname{Meas}^+(S)$; $\operatorname{Meas}^{1+}(S) := \{\mu \in \operatorname{Meas}^+(S) \colon \mu(S) = 1\}$, *probability measures*. We denote by $\delta_z \in \operatorname{Meas}_{\mathrm{c}}^{1+}(S)$ the *Dirac measure* at a point $z \in S$, i.e., with the support $\operatorname{supp} \delta_z = \{z\}$, $\delta_z(\{z\}) = 1$. We denote by $\mu\big|_{S'}$ the restriction of μ to $S' \in \operatorname{Borel}(\mathbb{C}_\infty)$.

Definition (of affine balayage). *Let $O \subset \mathbb{C}$ be an open subset, and $S_0 \Subset O$. Let $\mathcal{V}$ be a class of Borel-measurable functions on $O \setminus S_0$. We say that a measure $\mu \in \operatorname{Meas}^+(O)$ is an affine balayage of a measure $\nu \in \operatorname{Meas}^+(O)$ outside S_0 for the class $\mathcal{V}$ and write $\upsilon \preccurlyeq_{S_0,\mathcal{V}} \mu$ if there exists a constant $C \in \mathbb{R}$ such that*

$$\int_{O\setminus S_0} v \, d\nu \leq \int_{O\setminus S_0} v \, d\mu + C \quad \textit{for all } v \in \mathcal{V}.$$

provided that all integrals are well defined by values from the extended real line $\mathbb{R}_{\pm\infty}$.

Reminder, that a domain $D \subset \mathbb{C}$ have non-polar boundary ∂D if, in particular, $\partial D \subset \mathbb{C}_\infty$ contains a non-isolated point, or $\partial D \subset \mathbb{C}_\infty$ has a non-zero Hausdorff dimension [1, 5.4.1]. A domain with non-polar boundary necessarily possesses the Green function g_D [1], [2].

Theorem 1 ([3, Theorem 1]) *Let $D\neq\varnothing$ be a domain in $\mathbb{C}$ with non-polar boundary ∂D, $M\in \operatorname{sbh}(D)\cap C(D)$ be a function with the Riesz measure $\mu_M\in \operatorname{Meas}^+(D)$, and $u\in \operatorname{sbh}_*(D)$ with the Riesz measure $\upsilon_u\in\operatorname{Meas}^+(D)$. Then the following three statements are equivalent:*
[s1] *There is a subharmonic function $h\in \operatorname{sbh}_*(D)$ such that*

$$u+h\leq M \quad \text{on } D. \tag{1}$$

[s2] *For any non-empty subset $S_0\Subset D$ and a constant $b\in\mathbb{R}^+_*$, the measure μ_M is an affine balayage of the measure υ_u outside $S_0\Subset D$ for the class*

$$\operatorname{sbh}_0^+(D\setminus S_0;\leq b):=\Bigl\{v\in \operatorname{sbh}_0(D\setminus S_0)\colon v\geq 0 \text{ on } D\setminus S_0,\ \sup_{D\setminus S_0} v\leq b\Bigr\}$$

of subharmonic positive test functions, where

$$\operatorname{sbh}_0(D\setminus S_0):=\Bigl\{v\in \operatorname{sbh}(D\setminus S_0)\colon \lim_{D\ni z'\to z} v(z')=0 \text{ for all } z\in\partial D\Bigr\}.$$

[s3] *There are a non-empty subset $S_0\Subset D$ and a number $b\in\mathbb{R}^+_*$ such that the measure μ_M is an affine balayage of the measure υ_u outside S_0 for the class*

$$\operatorname{sbh}_{00}(D\setminus S_0)\bigcap \operatorname{sbh}_0^+(D\setminus S_0;\leq b)\bigcap C^\infty(D\setminus S_0)$$

of subharmonic positive finite infinitely differentiable test functions, where

$$\operatorname{sbh}_{00}(D\setminus S_0):=\bigl\{v\in \operatorname{sbh}(D\setminus S_0)\colon \text{there is } S_v\Subset D \text{ such that } v\equiv 0 \text{ on } D\setminus S_v\bigr\}.$$

An application of Theorem 1 to study the distribution of *subsequences of roots for holomorphic functions from weight classes* can be found in [3].

"Subharmonic" Theorem 1 has a similar "harmonic" counterpart. Consider some more complicated classes of test functions. Given $S\subset\mathbb{C}$ and $r\in\mathbb{R}^+$, a set

$$S^{\cup r}:=S\bigcup\bigcup_{z\in S} D(z,r).$$

is called a *outer r-parallel set* [8, Ch. I,§ 4] *for S.*

For $v\in L^1\bigl(\partial D(z,r)\bigr)$, we define the averaging value of v at the point z as

$$v^{\circ r}(z):=\frac{1}{2\pi}\int_0^{2\pi} v(z+re^{is})\,\mathrm{d}s.$$

Let $\varnothing\neq \operatorname{int} S_0\subset S_0\Subset D$ be a connected subset of domain D, and

$$0<r<\frac{1}{3}\operatorname{dist}(S_0,\partial D), \qquad -\infty<b_-<b_+<+\infty \tag{2}$$

are constants. A function $v\in \operatorname{sbh}_0(D\setminus S_0)$ is called a *subharmonic signed test function from a class* $\operatorname{sbh}_0^{\pm}(D\setminus S_0,r;b_\pm)$, if this function satisfies the following three conditions:

[**t**1] $\sup\bigl\{v(z)\colon z\in\partial S_0\bigr\}\leq b_+$;

[**t**2] $\inf\bigl\{v^{\circ r}(z)\colon z\in S_0^{\cup(3r)}\setminus S_0^{\cup r}\bigr\}\geq b_-$;

[**t**3] *there is a subset $S_v\Subset D$ such that $v\geq 0$ on $D\setminus S_v$.*

We will use a substantially narrower class $\operatorname{sbh}_{00}^{\pm}(D\setminus S_0,r;b_\pm)$ of *subharmonic signed finite test function* v satisfying condition [**t**1], but the condition $\inf\{v(z)\colon z\in S_0^{\cup(3r)}\setminus S_0\}\geq b_-$ instead of *weaker* condition [**t**2], and also a *finiteness condition*
[**t**0] *there is a subset $S_v\Subset S$ such that $v\equiv 0$ on $D\setminus S_v$*
instead of *weaker* condition [**t**3].

Theorem 2 (a special case announced in [6, Theorem 2]) *Let the conditions of Theorem 1 be fulfilled. Then the following three statements are equivalent:*
[**h**1] *There exists a function $h\in\operatorname{har}(D)$ such that $u+h\leq M$ as in (1).*
[**h**2] *For any non-empty connected subset $S_0\Subset D$ and constants from (2), the measure μ_M is an affine balayage of the measure υ_u outside $S_0\Subset D$ for the class $\operatorname{sbh}_0^{\pm}(D\setminus S_0,r;b_\pm)$.*
[**h**3] *There are a non-empty connected subset $S_0\Subset D$ and constants as in (2) such that the measure μ_M is an affine balayage of the measure υ_u outside S_0 for the class*

$$\operatorname{sbh}_{00}^{\pm}(D\setminus S_0,r;b_\pm)\bigcap C^\infty(D\setminus S_0).$$

Let $\varnothing\neq D\subset\mathbb{C}$ be a domain, and $M\in\operatorname{sbh}(D)\cap C(D)$,

$$\operatorname{Hol}(D,M):=\bigl\{f\in\operatorname{Hol}(D)\colon |f|\leq\exp M \text{ on } D\bigr\}.$$

Theorem 2 gives a criterion for a zero set for holomorphic functions $\operatorname{Hol}(M)$, which was partially announced but not proved in [6, Theorem 2].

Corollary. *Let $\mathsf{Z}:=\{\mathsf{z}\}_{\mathsf{k}=1,2,\dots}\subset\mathsf{D}$ be a sequence without limit point in a simple connected domain D with two different points in ∂D or in a finitely connected domain D with $\operatorname{clos}D\neq\mathbb{C}_\infty$, and with counting measure*

$$n_{\mathsf{Z}}:=\sum_k\delta_{\mathsf{z}_k},$$

where δ_z is the Dirac measure at z. The following three statement are equivalent:
[**z**1] *This sequence Z is exact zero set Zero_f taking into account multiplicity for a function $f\in\operatorname{Hol}(D,M)$, i.e., in terms of counting measures $n_{\mathsf{Z}}=n_{\operatorname{Zero}_f}$.*
[**z**2] *For any connected subset $\varnothing\neq S_0\Subset D$ and constants (2), there is a constant C such that*

$$\sum_k v(\mathsf{z}_k)\leq\int_{D\setminus S_0}v\,\mathrm{d}\mu_M+C\quad\textit{for all } v\in\operatorname{sbh}_0^{\pm}(D\setminus S_0,r;b_\pm).$$

[**z**3] *There are a connected subset* $\varnothing \neq S_0 \Subset D$*, constants as in* (2)*, and a constant* C *such that*

$$\sum_k v(\mathsf{z}_k) \leq \int_{D\setminus S_0} v \,\mathrm{d}\mu_M + C \quad \textit{for all } v \in \operatorname{sbh}_{00}^{\pm}(D\setminus S_0, r; b_{\pm}) \bigcap C^{\infty}(D\setminus S_0).$$

The work was financially supported by a Grant of the Russian Science Foundation (Project No. 18-11-00002, Theorems 1 and 2), and by a Grant of the Russian Foundation of Basic Research (Project No. 19-31-90007, Corollary).

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