What quantum symmetry should be

M. D. Sheppeard

August 2019

One usually hears that modern physics is governed by symmetries, either discrete or continuous. In quantum gravity, where spacetime is a derivative concept, many symmetries are also derivative. What is foundational are discrete statements in quantum logic, or the abstract characterisation of the conditions of an experiment for the observer. The preeminence of categorical techniques, and their relevance to motivic mathematics, suggests looking not for classical symmetries but instead for their Hopf algebras.

Consider the canonical example: a Lie algebra $\mathbb{G}$, given by its associative universal enveloping algebra $U(\mathbb{G})$. Such Hopf algebras may be deformed to give representation categories with a nontrivial braiding. But since quantum measurement has discrete sets of outcomes, we want to start with rings or finite fields, rather than the reals. The axioms of most interest work over a ring R.

Our ring R is not necessarily commutative. Recall that a left module M over R is equipped with a scalar multiplication [1] $R \times M \to M$ such that $1x = x$ for $x \in M$ and $(rx)y = r(xy)$ for $r \in R$. In higher dimensions, associativity will be weakened. Define an algebra over R to be a module A with multiplication $\mu : A \otimes A \to A$ and unit $\eta : R \to A$, where the tensor product is over R. There is a category $\mathbf{Alg}_R$ of all algebras over R with the obvious morphisms. Any module M defines a tensor algebra

$$T(M) = \oplus_{n=0}^{\infty} M^{\otimes n} \tag{1}$$

with multiplication $\mu : T(M) \otimes T(M) \to T(M)$ given by the isomorphism between $M^{\otimes p} \otimes M^{\otimes q}$ and $M^{\otimes (p+q)}$.

Now when M is itself an algebra A, as is the case for $U(\mathbb{G})$, we have a map

$$\mu : T(A) \to A \tag{2}$$

taking $a_1 \otimes a_2$ to $a_1 a_2$. This setting is the right one for us because the construction is *monadic* [2]. Recall that a monad on a category $\mathcal{C}$ is an endofunctor $T : \mathcal{C} \to \mathcal{C}$ along with a map $\mu : T(A) \to A$ such that the associative law holds: $\mu(1 \otimes \mu) = \mu(\mu \otimes 1)$.

Classical logic is governed entirely by the category **Set** and its power set monad, which takes the set of all 2^n subsets of an n point set. We begin to do quantum logic by building square matrices in a categorically correct manner, as follows.

Take a category with r objects and Hom sets from the category $\mathbf{Alg}_R$, called $\mathcal{A}_{ij}$ for $i, j = 1, 2, \cdots r$. A *matrix algebra* is defined to be a tensor category $\mathcal{A}$ with duals on matrix objects such that any objects A and B satisfy the axioms

1. $A \equiv \cup A_{ij}$ with $A_{ij} \in \mathcal{A}_{ij}$ and $\mathcal{A}_{ij}\mathcal{A}_{jk} \subset \mathcal{A}_{ik}$
2. $A^{**} \simeq A$
3. $(AB)^* \simeq B^* A^*$
4. $(A^*)_{ij} \simeq (A_{ji})^*$.

The duals here are inherited from an involution on the objects $\mathcal{A}_{ij}$. Except for the lack of strict equalities, these are precisely the axioms of a nonassociative matrix algebra in Vinberg's approach to homogeneous cones [3][4]. They are also essentially the axioms of a *tortile tensor category* [1][5] with a braiding, where $\otimes$ is the matrix product, and axiom 4 is the compatibility of duals with twist maps $A \to A$ given by matrix transpose. In other words, the categorification of nonassociative matrix algebras is tortile $\otimes$ categories, such as a category of ribbon tangles for quantum computation.

Such nonassociative matrices are applied to Lie algebras for quantum gravity in [6]. From the categorical perspective, the nonassociativity will weaken the usual law for a monad T, but in an interesting way. With an associator 2-arrow filling in the square $\mu(1 \otimes \mu) = \mu(\mu \otimes 1)$, the Mac Lane pentagon is drawn on five sides of a three dimensional cube. Braided categories are of course secretly three dimensional, and when we fill the pentagon with a 3-arrow we obtain pentagonal faces for the three dimensional associahedron [7]. Physical dimensions are closely connected to the axioms required for quantum computation. We can now begin to study motivic cohomology and homotopy using alternatives to triangulated 1-categories, where the higher dimensional quantum monad has both a geometric and algebraic interpretation.

References

[1] R. Street, *Quantum groups: a path to current algebra*, Cambridge, 2007

[2] S. Mac Lane and I. Moerdijk, *Sheaves and geometry in logic*, Springer, 2000

[3] E. B. Vinberg, Trans. Moscow Math. Soc. 1963, 340-403, 1963

[4] C. B. Chua, Siam J. Optim. 14, 2, 500-506, 2003

[5] M. C. Shum, J. Pure App. Alg. 93, 1, 57-110, 1994

[6] P. Truini, M. Rios and A. Marrani, arXiv:1711.07881, 2017

[7] M. D. Sheppeard, PhD thesis, University of Canterbury, 2007