The prime gaps between successive primes to ensure that there is atleast one prime between their squares assuming the truth of the Riemann Hypothesis

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Abstract: Based on Dudek's proof that assumed the truth of the Riemann's hypothesis, that there exists a prime in the interval $(x - (4/\pi) x^{1/2} \log x, x]$, we determine the size of prime gaps that must exist between successive primes, so that we can be sure that there is atleast one prime number between their squares.

Results:

Let "l" and "m" represent two successive primes. Based on Dudek's proof [1] where he assumed the truth of the Riemann Hypothesis, the prime gap in this case is a= (m-l) and it must be definitely smaller than $(4/\pi)(m-1)^{1/2} \log(m-1)$, since there exists no other prime between l and m.

$a < (4/\pi)(m-1)^{1/2} \log(m-1)$

Consider l^2 and m^2, what must their values be, so that there is atleast one prime definitely between them as suggested by Dudek[1], assuming Riemann's hypothesis.

l^2 must be equal to or smaller than $\{m^2 - (8/\pi) m \log m\}$, which is obtained by replacing "$x = m^2$" in the expression
"$x - (4/\pi) x^{1/2} \log x$"

Therefore the interval $(m^2 - (8/\pi) m \log m, m^2]$ must contain atleast one prime. Note that m^2 is composite, so the prime will be located within the interval.

Another way to write the gap between the squares is $(m^2 - l^2) = m^2 - (m-a)^2 = 2ma - a^2$

If this gap is greater than or equal to the minimum gap needed based on Dudek's results, then we can expect atleast one prime in between them.

$(8/\pi) m \log m \leq 2ma - a^2$

$a^2 \leq 2ma - (8/\pi) m \log m$ ……………………….. inequality (A)

$a^2 \leq m\{2a - (8/\pi) \log m\}$

Since m= l+a, we can be sure that m>a. However, ensuring
$a \leq 2a - (8/\pi) \log m$ will guarantee the left hand side of inequality (A) will be smaller than the right side.

$a \leq 2a - (8/\pi) \log m$
$(8/\pi) \log m \leq a$

So when two primes “l” and “m”, (and m > l), are separated by a prime gap “a” where,

$(8/\pi) \log m \leq a < (4/\pi) (m-1)^{1/2} \log(m-1)$

the two primes, l and m must be successive primes and there must be atleast a single prime between their squares assuming the Riemann Hypothesis to be true.

References:

1. Dudek, Adrian W. (2014-08-21), "On the Riemann hypothesis and the difference between primes", *International Journal of Number Theory*, **11** (3): 771–778