Coordinates and static sphere metric solutions

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Abstract

We show there is a scalar function calculated from a metric solution to the Einstein field equations for a static spherically symmetric sphere that depends on coordinates.

Consider a static spherically symmetric sphere of radius b. Let $\rho(r)$ be the mass density, $p(r)$ the pressure, and $U^\mu(r)$ the velocity four vector. We have $\rho(r) = p(r) = 0$ for $r > b$. Let the metric $g_{\mu\nu}(r)$ approach the Minkowski metric as $r \to \infty$ and satisfy the Einstein field equations

$$G_{\mu\nu} = 8\pi[pg_{\mu\nu} + (p+\rho)U_\mu U_\nu] \tag{1}$$

Consider the coordinate tranformation

$$t' = t \qquad r' = f(r) \qquad \theta' = \theta \qquad \phi' = \phi \tag{2}$$

where $df/dr > 0$, $f(r) = r$ for $r < b$, and $f(r)/r \to 1$ as $r \to \infty$. On making this coordinate transformation (1) becomes

$$G'_{\mu\nu} = 8\pi[p'g'_{\mu\nu} + (p'+\rho')U'_\mu U'_\nu] \tag{3}$$

Now

$$\rho'(r') = \rho(f^{-1}(r')) \qquad p'(r') = p(f^{-1}(r')) \qquad U'_\mu(r') = \frac{\partial x^\alpha}{\partial x'^\mu} U_\alpha(f^{-1}(r')) \tag{4}$$

In (3) use (4) and then replace x'^μ by x^μ and define

$$\tilde{g}_{\mu\nu}(r) = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(f^{-1}(r)) \tag{5}$$

We then have $\tilde{g}_{\mu\nu}(r)$ is also a solution of (1) that approaches the Minkowski metric as $r \to \infty$.

Consider the Kretschmann scalar function

$$K(r) = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \tag{6}$$

We have there are values of r such that

$$K_{\tilde{g}}(r) = K_g(f^{-1}(r)) \neq K_g(r) \tag{7}$$

where $K_{\tilde{g}}(r)$ is $K(r)$ calculated using the metric $\tilde{g}_{\mu\nu}(r)$ and $K_g(r)$ is $K(r)$ calculated using the metric $g_{\mu\nu}(r)$. We however expect $K_{\tilde{g}}(r) = K_g(r)$.

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