On the proposal of an Eddington ratio of natural energies, ε

Michael J. Bucknum[1] and Eduardo A. Castro
INIFTA, National University of La Plata, Suc. 4, C.C. 16, La Plata 1900, Buenos Aires, Argentina

Abstract: Eddington in 1923, first identified 4 dimensionless numbers, derived from combinations of the basic physical constants, which are known as the "Eddington constants". In formulating these dimensionless numbers, Eddington, a leading physicist of his time, claimed that they are characteristic of the structure and dynamics of the Universe at large, on the microscopical scale, and at the macroscopical scale. These 4 dimensionless ratios are labeled here, and elsewhere, as α (also called the fine structure constant), and β (the electron-proton mass ratio), and γ, (the ratio of electrical-to-gravitational force of the proton on the electron), and δ (a ratio involving the cosmological constant and other constants). Here, in this communication, is defined a 5th Eddington ratio, labeled as ε (a ratio of characteristic energies of diatomic and monatomic hydrogen). The uncanny fitting of these 5 fundamental ratios to simple formulas involving the mathematical constants e, the base of natural logarithms, π, the familiar circular constant, ϕ, the golden ratio and "2", the only even prime number, is described to some degree along with a tabulation of characteristics of these 5 ratios..

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(1, corresponding author mjbucknum@yahoo.com)

1. Introduction

From the 9 natural physical constants indicated in Table 1 below, it is possible to form at least 5 dimensionless combinations of these constants. A.S. Eddington, who created this field of dimensionless physical constants as related to his last great work, The Fundamental Theory, [1] proposed the 4 dimensionless "Eddington constants" in 1946 as part of his thesis on the overall organization of the Universe from a microscopic and macroscopic perspective. Eddington had indeed organized the data earlier by 1923 from knowledge of the crude values of the first 8 fundamental natural constants shown in Table 1 below [2]. From these first 8 listed natural physical constants [3], by 1929, [4] and until his death in 1944, he went forward on an ambitious plan to formulate what he called his "Fundamental Theory" which employed these 8 natural physical constants, and the Eddington constants, intermingled with a description that involved abstract, higher mathematics and which consisted of over 300 pages of writing. Eddington's writings in this area together with this volume, which was edited and published posthumously in 1946, have been cursorily reviewed by Kragh in 2011 [5]

Table 1

2. Eddington Ratios

The Eddington ratio identified as "v/c" or the ratio of the velocity of the electron in Bohr's 1st orbit of H to the speed of light, has been assigned symbol α since it was the first dimensionless constant of this type identified (Sommerfeld, 1916 [6]). Next came the electron-proton mass ratio given by the symbol β. It is thus comprised as a ratio of the electron mass, "m", to the proton mass, "M". The 3rd Eddington ratio is a dimensionless number formed from the quotient of the electrical-to-gravitational forces of the proton on the electron, and it is given the symbol γ. Eddington's last ratio, involving the cosmological constant (see Table 1) and several of the other constants in Table 1, is in fact dimensionless but it does not represent a common ratio of physical quantities, being as it is a ratio of masses multiplied by lengths. It is given the symbol δ. It is thus these 4 ratios that are dimensionless that formed part of Eddington's final project on the basic structure and dynamics of the Universe. There are other possible dimensionless ratios of basic physical constants that have come to light since Eddington's time. Barrow has covered the identification and various definitions of these constants. [7] Table 2 below thus identifies these 4 characteristic Eddington constants by their definitions as combinations of the first 8 natural constants in Table 1. We include in Table 1 a 9th natural constant, and in Table 2 a 5th dimensionless physical ratio employing this natural constant, as anticipating the discussion below where these quantities are formally introduced.

Table 2

3. Energy Ratio, ε

No mention has been made, up to this point, about the last entry given in Table 1, the entry for the natural physical constant identified as the Hooke's law constant, k_H, for the diatomic hydrogen molecule. This is a molecular constant, with a numerical value of 573. 219 Nm^{-1}, it thus enters the discussion here from the perspective of chemical physics. [8] The Hooke's law constant defines the vibrational energy degree of freedom manifold of the diatomic hydrogen molecule. We propose here to thus define a 5th Eddington ratio formed from (1) the zero-point vibrational energy of dihydrogen in an energy ratio with (2) the Rydberg binding energy of atomic hydrogen. Such a ratio is shown in Equation 1 below and reflected in the 5th row of Table 2 above.:

$$\varepsilon = \frac{E(zero-point\ energy)}{E(Rydberg\ energy)} \tag{1}$$

The zero-point vibrational energy of dihydrogen is given by $\frac{h\nu}{2}$ where "h" is just Planck's constant, as identified in Table 1. The product of "h" times "ν" in the numerator of the preceding quotient identifies here the characteristic vibrational energy (here of diatomic hydrogen). Here "ν" is defined in terms of the Hooke's law constant "k_H" in Equation 2 below.

$$\nu = \frac{1}{2\pi} \cdot \sqrt{\frac{k_H}{M/2}} \tag{2}$$

Where in Equation 2 the factor "M" is just the mass of the proton in kg units and the factor "k_H" is just the hydrogenic Hooke's law constant as 573.219 Nm^{-1}, both as identified and given in Table 1. [3, 8] By dividing the product "hv" by the factor "2" one yields the molecular zero-point energy. In this scheme, we identify this energy quantity as fundamental, as it is here related to the Hooke's law constant of dihydrogen, as is identified in Table 1 as a fundamental natural constant. It is presented in this way for the purpose of defining a novel Eddington-like dimensionless constant, identified as ε, as is shown by the ratio in Equation 1.

The denominator in Equation 1 is called the Rydberg energy. It is just the binding energy of the electron to the proton in the H atom. It is a spectroscopic quantity identified by Rydberg in the 19th century, and later further defined in terms of several natural constants (as listed in Table 1) by Bohr. [9, 10] Equation 3 thus defines the denominator of Equation 1 as the Rydberg energy in terms of the fundamental natural constants h, q, k_c, and m of Table 1. Equation 4 then defines the dimensionless constant of Equation 1, identified in this discussion as ε, in terms of 6 of the fundamental natural constants in Table 1 (m, M, h, k_c, q and k_H). In all of this, the fundamental nature of the ε constant, can be viewed as a ratio of a basic factor relating to the binding strength of two proton-electron pairs…to the binding strength of an electron for a proton…in an isolated pair.

$$\mathrm{Ry} \quad = \quad \frac{m \cdot (q)^4 \cdot (k_c)^2}{2 \cdot (h/2\pi)^2} \qquad (3)$$

$$\varepsilon \quad = \quad \frac{\sqrt{(k_H)} \cdot (h/2\pi)^3}{m \cdot \sqrt{\left(\frac{M}{2}\right)} \cdot (q)^4 \cdot (k_c)^2} \qquad (4)$$

Employing the numerical values of these natural constants from Table 1, one can calculate the value of ε as being equal to $\frac{2.00260}{1{,}000{,}000}$. Alternatively, from the natural frequencies (given in wavenumbers, i.e. cm^{-1}, by Herzberg, [8]) one can calculate ε more easily than in Equation 4, and thus there is Equation 5 below.

$$\varepsilon \; = \; \frac{1}{2} \cdot \frac{4.39520 \, x \, 10^3 \, cm^{-1}}{1.09737 \, x \, 10^9 \, cm^{-1}} \qquad (5)$$

It can readily be shown that the value of this Eddington-like constant, called ε by us, is within less than one percent (beyond 99.8701%) of being exactly the integer "2" divided by one million. The mathematical significance of the number "2" is simply that it is the first prime number other than the number "1", and it is the only even prime number. Table 3 below shows these 5 dimensionless constants, described in this communication as α. β. γ, δ and ε, by their definitions as various physical ratios, and their corresponding numerical values are given in Table 2, and the mathematical expressions that they approximate, and the percentage agreement they have at the initial value they take on to the corresponding mathematical approximants' values can be easily inferred from the corresponding data tables.

Table 3

4. Conclusions

The results shown in Table 3 are worthy of some final comments. That the Eddington ratios shown in Table 3 here are so closely equal to various approximants based upon the mathematical constants, namely "e", "π", "ϕ" and "2", is truly remarkable. [11, 12] It suggests that the underlying constants of physics bear a close connection with some of the more familiar mathematical constants. The reason why this is the case is left for further speculation by the reader of course. Eddington himself may have tried to address this curiosity in his learned work, cited earlier, called the Fundamental Theory. [1] One note to be made about the data in Table 3 is that we have not come up with any approximant for the Eddington ratio of the electrical-to-gravitational force of the proton on the electron, identified here as the Eddington ratio, γ. An attempt at fitting this ratio to some simple mathematical approximants, like the other 4 ratios in Table 3, was made. It turns out that γ very closely matches to the mathematical approximant "$\sqrt{2}\cdot\phi \times 10^{39}$" to within less than a percent difference. However, unlike the other 4 approximants, that are numerically less than a percent below the corresponding values of their Eddington ratios, the likely approximant for γ, identified here as "$\sqrt{2}\cdot\phi \times 10^{39}$" lies above its corresponding Eddington ratio. Because of this difference the approximant for γ in Table 3 is listed in parentheses, along with its agreement status (in parentheses) at just about less than a percent difference above the given Eddington ratio. With all respect to the experimental measurements that have gone into determining G to 4 significant figures [13], the authors suggest here that the relevant measurements might be reexamined, to see if a better alignment of the corresponding Eddington ratio, γ, with the numerical approximant of Table 3 might result. And thus, the authors here believe that, perhaps the gravitational constant, G, should be reexamined and remeasured [13] to see if it is possibly a little bit smaller than currently reported, [3] in order to get its corresponding Eddington ratio, γ, aligned with the other 4 Eddington ratios and their corresponding mathematical approximants.

It is also to be noted that we have taken some liberty to reassign the value of the cosmological constant [14], λ, in Table 1 as different than the currently accepted value of this constant, [3] which is presently not very accurately known and is also in some dispute. The value of λ that we propose, is approximately in the numerical range to lend itself to an analysis such as that presented in the theme of this work. And the proposed value of λ here may be suggestive, or closer to the value that may ultimately be measured experimentally.

It thus turns out that these Eddington ratios, and their approximants in terms of familiar mathematical constants, may provide some type of basis for developing further a more fundamental physical theory, in some way, that might provide a rational basis for why these mathematical approximants exist at all. In other words, the authors believe that a theory might be developed some day, that describes and explains the meaning of the mathematical approximants to the Eddington dimensionless physical ratios. Such a theory might have some important implications and connections to chemical physics [8] and perhaps to other chemical contexts, potentially, as is believed by the authors here. It is to be noted here as well, as a final comment, that Gamow [15], in citing and describing Eddington's work on fundamental constants [1], has stated in this connection that among the established theoretical physics community, there is a consensus that "any pure numbers that come up in theoretical physics contexts will ultimately be explained in pure mathematical terms…" [15]. The authors of the current work were motivated in part by Gamow's comments thus…and Eddington's groundbreaking writings.

5. References

[1] A S Eddington *Fundamental Theory* (UK : Cambridge University Press) (1946)

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[6] A Sommerfeld *Atomic Structure and Spectral Lines* (UK : Methuen) (1923)

[7] J D Barrow *The Constants of Nature: From Alpha to Omega* (UK: Jonathan Cape) (2002)

[8] G Herzberg *Molecular Spectra & Molecular Structure, I. Spectra of Diatomic Molecules* (USA : Krieger) (1989)

[9] M Born *Atomic Physics* (USA : Dover) (1989)

[10] In the denominator of Equation 3, the Rydberg energy in Bohr's formulation, it is more accurate to replace the factor "m", the mass of the electron, with the reduced mass of the electron-proton pair, identified as μ. Here $\mu = \frac{mM}{m+M}$ where "m" is the electron mass, and "M" is the proton mass. However, because $\frac{M}{m}$ is about 1836.15, the reduced mass, μ, is about

99.9455 “parts in 100” equal to the mass of the electron “m”. It is therefore a small, but significant, correction. It would also be more accurate to replace “m” occurring in Equation 4, for the dimensionless constant ε, with the reduced mass of the electron-proton system μ, as discussed above for Equation 3.

[11] M J Bucknum and E A Castro *J. of Math. Chem* **56** 651 (2018)

[12] M J Bucknum and E A Castro *J. of Math. Chem.* **56** 1360 (2018)

[13] Cavendish was the first to accurately measure Newton’s constant of gravitation denoted by the letter “G”. Cavendish, in about 1798, used a newly invented instrument called a “torsional balance” to determine the mass of the Earth, and to thereby identify the constant, G, as equal to 6.67 x 10^{-11} Nm^2/kg^2. For over 220 years, and after a number of sophisticated remeasurements of G, Newton’s gravitational constant is presently known with certainty to only 4 significant figures at 6.674 x 10^{-11} Nm^2/kg^2. Reports of the 5th and 6th significant figures, in Newton’s gravitational constant G, have some uncertainty associated with them. The NIST compilation of data on the physical constants, as identified in [3], also contains references to the various numerical constant measurements, including those of G. The NIST data is available online (www.nist.gov).

[14] Eddington denoted the cosmological constant as λ. He approximated its value as on the order of 10^{-52} m^{-2} in the meter-kilogram-second (m-k-s) unit system. It is beyond the scope of this work to go into a detailed analysis of λ. Many, and perhaps most, astrophysics reference works published after 2000 give an explanation, in words, of the cosmological constant and identify it as Λ in Einstein's equations of General Relativity. However, these reference works usually do not report any precision numerical value for Λ. The cosmological constant is defined in several different unit systems in the definitions of it reported in Wikipedia (https://en.m.wikipedia.org). Wikipedia reports some precision numerical values of Λ, where at the beginning of the Wikipedia article on "the cosmological constant" it states that Λ is of the order of 10^{-52} m^{-2}, in agreement with Eddington. Below this statement, in the same Wikipedia article, Λ is defined more precisely as in the following, $\Lambda = 1.1056 \times 10^{-52}$ m^{-2}. This latter value is equated to an expression for Λ which involves directly the Hubble constant H, and inversely the speed of light c, together multiplied by a parameter for the curvature of space-time Ω. This report in Wikipedia, on the cosmological constant, thus suggests that a precise numerical value of Λ is not currently well known. With the approximate nature of Λ in mind, the authors here have suggested an approximate value of

Eddington's λ that fits reasonably within the margins of the mathematical approximant involving it in Tables 2 and 3, which is identified as the Eddington constant "δ". The value of Eddington's λ given in Table 1 is therefore identified in parentheses as "(1.19 x 10^{-52} m^{-2})", in Table 1, to thus indicate the speculative nature of this numerical constant used in the current analysis.

[15] G Gamow *Proceedings of the NAS* **59** 313 (1968)

6. Tables of Data

Table 1: Identities and Values of Physical Constants

Symbol	Name	Value
h	quantum of action	6.62607×10^{-34} J·s
c	speed of light	2.99792×10^{8} m/s
q (= e)	elementary charge	1.60217×10^{-19} C
m	electron mass	9.10938×10^{-31} kg
M	proton mass	1.67262×10^{-27} kg
G	Newton's law constant	6.67408×10^{-11} N·m^2/kg^2
k_c	Coulomb's law constant	8.98755×10^{9} N·m^2/C^2
λ	cosmological constant	(1.19×10^{-52} m^{-2})
k_H	Hooke's law constant	573.219 N/m

Table 2: The 5 Proposed Eddington Constants

Symbol	Formula	Value
α	$\dfrac{k_c \cdot q^2}{\left(\frac{h}{2\pi}\right) \cdot c}$	8.54245 x 10^{-2}
β	$\dfrac{m}{M}$	5.44617 x 10^{-4}
γ	$\dfrac{k_c \cdot q^2}{G \cdot m \cdot M}$	2.26872 x 10^{39}
δ	$\dfrac{c}{\left(\frac{h}{2\pi}\right)} \cdot \sqrt{\dfrac{m \cdot M}{\lambda}}$	2π ·1.61895 x 10^{39}
ε	$\dfrac{\sqrt{(k_H)} \cdot (h/2\pi)^3}{m \cdot \sqrt{\left(\frac{M}{2}\right)} \cdot (q)^4 \cdot (k_c)^2}$	2.00260 x 10^{-6}

Table 3: Eddington symbols, ratios and approximants

Eddington symbol	Eddington ratio	Approximant	Agreement (%)
α	v/c	$\frac{e \cdot \pi}{100}$	99.9681
β	m/M	$\frac{\sqrt{3} \cdot \pi}{10,000}$	99.9123
γ	F_{elec}/F_{grav}	(“$\sqrt{2} \cdot \phi$” x 10^{39})	(99.1467)
δ	(m·l)’/ (m·l)”	“$2\pi\phi$” x 10^{39}	99.9411
ε	E_{vib}/E_{elec}	$\frac{2}{1,000,000}$	99.8701