How Bayesian probability might help provide a *realist* interpretation of the quantum formalism

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Abstract We offer a realist interpretation of non-relativistic quantum mechanics in which dynamical properties are properly possessed by the system in question, and are supposed to have definite values at any time. Like the QBists, we employ Bayesian probability, but we adopt something closer to the Bayesian statistics of E. T. Jaynes than to the *subjective* Bayesian statistics of B. de Fenetti employed by the QBists. Accordingly, we view calculated Bayesian probabilities as rational degrees of expectation of *dynamical property values* rather than as personal degrees of expectation of *future* (*measurement*) *experiences*. Probabilities are, for us, based on knowledge of the value of some *dynamical property* of the system, not on knowledge of *previous experiences* unassociated with system dynamical properties. As some Bayesians might, we take a probability equal to 1 not generally to indicate *certainty* but only (full) expectation; and we disallow probabilities of conjunctions of propositions claiming incompatible properties. Then, by reinterpreting and adding a little to the quantum formalism, we argue that we can maintain the advantages of the QBist interpretation. So, for us (as for the QBists), there is no unexplained collapse of the wave function, no need for 'spooky action at a distance', and no problem raised by the double slit experiment, the Kochen-Specker paradox or Bell type theorems. By holding on to a *realist* perspective, modelling (of ideal measurements, of system preparation processes etc.) is possible, and we can claim certain dynamical laws of quantum mechanics without leading to contradiction.

Keywords Quantum Mechanics, uncertainty principle, Bayesian probability, Realism, QBism.

1 Introduction

In this paper we offer a new realist and probabilistic interpretation of pure-state non-relativistic quantum mechanics[1].

To understand the sense in which it is realist we need to clarify what we consider to be properties of a quantum mechanical system – properties that we take to be real, and actually possessed by the system. We take a 'property' of a system to be an aspect of it quantified by one or other of the eigenvalues of any one complete set of commuting observables relating to the system. The quantified value of a property (in certain units) is thus a single number when the complete set contains just one member, but more generally it is a set of numbers each associated with a particular observable of a complete set. Even so, we

[1] Throughout this paper we will assume non-relativistic conditions apply. So by 'quantum mechanics' we will always mean *non-relativistic* quantum mechanics.

will always refer to the ‘value’ of a property (meaning a single number or a set of numbers) much as one speaks of the ‘value’ of a vector –meaning all three (or more) of its components.

We note that the term ‘system’ used here may refer to a particular aspect of a larger system, an aspect of which we could have pure knowledge. For example, in a single electron system, we may single out the electron’s orbital motion as a ‘system’, concerning which we might (under certain circumstances) have pure knowledge (the electron’s spinning motion being then independent of its orbital motion and not entering into consideration at all). We may speak of properties of this ‘system’ (e.g. the electron’s position in space or its momentum in momentum space) as properties possessed by the ‘system’ (i.e. by the electron’s orbital motion). We may similarly single out the electron spinning motion as an aspect of the electron system, and speak of the properties of this ‘system’ (e.g. its spin component in the *z*-direction of one particular Cartesian coordinate frame, or its spin component in the *z*-direction of another Cartesian coordinate frame). These various properties of the electron system are supposed to be real, properly possessed properties, in any circumstances, whether or not the orbital and spinning motions are independent and regardless of whatever knowledge we may or may not hold about the dynamics of the electron’s motion.

Our interpretation is *realist* in the direct sense that we take any property (as defined above) to be an actual real property of a quantum mechanical system – a property which at any time is supposed to have a definite value (whether we know that value or not). It is *probabilistic* in that we view the pure-state quantum formalism as providing the means to calculate probability distributions over the possible values of any one property supposing knowledge of the value of another.

Our interpretation of the quantum formalism has something in common with the recent QBist interpretation (well described in [1]) yet differs markedly from it. In common with the QBist interpretation, we employ Bayesian statistics, but we employ a form of rational Bayesian statistics closer to that of Jaynes [2] than to the Bayesian statistics of Finetti [3] used by the QBists. Crucially, as we have said, we attach full reality to all properties associated with complete sets of observables at all times, whereas the QBists deny (or do not commit to) the reality of such properties. The probabilities derived in our interpretation are our logical degrees of expectation *of actual values of system properties*. They are not (or not just) personal degrees of expectation of our future (measurement) experiences as the QBists would say. We are not QBists for the same reason that Marchildon [4] is not. That is, we do not see physics as merely a set of rules relating our experiences, but see it as having, as its central aim, the discovery of the nature of the real physical world and our place in it. We see physics as performing the job of ‘saving the appearances’. Without contradicting this view of physics, we do however believe a quantum state (or wave function) is representing our *knowledge* regarding the physical state of the system rather than representing the physical state itself. We see wave functions as providing (in the form of their squared moduli) probability distributions over the possible values of system properties given our knowledge of the value of a particular system property.

In most interpretations of quantum mechanics (including the Copenhagen and QBist interpretations) the uncertainty principle is, we argue, taken further than it should be. In its original or simplest form it claims that some knowledge (like knowledge of the position *and* momentum of a particle to arbitrarily fine precision) is unobtainable in principle owing to the very nature of matter, and, therefore, of any measuring apparatus we might employ to try to obtain that knowledge. However, in its (let us say) 'more developed' form it is taken further to imply that dynamical properties themselves (like a particle's position or a particle's momentum) are generally non-existent or do not have precise values. Hence an electron can be 'in two or more places at once', or 'nature herself generally does not know where an electron is', etc.

We take this 'more developed form' to constitute a Mind Projection Fallacy, and instead of denying or blurring the *possession* of properties, we will take the uncertainty principle in its simplest sense - that simultaneous *knowledge* of more than one possessed property is sometimes impossible. That way we keep the classical notion of ordinary possession of exact properties by systems, and admit only to not being able to *know* or accurately measure every combination of them at once. Most importantly however, we *do* take the simplest form of the uncertainty principle as signalling the need to change the way we *reason* about the quantum world as opposed to the classical world. That is, the rules of (knowledge based) Bayesian probability (as an extension of logic) should, in quantum theory, be changed from the form they take in relation to classical physics (or ordinary life). Of course, in classical physics the uncertainty principle is of no significance on account of the relatively imprecise nature of our knowledge and of the measurement means by which that knowledge is obtained, and classical probability is the appropriate theory of probability to use then.

The detailed nature of quantum mechanical systems is unknown at present, and this prevents us forming clear physical pictures of processes in quantum mechanics. The only thing we know for certain is that some system properties have classical analogues and that in the classical limit these properties follow the classical laws of motion. Quantum mechanical processes can therefore only be pictured clearly in the classical limit.

However, rather than proceeding in a purely formal manner with regard to quantum mechanical properties (a procedure which has its problems, in particular, with regard to continuity to the classical limit) we will commit ourselves to a certain physical picture of the quantum world, all be it incomplete. This picture will help fix ideas, at least for the present.

To form this picture, we take particles to be real material points (as in classical mechanics) but we drop the differentiability of particle position coordinates with respect to time normally assumed in classical mechanics.

We thus claim that quantum mechanical systems are made up of particles (material points) in the classical sense, except that their motions, though continuous, are *irregular* even at the smallest scales; that is, they move rather like pollen particles in Brownian motion. So, with regard to a system of particles, the representative point in configuration space accordingly moves in an irregular fashion also.

The continuous but irregular motion (of a single particle or of a representative point in configuration space) is, we suppose, present even over *vanishingly short* times. The motion over vanishingly short times can be thought of as occurring in addition to the drifting motion over any ordinary (non-vanishing) time period. The momentary values of the quantum mechanical spin components and momentum components of a particle (in relation to any one particular Cartesian coordinate system) are, we suppose, associated with the irregular motion going on during vanishingly short times. (In fact, any one of the inferred momentary dynamical properties of a system claimed in section 5.4, is supposed to be associated in one way or another with the irregular motion going on during vanishingly short times.) These properties will need more than the simple differential calculus to describe mathematically. We make no attempt here to provide such a description, but simply postulate that one day it might be found possible to characterise the irregular motions in vanishingly small times, and thus to define particle velocities, spins, and all dynamical properties in a definite mathematical way. This might account for why particle momenta components take continuous values while particle spin components can only be multiples of $\frac{1}{2}$ etc.

For the present (i.e. in the theory we offer now) we simply claim that the momentary components of a particle's spin or of its linear momentum are *internal* properties of the particle, which differ, and take different values, depending on the Cartesian coordinate frame employed. We claim that *all* the momentary dynamical properties of a system (including the inferred properties claimed in section 5.4) are generally associated with internal properties of the particles and differ depending on the Cartesian coordinate frame employed. As we have said, particle positions change in a continuous but irregular way, and we distinguish between internal properties of particles (properties of their motion in infinitesimal times) and their drifting motions through space (during non-infinitesimal times).

As in classical physics, particles may be moving in external (scalar or vector) potential fields associated with macroscopic sources. For example, they may be moving in the electric field (electric potential gradient) between electrodes, or in the magnetic field (the curl of an electromagnetic vector potential) due to an electromagnet. Charged particles (e.g. electrons) generate electric potentials and hence electric fields which they carry around with them. We take these fields to be quasi-static (see section 7.1). External potential fields and the potential fields due to particles themselves are at any time continuous and differentiable with respect to spatial coordinates. External potential fields may also be considered to be differentiable with respect to *time*, but this is not true of the (quasi-static) fields of quantum mechanical particles themselves owing to their irregular motion.

The internal properties of a particle are, we suppose, influenced by the spatial *derivatives* of external potentials in the particle's vicinity whenever the particle is sensitive to the kind of potential in question. In particular, under an active external scalar potential, particle momentum is, we suppose, changed at a rate proportional to the spatial gradient of the potential field at the point occupied by the particle as in classical mechanics. Likewise, the components of a particle's spin are supposed to undergo actual precession about the local external magnetic field at the rate quoted in quantum mechanics text books.

In this paper we postulate no detailed laws governing the effects of potentials on the *drifting* motion of particles, we claim only that a particle's drifting motion is in some way affected by the values of the potentials in its immediate vicinity, and we rely on the Schrödinger equation to calculate our probabilities for particle position at any time. This suggests, as is well known, an apparent tendency of a particle to drift from regions of high potential to regions of low potential. Because we suppose particle momentum is determined by the local spatial derivatives of the potentials, it is understandable, then, that there is a statistical correlation, in the classical limit, between the *internal* property of a particle's momentum on the one hand, and the particle's mass times the velocity of the particle's wave packet on the other.

Several apparent problems with property possession have been raised in connection with the double slit experiment, the Kochen-Specker paradox, Bell type theorems and so on. These have led many researchers to reject entirely the actual possession (by systems) of real dynamical properties. However, we reinterpret (and add to) the formalism of pure-state quantum mechanics in a way that renders those problems, and others, non-existent. To achieve this it will be necessary to state more precisely than usual (i) what probabilities mean and how they are to be calculated, (ii) when and how the collapse of a wave function can be carried out, and (iii) what constitutes the classical limit.

Many of the ideas presented in this paper are similar to those employed in an earlier work [5] self-published by the author. In that earlier work certain rather general laws relating to the drifting motion of particles are included. (These are, for example, to do with the isotropy and homogeneity of space and time in relation to motion in general, and to the absence of a direct effect of local potential gradients on drifting motion.) Using these laws, we *derive* the usual quantum formalism (including the Schrödinger equation) on the basis of a proposed Bayesian (complex-valued) probability theory peculiar to quantum mechanics. In the present paper we provide a shorter (self-contained) theory relying on the usual quantum formalism for calculating ordinary (real-valued) probabilities. We thus have no need, in this paper, to formulate complex-valued probability theory, or to employ physical laws relating to the drifting motion of particles. Some of the illustrative examples in this paper are different from those in [5]. At certain points in the paper we make comparisons with the theory in [5], and take the opportunity to add to, improve and correct some of the ideas in [5].

2 Probability theory in general

We adopt, with Jaynes [2], a rational Bayesian view of probability and start out by claiming that the probability of an event (given our limited knowledge of the process in question) is our *degree of belief* that that event occurs. We suppose certain rules of probability hold and can be used to calculate probabilities. Probability theory is thus treated as an extension of logic. Or as Jaynes would say, probability is 'the logic of science'.[2]

[2] The expression 'the logic of science' (meaning probability theory) seems to have been coined by Jaynes and used in the title of his famous book on Bayesian probability [2] published shortly after his death. He uses the term 'the logic of science' to reflect the view of probability taken by Clerk Maxwell who, when writing to a

We will, however, differ from Jaynes in one respect. For we will suppose that if a calculation gives the probability of an event equal to 1, we should take that to mean we *expect* the event to happen, rather than to mean it must *certainly* happen given of our knowledge of the process and our knowledge of physical law. That is, an event whose probability is 1 might not be *certain* to occur, but will be *expected* to occur given our knowledge and our logical deduction (of its probability) from that knowledge.[3] Similarly, an event whose probability is calculated to be 0 is (fully) expected *not* to occur.

In support of making this change (from certainty to expectation) we note the following two examples of common occurrence of probabilities equal to 1.

(i) With regard to a known probability density $p(x)$ which is a differentiable function of a continuous variable x, the probability that $1 \le x \le 4$ might have a definite (non-zero) value equal to $\int_1^4 p(x)dx$, and $p(x)$ may be finite and non-zero for $0 \le x \le 8$, but the probability that $x = \pi$ (say) is then zero, and the probability that x differs from π is equal to 1. This does not, however, mean we are *certain* that x differs from π. It means only that we *expect* x to differ from π.

(ii) It can sometimes be shown (given our knowledge) that the probability of a particular relative frequency of an outcome of a process is close to 1, and equal to 1 in a certain limit. According to our view of probability, this does not mean we are then *certain* of that particular relative frequency even in the limit. It means only that we *expect* that particular relative frequency in the limit. This is as it should be, for there may be nothing *physically causing* this frequency, and if the frequency were a *logical* consequence of our knowledge, and therefore *certain*, contradictions would sometimes arise (see the third paragraph of Appendix A).[4]

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friend in 1850, said 'They say that Understanding ought to work by the rules of right reason. These rules are, or ought to be contained in Logic, but the actual science of logic is conversant at present with things either certain, impossible, or *entirely* doubtful, none of which (fortunately) we have to reason on. Therefore the true logic for this world is the Calculus of Probabilities, which takes account of the magnitude of the probability (which is, or which ought to be in any reasonable man's mind). This branch of Math., which is generally thought to favour gambling, dicing, and wagering, and therefore highly immoral, is the only "Mathematics for Practical Men," as we ought to be. Now as human knowledge comes by the senses in such a way that the existence of things external is only inferred from the harmonious (not similar) testimony of the different senses, Understanding, acting by the laws of right reason, will assign to different truths (or facts or testimonies, or what shall I call them) different degrees of probability. Now as the senses give new testimonies continually, and as no man has ever detected in them any real inconsistency, it follows that the probability and *credibility* of their testimony is increasing day by day, and the more man uses them the more he believes them. … When the probability … in a man's mind of a certain proposition being true is greater than being false, he believes it with a proportion of faith corresponding to the probability, and this probability may be increased or diminished by new facts.' (p.80 of [6])

[3] Note that 'expected' is not here meant in the way it is in the expression 'expected value' (of a random variable) used in statistics. To avoid confusion, we will sometimes say of an event whose probability equals 1, that that event is '(fully) expected'.

[4] As well as *expecting* certain frequencies we may also *expect* certain resulting mean values (of physical quantities) to be present. These mean values, too, may not be physically determined or logically implied. They may therefore not be *certain* either.

Note that we do not identify probabilities with relative long term frequencies in many trials (neither of course would Jaynes, the QBists or other Bayesians), but most physicists tend to do so, and in this we suppose they are mistaken.

Measurements of long term relative frequencies, however, play an important (or even vital) role in the validation and development of quantum mechanics (as in particle scattering experiments for example), and in the validation and development of a probabilistic theory of any process (quantum mechanical or not). For so long as *expected* relative frequencies agree with *observed* long term relative frequencies we are happy. When marked and recurring differences occur, we feel obliged to rethink the model assumptions on which our probability predictions were based, and to modify these till agreement is restored.

On account of the alteration in the meaning of probability 1, we had better change from regarding probabilities as degrees of belief to regarding them as *degrees of expectation*. Otherwise a probability calculated to equal 1 would mean *full belief* which, in the scientific context, is the same as certainty. So, for consistency, we had better (and we will always henceforth) refer to probabilities as degrees of expectation.[5]

Finally we note that, when we claim, from observations, to hold particular knowledge (of the physical world), and base our calculated (Bayesian) probabilities on that knowledge, we are in fact never absolutely certain of the truth of the propositions expressing that knowledge. Therefore we should, and will henceforth, take the 'knowledge' on which probabilities are based to be represented by propositions whose truth we have come to (*fully*) *expect* rather than to know for certain.[6]

The need for reconsideration of the meaning of probability and of its role in science, and in particular in quantum theory, has been forcibly expressed by Appleby [7], who makes a detailed study of the various interpretations of probability, and comes to the conclusion (like us) that a Bayesian approach to probability is more satisfactory for various reasons and seemingly more fitting in relation to quantum theory where probability plays a fundamental role. (See also Jaynes [8] and Marlow [9] for support of this view).

3 Incompatible properties and non-existent probabilities

In quantum theory we suppose, as we have said, that there are limitations to what we can claim to know (i.e. to (fully) expect). We will say that two or more properties of a system (properties present at the beginning of, or at any time into the natural evolution of the system)

[5] In [5] the squared moduli of complex-valued probabilities were referred to as 'degrees of belief'. It would have been better there too to have referred to them as 'degrees of expectation'.

[6] This differs from the position taken in [5] where knowledge (of physical properties) was (ideally) assumed to entail *certainty* of the propositions claiming those properties. Relaxation of the meaning of knowledge (to *expectation* of the relevant propositions rather than *certainty* of them) seems essential for the consistency of the interpretation we are offering both here and in [5] (see, in particular, the end of section 6.6 of the present paper).

are ‘incompatible’ when it is not possible, on account of the uncertainty principle, to know them both at the start, nor to perform measurements to get to know them both at a time thereafter.[7] (We will see, in section 6.3, that it is, in our interpretation, sometimes possible to get to know incompatible properties *retrospectively*.)

We will also stipulate (in common with the QBists) that joint probability distributions over incompatible properties do not exist. So for example, with regard to a particle with spin, we cannot speak of the probability that the particle has, at a certain time, a particular z component of spin *and* a particular x component of spin.

This limitation on the existence of probabilities is not as obvious for us as it is for the QBists who relate probabilities to measurement outcome experiences rather than to possessed values. Since it is not possible, for example, directly to measure the z component *and* the x component of a particle’s spin at one time, experience of such measurement outcomes is not possible, so it is naturally meaningless to speak of a probability for such an experience.

However, since we are supposing those spin components are real properties, and we are assigning probabilities to real properties, one might wonder why *we* cannot allow joint probabilities over them.

To explain this, we argue that rational thought is ‘economical’. That is, it *need not* and *does not* bother with concepts whose truth could never be tested. Since we clearly cannot, for example, directly measure the z component *and* the x component of a particle’s spin at one time in order to test (in repeated trials) any supposed joint probability distribution $p(\sigma_z, \sigma_x)$ over those variables, there is no need for that joint distribution. Use of the product rule $p(\sigma_z, \sigma_x) = p(\sigma_z)\, p(\sigma_x | \sigma_z)$ to *derive* $p(\sigma_z, \sigma_x)$ is not possible either, for, under our rational Bayesian interpretation, $p(\sigma_x | \sigma_z)$ is the probability of σ_x having acquired knowledge of the value of σ_z (rather than under the mere supposition that σ_z has a certain value). Acquisition of knowledge of the value of σ_z likely alters the value of σ_x rendering the formula $p(\sigma_z, \sigma_x) = p(\sigma_z)\, p(\sigma_x | \sigma_z)$ inappropriate. So joint probabilities over incompatible variables are both untestable and incalculable, and are therefore rightly regarded as non-existent.

As well as properties (as defined in section 1) we have of course all manner of ‘attributes’ of a quantum mechanical system (or set of such systems) which are *functions* of the properties of the system(s).[8] These attributes include functions of incompatible properties of the

[7] Different properties of a system *are* sometimes compatible. For example, the momentum of a single *free* particle at one time is compatible with its momentum at another time. Knowledge of both is possible at the outset because momentum is conserved. Also, at any one time, the z components of spin of a particle in two Cartesian coordinate systems with a common z axis are compatible, one spin component being physically correlated with (but not the same property as) the other.

[8] We include no assumption (in the quantum formalism) to the effect that *any* attribute (function of properties) has an operator which is the same function of the operators of the properties as the attribute itself is of the properties themselves. That assumption is simply not needed in quantum theory. The sameness of the functions is in fact already demonstrable in (but only in) the case of functions of the *compatible* properties of a system in a particular representation (so, for example the kinetic energy operator for a particle is related to the operators of

system(s). Two or more attributes are taken to be compatible or incompatible according as their respective property arguments are compatible or incompatible.

Attributes too are taken to be real and properly possessed by a system (or by a set of systems), but probabilities over the possible values of an attribute, under any state of pure knowledge of system dynamics, are existent only when the attributes are *compatible*, for only then might it be possible to confirm the probabilities by way of measurements in repeated trials. Henceforth we will take 'attributes' to include 'properties' (in the sense that a property can be regarded as a simple function of itself). We will take 'system' to include a *number* of physically independent (non-interacting) systems (for example a pair of particles no longer interacting, or an ensemble of systems set up to perform repeated measurements, etc.).

So, in our interpretation, probabilities refer to *observable* attributes of systems given our knowledge. More precisely, if an attribute is unobservable *on account of the uncertainty principle* [9], there is simply no probability for it.[10]

Finally we note the following rules relating to expectation and incompatible attributes. These rules are consequences of the non-existence of certain probabilities noted above, and reflect the necessity to change our way of reasoning about expectations on account of the uncertainty principle. (This does not mean we need to change the rules of propositional logic. We can leave those rules well alone.)

(i) If we calculate a probability 1 for proposition A claiming the presence of attribute A of a system at the beginning or at some time into the natural evolution of the system, and if we calculate a probability 1 for proposition B claiming the presence of attribute B of the system at the beginning or at some time into the natural evolution of the system, then we *expect* the truth of A and we *expect* the truth of B, but we can only claim to expect the truth of the conjunction AB (i.e. of the presence of the attribute 'A and B') when attributes A and B are compatible. If they are incompatible no such claim can be made, and this is related to the fact that the probability of AB is then non-existent and is not demonstratively equal to 1.

The same applies to any number of propositions $A, B, C, ...$ claiming attributes of a system, or of an ensemble of systems, in natural evolution. If the truth of each of these attributes is *expected*, their conjunction is *expected* only when the attributes are compatible.

(ii) Suppose, as a result of physical law, that proposition Y (claiming attribute Y) implies proposition X (claiming attribute X). Suppose, also, that we (fully) expect Y to be

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its momentum components by $\hat{K} = (1/2m)(\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2)$, just as the value of the kinetic energy attribute is, by definition, related at any time to the values of the momentum components by $K = (1/2m)(p_x^2 + p_y^2 + p_z^2)$.)

[9] as is any attribute that is a function of two or more incompatible properties

[10] This does not however prevent there being a probability of some other (observable) attribute different from the original attribute but physically correlated with it under our knowledge of dynamical properties of the system in question. See, for example the final paragraph section 6.3.

true. Then the probability of X under our knowledge Y should be set equal to 1 only when X and Y are compatible attributes. If they are incompatible that probability assignment may not be correct, i.e. *expectation* of X does not automatically follow from *expectation* of Y. For, employing Jaynes' [2] propositional logic notation, the formal proof that $P(X|Y)=1$ when $Y \Rightarrow X$ is conducted as follows: we have $X \equiv X(Y+\bar{Y}) \equiv XY + X\bar{Y}$ so $P(X|Y) = P(XY + X\bar{Y}|Y) = P(XY|Y) + P(X\bar{Y}|Y) = 1+0$. However, this proof involves non-existent probabilities when X and Y claim incompatible attributes and is therefore invalid then.[11] (We employ rule (ii) in section 6.6.)

4 Laws of probability in pure-state quantum mechanics

The laws of probability we should adopt for calculating rational Bayesian probabilities based on our knowledge depend on the circumstances.

Jaynes [2] has carefully formulated probability rules for applications of rational Bayesian probability in ordinary everyday circumstances and in classical physics. These include the usual sum and product rules, Bayes rule, and various rules for calculating prior probabilities (the principle of indifference, the method of transformation groups and the method of maximum entropy). He has done this in a way that transcends any other account of rational Bayesian theory, and this makes his book a very important contribution to knowledge. (See also his many very interesting papers, for example [8], [10] and those collected in [11].) In Appendix A we note just a few of Jaynes' contributions to the Bayesian probability debate that are relevant to this paper.

Jaynes has also formulated (new) rules of rational Bayesian probability in connection with the theory of quantum mechanical *mixed* states (see sections 7 and 10 of [12]) and has thus formulated quantum mechanical mixed state theory in a rather unique way, giving it particular clarity and simplifying the derivations of distributions in quantum mechanical statistical thermodynamics. This has involved use of a principle of maximum (information) entropy defined (in section 7 of [12]) in terms of the weights in a particular array representation of the general mixed state of a quantum mechanical system.[12]

However, Jaynes did not try to formulate (new) laws of Bayesian probability for application to *pure-state* quantum mechanics. Yet this, together with the formulation of (new)

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[11] Note that, the 'First Law of Extreme Values' of complex-valued probabilities, claimed on p.9 of [5], needs qualification in this regard –a point not realised when [5] was written.

[12] According to Jaynes, any mixed state of knowledge is represented (in the x_i representation) by an array $\{\psi_1(x_i),\ldots\psi_m(x_i); w_1,\ldots w_m\}$ of m wave functions and associated positive weights (which sum to 1), subject to an equivalence relation whereby two arrays represent the same mixed state whenever their density matrices (defined as $\rho_{jk} = \sum_{n=1,..m} w_n \psi_n(x_j)\psi_n^*(x_k)$) are equal. See Chapter XIV of [5] for an account of, and development of Jaynes' theory of mixed states.

dynamical laws of quantum mechanics, seems to be necessary if we want to reinterpret the formalism of pure-state quantum mechanics in a rational Bayesian manner.[13]

However in this paper we take a shortcut and simply suppose (as the QBists do) that (real-valued) probabilities are calculated directly, using the usual formalism of pure-state quantum mechanics; but, in keeping with our realist approach, we *reinterpret* that formalism in the following manner.

5 Reinterpretation of the formalism of pure-state quantum mechanics

Our interpretation of the quantum formalism differs from the traditional interpretation and from that of the QBists. It is as follows.

We take the state (ket) vector in Hilbert space to represent our supposed 'pure-state' of knowledge of the dynamical properties of the system in question, not the physical state of the system (and not just our expectations regarding our 'future experiences' relating to measurements made on the system). In the 'Schrodinger picture' this means that any *wave function* at any one time (representing the system's (time evolving) ket at that time) *also* represents our supposed pure-state of knowledge of the (time evolving) system's dynamical properties, not the physical state of the system at that time (and not expectations of future experiences).

Any pure-state of knowledge stays the same throughout the natural evolution of the system and the wave function (in the Schrödinger picture) evolves only because it is a function of variables whose meaning changes with time. (Those variables might, for example, be the coordinates of the particles of the system at time t, with time t appearing as a parameter in the wave function). As a result, the time dependent Schrödinger equation is, for us, not a law of motion of the physical state of the quantum mechanical system, it is a law of evolution of the probability amplitude expressing our fixed pure-state of knowledge of that system on account of the changing meaning of the wave function's variables. In the Heisenberg picture the state ket, and its wave function in a particular representation, stay constant, and either of them describes, in a time independent fashion, our pure-state of knowledge of the process the system is undergoing.

In itself, the change in interpretation just described, i.e. the change from 'state of the system' to 'state of our knowledge of the system', is not new. It was the view taken, for example, by Peierls [13].

Given the composition of a system (the particles making it up, the potentials present etc.), a pure-state of knowledge of the system's dynamics is a state of knowledge that is

[13] In [5] this task is tackled by taking probabilities to be complex-valued and by supposing implication (of one proposition by another) carries a phase. A complex-valued probability theory is developed in [5] with its own sum and product rules and its own Bayes rule, and its own rules for calculating prior probabilities (generalisations of the prior probability rules adopted by Jaynes and mentioned above). This theory is used to calculate complex-valued probabilities based on certain general physical laws claimed in [5], and the results are demonstratively consistent with the usual quantum formalism when the (real-valued) probabilities calculated in that formalism are taken equal to the squared moduli of the corresponding complex-valued probabilities.

maximal given the uncertainty principle. It is knowledge on the limiting boundary of possible knowledge of the system's dynamics. This is what is special about 'pure-states of knowledge' as opposed, for example, to 'mixed states of knowledge' in general, and to no knowledge at all about a system's dynamics (which qualifies as a particular mixed state of knowledge). Which states of knowledge constitute *pure-state*s of knowledge are well known to us through experience, and the claim that certain states of knowledge are both possible and pure, forms part of our interpretation of the formalism of pure-state quantum mechanics.

When working in the Schrödinger picture (as we will do throughout this paper) we take the squared modulus of a (normalised) wave function (in any one representation) to be our probability distribution over the various properties quantified by the variables of that wave function. In line with our realist approach, one such property must of course be the actual one present and the squared modulus of a (normalised) wave function gives the probability of each possibility. The squared modulus of a wave function is not the probability of 'realising' the 'potential' property *on measurement* as is traditionally claimed, though it is of course equal to the probability of finding this property on measurement if the measurement in question could in principle be made. For example, instantaneous measurement of a charged particle's precise position is a possibility in quantum mechanics (see section 7.2) and the modulus squared of a single particle wave function (in the position representation referring to a particular time) gives both the probability density for the particle to actually *be* at a point in space at a particular time, *and* the probability density for finding it there on measurement.

We take the whole formalism of pure-state quantum mechanics, with its operators, eigenfunctions, eigenvalues, and commutation laws etc. to be *a set of rules for deriving rational Bayesian probability distributions* (as the squared moduli of wave functions in any representation) whenever our knowledge of the quantum mechanical process in question is *pure*. The QBists, too, see the quantum formalism as a procedure for probability calculation, but we differ from the QBists both with regard to what the probabilities refer to (for us - physical properties, for them – personal experiences) and with regard to the role of the formalism itself. To the QBists the formalism is the logical calculus of subjective probabilities, while, for us, it is a set of rules (for probability calculation) arising from a *mix* of laws of probability (viewed as laws of thought) *and* physical laws governing the dynamics of quantum mechanical processes. These laws are not clearly separated in the formalism,[14] but we do not attempt, and have no need, to unscramble this mix in the present paper.[15]

We do need, however, to incorporate a little more into the formalism of pure-state quantum mechanics. These additions are given in subsections 5.1, 5.2 and 5.3.

[14] Jaynes (in [10]) has commented on this as follows: "We believe that to achieve a rational picture of the world it is necessary to set up another clear division of labour within theoretical physics; it is the job of the laws of physics to describe physical causation at the level of ontology, and the job of probability theory to describe human inferences at the level of epistemology. The Copenhagen theory scrambles these very different functions into a nasty omelette in which the distinction between reality and our knowledge of reality is lost."

[15] We *do* try to unscramble the mix in [5] which has that task as its main objective.

5.1 Sum and product rules

With regard to any one of the derived probability distributions mentioned above we suppose the usual sum rule and the usual product rule apply.

Suppose (only for simplicity of formulation) that our Hilbert space is of finite dimension N. Suppose, also, that x_i ($i = 1,...N$) are the (mutually exclusive) propositions claiming the possible values x_i of the property x employed in a particular representation. Then, if $\Phi(x_i|Y)$ is our wave function under a pure-state of knowledge Y (acquired before the process in question starts), the probability of x_i is $P(x_i|Y) = |\Phi(x_i|Y)|^2$. The probability of the attribute claimed by the disjunction 'x_3 or x_5 …etc.' is

$$P(x_3 + x_5 + ..|Y) = P(x_3|Y) + P(x_5|Y) + ... \tag{1}$$

where, following the notation used by Jaynes in [2], '+' on the LHS means 'or'.

Also, for any system attribute claimed by a disjunction A of the x_i ($i = 1,...N$), we have, for the probability of the conjunction of x_i and A the product rule

$$P(x_i A|Y) = P(A|Y)P(x_i|AY), \tag{2}$$

AY being the conjunction of propositions A and Y.

In the product rule (2) we take it that knowledge of the truth of A can be acquired in such a way that AY is, like Y, a pure-state of knowledge so that $P(x_i|AY) = |\Phi(x_i|AY)|^2$. Only then can the product rule apply.

We can of course deduce from (1) and (2) the more general sum and product rules:

$$P(A + B|Y) = P(A|Y) + P(B|Y) - P(AB|Y),$$
$$P(BA|Y) = P(A|Y)P(B|AY),$$

where A and B are any disjunctions of the x_i ($i = 1,...N$) and the same requirement regarding AY applies.

Hence the ordinary probability rules apply to the sample space whose atomistic propositions are the propositions x_i ($i = 1,...N$) of any one representation which refers to one particular time. The only difference lies in the limitations imposed on conditioning (from Y to AY) as noted.

5.2 Rule for collapsing a wave function

Next we introduce a rule for collapsing a wave function. This relates to the product rule (2).

Suppose we hold pure knowledge represented by a time dependent wave function $\Phi(x_i|Y)$ based on pure knowledge Y acquired, initially, as a result of system preparation at time t_0. Then it may be possible at time t, (with $t > t_0$), to acquire knowledge A that the true x_i (at time t), is one of certain set A of the propositions x_i. (Here A is the disjunction of the x_i in A.) Acquisition of this new knowledge might be achieved without disturbing the process of system evolution (or at least without changing which of the propositions x_i is actually true at time t) and might be achieved in such a way that knowledge AY is pure.

In that case, we claim that on acquisition of knowledge A, our wave function $\Phi(x_i|Y)$ at time t should be replaced by the wave function $\Phi(x_i|AY)$ at time t given by

$$\Phi(x_i|AY) = \begin{cases} \dfrac{\Phi(x_i|Y)}{\sqrt{P(A|Y)}} e^{-i\alpha} & x_i \in \mathrm{A} \\ \\ 0 & x_i \notin \mathrm{A} \end{cases} \tag{3}$$

where $P(A|Y)$ is the probability (under knowledge Y) of A at time t and α is an indeterminate constant phase. Thereafter the new wave function evolves, as usual, according to the Schrödinger equation.[16]

This is our law of wave function collapse under pure-states of knowledge, and it is clearly consistent with the product rule (2) of probability. We are adding this law to the usual quantum formalism viewed as a set of rules for calculating probabilities.

As such, collapse of the wave function is not some (unaccounted for) physical process, but is rather a rule of logic. Collapse of the wave function is a logical consequence of the acquisition of more knowledge, not something caused physically.

Examples of situations in which wave function collapse may be applied are given in sections 6.1, 7.2 and 7.3.

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[16] Whether or not the acquisition of knowledge A disturbs the particle's subsequent motion, it is always the case that our *narrower* expectation distribution (3) over the possible x_i values at time t, is accompanied by a *wider* expectation distribution over the possible values of at least one other (basic or inferred) property at time t.

5.3 The classical limit

Under certain states of pure knowledge, a particle's wave function (in the position representation) takes the form of a compact wave packet moving in space.

Or it takes the form of a wave spread out in space with the de Broglie wavelength still very small compared to the region of space occupied by the wave.

Or these kinds of 'waves' may be in configuration space (in the case of a many particle system).

In other states of pure knowledge the wave function is a linear combination of such 'waves'. For example the stationary state wave function of a particle in a box is (for large quantum numbers) a standing wave made up of plane waves interfering to form an intricate pattern. Or inside a particle interferometer, the wave function of a single particle may be formed of two wave packets, one in one branch of the interferometer and one in the other.

In all these cases we postulate (as part of the quantum formalism) that the particle (or each particle of a many particle system) is in fact moving in an orbit close to a classical orbit, or at least that we should *expect* it to be so doing. We speak of the wave functions in such cases as being 'quasi-classical'. Then, as the de Broglie wavelength tends to zero, we speak of approaching 'the classical limit' of full expectation of classical motion.

Related to this postulate is the *correspondence principle* which, for us, takes the following form.

Correspondence principle

> Under 'quasi-classical' states of pure knowledge of a process in quantum mechanics when the particles are expected to move (to classical accuracy) in classical orbits, the probabilities of propositions concerning properties with a classical analogue (calculated using the rules of quantum mechanics and averaged if necessary over classically small domains) are interpretable as classical probability distributions over classical properties consistent with the laws of classical mechanics and classical probability.

Here 'classical probability' refers to the Bayesian probability theory (as Jaynes has formulated it) for use in ordinary life and in classical physics.

When the correspondence principle applies, and whenever we can, if we wish, interpret our knowledge of the dynamical properties of the process in a classical manner, the quantum mechanical formalism always comes up with probabilities the same as the probabilities found using classical Bayesian probability theory applied to a classical model of the process.[17]

[17] For example, the stationary state wave function of a particle in a box becomes quasi-classical at high values of the quantum numbers associated with the squares of the components of momentum in directions parallel to the three mutually perpendicular edges of the box. The squared modulus of this wave function averaged over classically small regions gives a uniform probability density over particle position in the box –exactly the same as the probability density given by Bayesian probability applied to motion of a classical particle in the box under knowledge only of the squares of the values of its three components of momentum.

The classical limit is approached in the more general formalism of mixed-state quantum mechanics when each wave function in the array representing our mixed state of knowledge is quasi-classical. The correspondence principle holds then also.

As we approach the classical limit of certain mixed states of knowledge, the uncertainty principle no longer limits our possible knowledge of dynamical properties with classical analogue, i.e. we may know (fully expect) all these properties *to classical accuracy*, and hence maybe well enough for all practical purposes. Particles then effectively move classically, just as if Newtonian mechanics held exactly. As is well known, classical mechanics is effectively applicable when all the particles of a system have high enough masses (i.e. very much higher than the mass of an electron for example).

That completes our 'additions' to the formalism of pure-state quantum mechanics. We now return to the question of our interpretation of that formalism.

5.4 Inferred properties

In the formalism of pure-state quantum mechanics any complete set of orthonormal ket vectors in the system's Hilbert space (or of their wave functions in a particular representation) is traditionally said to be associated with an 'observable' (generally short for a 'complete set of commuting observables'). That is, any complete set of orthonormal wave functions is associated with a process which if set in motion at a particular time would lead at once to the collapse of the system's wave function into one or other of the wave functions of the complete set, with a macroscopic indicator showing us which. Such a process (not itself governed by the Schrodinger equation nor by any part of the quantum formalism) is thought of as performing a 'measurement' of the 'observable' without implying this 'measurement' is measurement of a property of the system properly *possessed* by it at the time in question. The property is said, at most, to be only somehow 'potentially' present before the 'measurement'.

The QBists make a similar claim and see such a measurement as an intervention on behalf of the individual who participates in the process. In both the traditional and QBist interpretations of quantum mechanics, measurements take on a generally abstract quality. They are supposed possible when the physical manner in which they might be carried out is often unclear. For us, measurement is only possible if the means for doing it are demonstrable, i.e. if the measurement process can be modelled. We give examples of such modelling in section 7.

In our interpretation of the formalism, we claim any complete set of normalised orthogonal kets, or their wave functions $\phi_j(x_i)$ ($j = 1,...N$) in any one representation, is associated with an 'inferred' dynamical property P *possessed* by the system and taking possible values P_j ($j = 1,...N$). The possible values of this property are distinct and equal in number to the dimension N of the Hilbert space. Which of the possible values P_j of P actually applies generally depends on the time. However, if wave function $\phi_j(x_i)$ applies at time t we should (fully) *expect* P to be taking value P_j at that time. This 'knowledge' of P_j at time t constitutes pure knowledge on its own, and accordingly we can set

$$\phi_j(x_i)e^{i\beta} = \Phi(x_i|P_j)$$

where P_j is the proposition claiming P_j at time t, and β is an arbitrary constant phase. That is, we can regard our wave function as expressing our knowledge (expectation) of the truth of P_j, rather than our original knowledge (whatever that was) that led in the first place to the wave function $\phi_j(x_i)$ applying at time t.

Under any one state of pure knowledge Y, the value of an inferred property P at any one time is generally unknown, and may not be instantly measurable,[18] but we claim that the probability distribution $P(P_j|Y)$ over the P_j can be obtained by expanding $\Phi(x_i|Y)$ at time t in the functions $\phi_j(x_i)$ ($j=1,...N$) and equating $P(P_j|Y)$ to the squared modulus of the coefficients. Thus

$$P(P_j|Y) = |a_j(t)|^2 \tag{4}$$

where

$$\Phi(x_i|Y) = \sum_{j=1}^{N} a_j(t)\phi_j(x_i)\,. \tag{5}$$

Since each of the orthogonal (time independent) wave functions $\phi_j(x_i)$ ($j=1,...N$) need only be specified to within a constant phase factor, the phases of the $a_j(t)$ are indeterminate, but this is of no consequence because their squared moduli *are* determinate -they do not change if the absolute phases of the $\phi_j(x_i)$ (for $j=1,...N$) are varied.

[18] It can sometimes (perhaps always) in principle be measured (after a delay), by supposing the system potentials are suitably changed from time t onwards so that at a much later time t' the system wave function is quasi-classical (outward travelling spherical wave-packet in configuration space) and the particles of the system are expected to be freely moving in classical orbits. By observing the positions of the particles (to classical accuracy) at time t', the value of P_j at time t can be inferred. That is, there is a (Bayesian statistical) correlation between the value of P_j at time t and the classical particle positions at time t'. The system potentials employed, from time t, to effect this measurement are such that if the system wave function is $\phi_j(x_i)$ at time t, then the particles will have definite expected positions (to classical accuracy) at time t'. In particular when the supposed system potentials from time t are just *zero* (so the particles move freely from time t onward) the particle positions at the much later time t' are correlated with the particle's *momenta* at time t, as explained in Chapter 5 of [14]. We do not, however, take the suggestion of Feynman and Hibbs (p.96 loc. cit.) that momentum (or other properties) can be reduced to mere position occupation at a later time.

5.5 The possible implications of probability 1 in quantum mechanics

If we hold pure knowledge Y of a process, that is if we (fully) expect proposition Y to be true, and if the probability of an event under this knowledge is calculated (using the quantum formalism) to equal 1, it means (as we have claimed is always the case) that we (fully) *expect* the event to happen. Expectation of an event is a state of mind, and under it, we might claim, in the quantum mechanical context, any one of three possibilities. That the event is (i) a *pure-logical* consequence of the proposition Y expressing our knowledge, or (ii) *certain* to occur, supposing the truth of proposition Y, on account of some physical law governing the system in question independently of our knowledge of the system's dynamical properties, or finally, we might feel we can make neither of these claims and must sit content with (iii) just (fully) *expecting* the event to occur.

In any particular case (when a probability of an event is calculated to equal 1) we may of course only make the first or second claim above if that leads to no contradiction with anything else we know or wish to claim. The possibility of *sometimes* being able to make the second claim above allows us (on the basis of our realist approach) to postulate dynamical laws not normally supposed to be active.

5.6 Probability in relation to a number of physically independent systems

Finally we note the rules of probability for a number of physically independent quantum mechanical systems where we have a pure-state of knowledge of each. These rules are the same as those normally applying in the formalism of pure-state quantum mechanics.

That is, if we take any one representation x_i $(i = 1,...N_1)$ in the first system with the x_i referring to the property x at any one time t_1 during its natural evolution, take any one representation y_j $(j = 1,...N_2)$ in the second system referring to a time $t_2, \ldots$ etc. , then there is a joint probability distribution over the conjunctions $x_i y_j ...$ $(i = 1,...N_1,\ j = 1,...N_2,\ \ldots)$ given by

$$P(x_i y_j ... | Y_1 Y_2 ...) = |\Phi(x_i | Y_1)|^2 |\Phi(y_j | Y_2)|^2 ...$$

where $Y_1, Y_2,...$ represent our pure-states of knowledge of each system, and $\Phi(x_i|Y_1)$, $\Phi(y_j|Y_2)$ $\ldots$ are the corresponding wave functions for each system. Also, as in section 5.1, the usual sum and product rules apply in the sample space whose atomistic propositions are the conjunctions $x_i y_j ...$ $(i = 1,...N_1,\ j = 1,...N_2,\ \ldots)$.

In particular, if the representations employed in each system refer to the same time, then all the systems taken together form a system of which we have pure knowledge

$Y = Y_1 Y_2 ...$; and the propositions $x_i y_j ...$ ($i = 1,...N_1$, $j = 1,...N_2, \ldots$) serve as a representation for the whole system, our wave function for the whole system being[19]

$$\Phi(x_i y_j ...|Y) = \Phi(x_i|Y_1)\Phi(y_j|Y_2)....$$

Or, the systems may form an ensemble of identical (non-interacting) quantum mechanical systems evolving after preparation at the same time or after preparation at different times, each under the same pure-state of knowledge. Also, the representations employed may refer to the same property in each system at the same time into its natural evolution after preparation. Such an ensemble may be set up for the purpose of testing (by measurements) an expected long term frequency or an expected long term mean value of an attribute of the system.

In any above case, the individual systems are not only physically independent (i.e. non-interacting) but are claimed also to be *logically* independent on account of our knowledge of each being pure. That is, we claim that knowledge of the results of measurements on any number of the systems should have no effect on our probabilities for properties of any other of the systems. This ‘absolute’ logical independence is peculiar to pure-state quantum mechanical probability theory and is the thing that makes probabilities in quantum mechanics more fundamental than probabilities in classical mechanics or ordinary life where whether or not probabilities (of outcomes in repeated trials for example) are logically independent is generally hard to know for sure.

That concludes our reinterpretation of (and additions to) the usual quantum formalism.

6. Resolution of paradoxes

We now look at various paradoxes or contradictions that are supposed to arise if one tries to adopt a realist point of view as we are doing. We show how these paradoxes or contradictions disappear under the rational Bayesian approach we adopt to probability in quantum mechanics and under our additions to, and reinterpretation of, the formalism of pure-state quantum mechanics.

The examples considered here should serve to show why many similar paradoxes or refutations of realism in the literature will also disappear.

[19] Of course, as well as different systems, the separate wave functions may relate instead to different aspects of the one and the same system whenever these aspects evolve independently (e.g. to spin components and to positions of non-interacting particles of a multi-particle system in a uniform magnetic field). This enables us sometimes to restrict attention to limited aspects of a system (e.g. only to the spin components of the particles in a singlet state) and to regard our knowledge as pure knowledge of those aspects.

6.1 The double slit or particle interferometer argument

It is often argued that if we try to hold on to the idea that a quantum mechanical particle (an electron for example) has a definite position at any time and moves continuously through space, then the interference of probabilities present in a particle double slit experiment or simple particle interferometer is inexplicable.

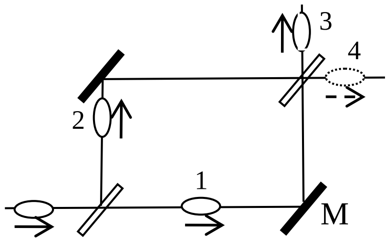

Figure 1. A simple particle interferometer. The incoming wave packet is split into packets 1 and 2 by the first beam splitter. These are reflected by mirrors M and are each partly transmitted and reflected at the final beam splitter to form out-going packets 3 and 4. Under fine-tuning packet 4 is cancelled out.

With reference to the simple particle interferometer, the argument goes through 6 steps as follows. (i) The particle has a certain normalised wave function ψ which after passage through the first beam splitter of the interferometer (Figure 1), divides into two (half-normalised) wave functions, say ψ_1 and ψ_2, setting off in different ways (way 1 and way 2) from the first beam splitter, each wave function evolving according to the Schrödinger equation thereafter, and (ii) these in time interfere (like real waves would) so that the probability density for the particle being at some point (i.e. $|\psi_1+\psi_2|^2$ at a later time) is not the same as $|\psi_1|^2+|\psi_2|^2$ in regions where ψ_1 and ψ_2 overlap (as in packets 3 and 4 in Figure 1). (iii) If the particle moves continuously through space it must either set off from the first beam splitter in way 1 or in way 2. (iv) If it sets off in way 1 the probability density for the particle being at any point thereafter is $2|\psi_1|^2$ and if it sets off in way 2 it is $2|\psi_2|^2$ and (v) as the probabilities for the particle to set off one way or the other are each $\frac{1}{2}$, (vi) the net probability density should be $\frac{1}{2}\times 2|\psi_1|^2+\frac{1}{2}\times 2|\psi_2|^2=|\psi_1|^2+|\psi_2|^2$ not $|\psi_1+\psi_2|^2$. Hence we have a contradiction, and the claim of continuous particle motion must be false.

Steps (i), (ii), (iii) and (v) of this argument are fine. The particle is expected, with equal probability, to set off one way or the other from the first beam splitter, and the probability density correctly calculated by quantum mechanics is $|\psi_1+\psi_2|^2$ thereafter. The problem lies in steps (iv) and (vi). For according to our rational Bayesian approach to probability, we cannot speak of the probability of an event *supposing* the particle sets off in way 1 (or way 2) from the first beam splitter. We can only speak of the probability of an event *knowing* the particle sets of in way 1 (or way 2). Even if the acquisition of knowledge as to which way the particle goes might be achieved without disturbing the particle, it is still necessary that we acquire that knowledge before we can condition our probability distribution from $|\psi_1 + \psi_2|^2$ to $2|\psi_1|^2$ (or from $|\psi_1 + \psi_2|^2$ to $2|\psi_2|^2$). This is because we take probability to be set by *knowledge* not by *physical conditions*.

In the case of the interferometer, where the wave packets are moving in separate paths inside the interferometer, we are close to a classical limit, the de Broglie wavelength being very small compared to the packet dimensions. As explained in section 5.3, we should therefore expect the particle to be moving close to one or other of the paths followed by the packets (without being absolutely sure it must). We can find out which way the particle goes by placing a particle detector in one path. If this particle detector finds no particle, then (without having interfered with the particle's motion) we know (fully expect that) the particle went the other way. We can then use our new knowledge to collapse our wave function (in the way explained in section 5.2) from $\psi_1 + \psi_2$ to $\sqrt{2}\psi_1$ (or from $\psi_1 + \psi_2$ to $\sqrt{2}\psi_2$). There is no contradiction, and position measurements conducted in many trials following null-detection will confirm the correctness of our reasoning.

Now suppose the interferometer is fine-tuned so that (when no null-detection is performed inside the interferometer) the particle is found always to exit one way out the interferometer. It might then seem strange that an observation, a null-detection (on one path inside the interferometer) which causes no disturbance, can result in the particle sometimes exiting the interferometer in a way it would seem never to do without that observation. However, that is evidently the way things are and there is no actual contradiction. For without observation, even if the probability the particle exits one way is calculated to equal 1, this need not mean the particle is *certain* to exit that way. As explained in section 2, probability 1 can mean only that we *expect* the particle to exit that particular way given our knowledge. So by passing from not performing to performing the null-detection we are simply passing from one degree of expectation (for exiting the particular way) to another, not from certainty to lack of certainty. After null detection and while the particle remains in the interferometer, our expectation distribution over particle position sharpens (reduces to one wave packet rather than two), but becomes less sharp (two wave packets rather than one) when the particle exits the interferometer. We see that application of the rules of quantum Bayesian probability to quantum mechanical processes *can* lead to results of a kind that cannot be replicated in applications of the rules of classical Bayesian probability to classical mechanical processes, but that does not imply there is something wrong with our quantum Bayesian probability theory or our realist perspective.

If the reader is thinking that the fact that (in repeated trials without null detection) the particle exits the interferometer the same way over and over again, is proof that it is *certain* to do so, they are relying on philosophical *induction*. The reliability of induction has, however, long been disputed by philosophers, and we now have further reason to believe it is not always applicable. We suppose it is not valid here, and consequently we are not forced to think that null detection in one branch must sometimes affect particle motion.

The important thing is that, when we have knowledge of a process (that constitutes pure knowledge), and when we have applied the rules of quantum Bayesian probability consistently, we get agreement between theory and observation in as much as our (full) expectations (of individual events or of relative long term frequencies) are always confirmed. Happily this is generally the case, and if sometimes it is not, we should put that down to not having the correct quantum mechanical model of the process under study. We then have to adjust that model in some way, to take account of effects we wrongly thought could be neglected, etc.

6.2 Bell type inequalities

Certain Bell type inequalities relate to a system composed of a pair of spin one-half particles produced in a singlet spin state and sent flying out from a source S in opposite directions each towards apparatus designed to measure particle spin components in any direction (of our choice) in a plane P perpendicular to their line of motion (Figure 2). Let the directions chosen be denoted $\boldsymbol{a}$ for particle 1 and $\boldsymbol{b}$ for particle 2, and let the experiment be repeated a large number times, say M times. If a_i denotes the result of the measurement on particle 1

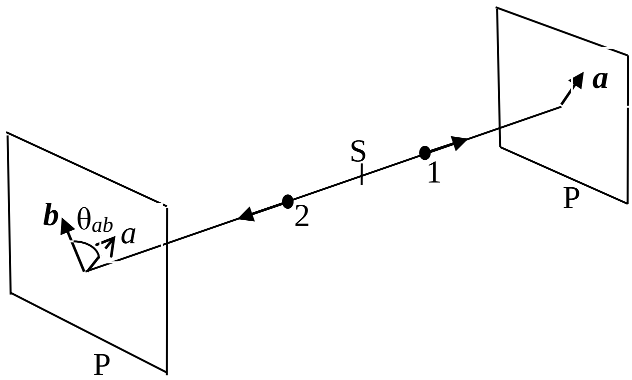

Figure 2. Spin one-half particles in a singlet state immerging from source S, and the directions in which their spins are measured.

in the i^{th} trial, being $+1$ when spin is 'up' in the $\boldsymbol{a}$ direction and -1 when it is 'down', and if b_i denotes in the same way the result of the measurement on particle 2, then quantum mechanics predicts the correlation coefficient between a_i and b_i, or the mean value[20]

$$\frac{1}{M}\sum_{i=1}^{M} a_i b_i \,,$$

is equal to $-\cos\theta_{ab}$ where θ_{ab} is the angle between the directions $\boldsymbol{a}$ and $\boldsymbol{b}$. So at least, in the limit as $M \to \infty$ we expect

[20] See p.84-85 of [15] for the proof that this mean value is equal to the correlation coefficient, and p.41 of [15] for the proof of (6) using quantum mechanics. [15] also provides a general account of Bell type inequalities and their implications for property possession and locality on the basis of certain assumptions.

$$\frac{1}{M}\sum_{i=1}^{M} a_i b_i = -\cos\theta_{ab} \qquad (6)$$

Adopting our realist view, we take the a_i and the b_i to have definite values quantifying properties of the particles in the i^{th} trial (whether or not those values are measured) and we take the LHS of (6) to be an attribute of the M trials having a definite value (independent of any measurements). The RHS of (6) is (from quantum theory) the value (fully) expected for that attribute (at least in the limit $M \to \infty$).

If we consider another pair of directions $\boldsymbol{a}'$ and $\boldsymbol{b}'$ in the planes P then each of the following equations is expected to hold,

$$\frac{1}{M}\sum_{i=1}^{M} a_i b_i = -\cos\theta_{ab} \qquad (7)$$

$$\frac{1}{M}\sum_{i=1}^{M} a_i b'_i = -\cos\theta_{ab'} \qquad (8)$$

$$\frac{1}{M}\sum_{i=1}^{M} a'_i b_i = -\cos\theta_{a'b} \qquad (9)$$

$$\frac{1}{M}\sum_{i=1}^{M} a'_i b'_i = -\cos\theta_{a'b'} \qquad (10)$$

where the four angles θ_{ab}, $\theta_{ab'}$,… etc. are supposed given. Associated with each of these equations we have, respectively, the propositions A_{ab}, $A_{ab'}$, $A_{a'b}$ and $A_{a'b'}$ claiming that certain attributes of the set of trials (expressed by the LHSs of (7)-(10)) have certain values (the values on the RHSs of (7)-(10)).

Now while we might, in principle, test the truth of any one of the propositions A_{ab}, $A_{ab'}$, $A_{a'b}$ and $A_{a'b'}$, we cannot test the truth of their conjunction. Measurement of a_i *and* a'_i for example, is disallowed by the uncertainty principle. So while the truth of each of the propositions is (fully) expected (at least in the limit $M \to \infty$), the truth of their conjunction is not (see rule (i) of section 3). It is this change in our way of reasoning that saves us from the contradiction that would otherwise arise.

If we *where* to expect the truth of the conjunction of the propositions A_{ab}, $A_{ab'}$, $A_{a'b}$ and $A_{a'b'}$, then we would have, by (7)-(10), to expect the attribute

$$\frac{1}{M}\sum_{i=1}^{M} a_i b_i + \frac{1}{M}\sum_{i=1}^{M} a_i b'_i + \frac{1}{M}\sum_{i=1}^{M} a'_i b_i - \frac{1}{M}\sum_{i=1}^{M} a'_i b'_i$$

of the set of trials to have the value

$$= -\cos\theta_{a'b'} - \cos\theta_{a'b} - \cos\theta_{ab'} + \cos\theta_{ab} \,.$$

Now, as explained on p.84 of [15], since the a_i and the b_i and their primed forms can only take values ± 1 we have

$$\left| \frac{1}{M}\sum_{i=1}^{M} a_i b_i + \frac{1}{M}\sum_{i=1}^{M} a_i b_i' + \frac{1}{M}\sum_{i=1}^{M} a_i' b_i - \frac{1}{M}\sum_{i=1}^{M} a_i' b_i' \right| \le 2 \,.$$

Hence we would have to expect

$$\left| \cos\theta_{a'b'} + \cos\theta_{a'b} + \cos\theta_{ab'} - \cos\theta_{ab} \right| \le 2 \,.$$

This inequality is, however, demonstratively false for some values of the angles. Since something *certainly* false cannot be *expected*, we would have a contradiction.

As we have said, we avoid this contradiction by disallowing expectation of the truth of any proposition claiming an attribute unobservable on account of the uncertainty principle. We are thus not obliged to give up the idea of possession of spin components by the particles, nor are we obliged to believe that measurement of a spin component of one particle must be instantly changing spin components of the other through some kind of 'spooky action at a distance'.

6.3 Validity of the EPR argument

In relation to section 6.2, we note that, according to our realist approach, the argument of Einstein, Podolsky and Rosen (as applied to a singlet spin state) may be perfectly correct. Equal and opposite particle spin components in any one direction $\boldsymbol{a}$ (as expected by quantum theory) may, according to us be a natural property of the particles of the singlet system, ensured by its preparation or the way the particles were formed.[21] Then if $\boldsymbol{a}$ and $\boldsymbol{b}$ are in the same direction we have, for certain, the relation $a_i = -b_i$ between particle spin components. This means measurement of a spin component of particle 1 enables us to (fully) expect a particular spin component of particle 2 in the same direction without affecting that spin component (or any other spin component of particle 2). If we follow up this measurement by a measurement of a spin component of particle 2 in another direction $\boldsymbol{c}$, and assume all spin

[21] This is an example of the possibility of us establishing a physical law by adopting option (ii) in section 5.5. The probability (in the singlet state) of equal and opposite spin components in any direction equals 1, and we here claim that equal and opposite spin components (in any direction) is due to a physical law (of spin component addition) and to the fact that the total spin component of the particle pair (in any direction) is known to be zero because of the way the particles are prepared.

components stay constant when not measured[22], we can get to expect that two particular spin components of particle 2 applied at times before the measurement performed on particle 2. This is an example of the acquisition of knowledge (full expectation) of incompatible properties retrospectively.

The above EPR argument therefore seems, from our point of view, to provide valid support for the real existence of properties (or the actual possession of properties by systems including the simultaneous possession of incompatible properties).

We note, however, that the uncertainty principle still acts to restrict what we can get to know retrospectively. We cannot for example get to know the spin component of particle 2 in a third direction because we have, by now, disturbed the spin components of both particles in all other directions without being in any way able to take account of, or correct for, the changes we have caused.

In a system composed of a pair of spin one-half particles produced in a singlet spin state, we have a joint probability distribution over the spin components of particles 1 and 2 in given directions $\boldsymbol{a}$ and $\boldsymbol{b}$, and, because of the natural relation $a_i = -b_i$ holding when directions $\boldsymbol{a}$ and $\boldsymbol{b}$ are the same, the property of the particles in which they have particular values of spin components in any one pair of directions $\boldsymbol{a}$ and $\boldsymbol{b}$ is fully correlated with the attribute according to which particle 2 on its own has a particular pair of spin components (in directions $\boldsymbol{a}$ and $\boldsymbol{b}$). The property in question is *directly* measurable, but the attribute in question is not, and while there is a probability for the property, there is no probability for the attribute. For as we have said, we do not allow joint probabilities of incompatible properties.

6.4 The system considered by Greenberger et al

An argument (due to Greenberger et al [16]) which claims to show the impossibility of possession of properties by systems, is one that leads to contradictions between propositions claiming properties of a *single* system rather than properties of an ensemble of identical systems undergoing independent trials.

In the system considered, four distinguishable spin one-half particles are emerging in different directions from a source where they have been prepared. They may each be passed into apparatus designed to measure their spin components in directions of our choice. We suppose we have pure knowledge Y of their spinning motions represented by the wave function

$$\Phi(\sigma_1\sigma_2\sigma_3\sigma_4|Y) = \frac{1}{\sqrt{2}}(\delta_{\sigma_1\frac{1}{2}}\delta_{\sigma_2\frac{1}{2}}\delta_{\sigma_3-\frac{1}{2}}\delta_{\sigma_4-\frac{1}{2}} - \delta_{\sigma_1-\frac{1}{2}}\delta_{\sigma_2-\frac{1}{2}}\delta_{\sigma_3\frac{1}{2}}\delta_{\sigma_4\frac{1}{2}}) \qquad (11)$$

[22] This assumption is supported by the fact that measurement of a free particle's spin component in any one chosen direction is reproducible any number of times (see section 7.3). After one such measurement the probability for the same result of the next is 1, and a physical law of conservation of a free particle's spin components can be (and is) claimed by us using option (ii) in section 5.5.

using the z components of spin basis $\sigma_1\sigma_2\sigma_3\sigma_4$ in Cartesian coordinate system O.[23] This basis takes the possible values

$$\begin{aligned}\sigma_1\sigma_2\sigma_3\sigma_4 = {} & \tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}, \quad \tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,-\tfrac{1}{2}, \quad \tfrac{1}{2}\,\tfrac{1}{2}\,-\tfrac{1}{2}\,\tfrac{1}{2}, \quad \tfrac{1}{2}\,\tfrac{1}{2}\,-\tfrac{1}{2}\,-\tfrac{1}{2}, \\ & \tfrac{1}{2}\,-\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}, \quad \tfrac{1}{2}\,-\tfrac{1}{2}\,\tfrac{1}{2}\,-\tfrac{1}{2}, \quad \tfrac{1}{2}\,-\tfrac{1}{2}\,-\tfrac{1}{2}\,\tfrac{1}{2}, \quad \tfrac{1}{2}\,-\tfrac{1}{2}\,-\tfrac{1}{2}\,-\tfrac{1}{2}, \\ & -\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}, \quad -\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,-\tfrac{1}{2}, \quad -\tfrac{1}{2}\,\tfrac{1}{2}\,-\tfrac{1}{2}\,\tfrac{1}{2}, \quad -\tfrac{1}{2}\,\tfrac{1}{2}\,-\tfrac{1}{2}\,-\tfrac{1}{2}, \\ & -\tfrac{1}{2}\,-\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}, \quad -\tfrac{1}{2}\,-\tfrac{1}{2}\,\tfrac{1}{2}\,-\tfrac{1}{2}, \quad -\tfrac{1}{2}\,-\tfrac{1}{2}\,-\tfrac{1}{2}\,\tfrac{1}{2}, \quad -\tfrac{1}{2}\,-\tfrac{1}{2}\,-\tfrac{1}{2}\,-\tfrac{1}{2}\end{aligned}$$

The wave function is thus a *superposition* of two states in which the particles would be said to have certain known spin components.

When we refer spin components of particles $1,2,3,4$ respectively to fixed coordinate systems $\mathrm{O}_1,\mathrm{O}_2,\mathrm{O}_3,\mathrm{O}_4$ of orientations generally different from that of our original coordinate system O, our wave function changes to

$$\Phi(\sigma_1'\sigma_2'\sigma_3'\sigma_4'|Y) = \sum_{\sigma_1\sigma_2\sigma_3\sigma_4}\Phi(\sigma_1'\sigma_2'\sigma_3'\sigma_4'|\sigma_1\sigma_2\sigma_3\sigma_4)\Phi(\sigma_1\sigma_2\sigma_3\sigma_4|Y)$$

where

$$\Phi(\sigma_1'\sigma_2'\sigma_3'\sigma_4'|\sigma_1\sigma_2\sigma_3\sigma_4) = \Phi(\sigma_1'|\sigma_1)\Phi(\sigma_2'|\sigma_2)\Phi(\sigma_3'|\sigma_3)\Phi(\sigma_4'|\sigma_4) \qquad (12)$$

in which each factor on the RHS has the form

$$\Phi(\sigma'|\sigma) = \begin{pmatrix} \cos\frac{\alpha}{2}e^{i(\beta+\gamma)/2} & i\sin\frac{\alpha}{2}e^{-i(\beta-\gamma)/2} \\ i\sin\frac{\alpha}{2}e^{i(\beta-\gamma)/2} & \cos\frac{\alpha}{2}e^{-i(\beta+\gamma)/2} \end{pmatrix}$$

α,β and γ being the Euler angles (as defined in paragraph 6-6 of [17]) of coordinate rotation from O. Let the Euler angles for each coordinate system $\mathrm{O}_1,\mathrm{O}_2,\mathrm{O}_3$ and O_4 be denoted $\alpha_1,\beta_1,\gamma_1;\alpha_2,\beta_2,\gamma_2;\alpha_3,\beta_3,\gamma_3$ and $\alpha_4,\beta_4,\gamma_4$, and let

$$\begin{aligned}&\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \tfrac{\pi}{2}, \\ &\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 0\end{aligned}$$

so that in the matrix notation

[23] For simplicity we omit commas between the variables $\sigma_1,\sigma_2,\sigma_3$ and σ_4.

$$\Phi(\sigma_1'|\sigma_1) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\beta_1/2} & ie^{-i\beta_1/2} \\ ie^{i\beta_1/2} & e^{-i\beta_1/2} \end{pmatrix}$$

and similarly for the other factors on the RHS of (12).

Using these equations we find, for the wave function (11) in the $\sigma_1'\sigma_2'\sigma_3'\sigma_4'$ representation and in column vector form, the results

$$\Phi(\sigma_1'\sigma_2'\sigma_3'\sigma_4'|Y) = \frac{1}{4\sqrt{2}} \begin{pmatrix} 0 \\ 2i \\ 2i \\ 0 \\ -2i \\ 0 \\ 0 \\ 2i \\ -2i \\ 0 \\ 0 \\ 2i \\ 0 \\ -2i \\ -2i \\ 0 \end{pmatrix} \quad \text{or} \quad \frac{1}{4\sqrt{2}} \begin{pmatrix} -2i \\ 0 \\ 0 \\ 2i \\ 0 \\ -2i \\ -2i \\ 0 \\ 0 \\ -2i \\ -2i \\ 0 \\ 2i \\ 0 \\ 0 \\ -2i \end{pmatrix} \quad \begin{array}{cccc} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{array}$$

whenever $\beta_1 + \beta_2 - \beta_3 - \beta_4$ is equal to 0 or π respectively.

In the first case the probability is only non-zero for *odd* numbers of negative spin components among the $\sigma_1'\sigma_2'\sigma_3'\sigma_4'$, while in the second case the probability is only non-zero for *even* numbers of negative spin components.

Whenever $\beta_1 + \beta_2 - \beta_3 - \beta_4$ is equal to 0 the probability for an odd number of negative spin components (a particular attribute of the system) is

$$P(\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}{-\tfrac{1}{2}} + \tfrac{1}{2}\tfrac{1}{2}{-\tfrac{1}{2}}\tfrac{1}{2} + \tfrac{1}{2}{-\tfrac{1}{2}}\tfrac{1}{2}\tfrac{1}{2} + \tfrac{1}{2}{-\tfrac{1}{2}}{-\tfrac{1}{2}}{-\tfrac{1}{2}} + {-\tfrac{1}{2}}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2} + {-\tfrac{1}{2}}\tfrac{1}{2}{-\tfrac{1}{2}}{-\tfrac{1}{2}} + {-\tfrac{1}{2}}{-\tfrac{1}{2}}\tfrac{1}{2}{-\tfrac{1}{2}} + {-\tfrac{1}{2}}{-\tfrac{1}{2}}{-\tfrac{1}{2}}\tfrac{1}{2}|Y) = 1$$

… (13)

Whenever $\beta_1 + \beta_2 - \beta_3 - \beta_4$ is equal to π the probability for an even number of negative spin components is

$$P(\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}+\tfrac{1}{2}\tfrac{1}{2}-\tfrac{1}{2}-\tfrac{1}{2}+\tfrac{1}{2}-\tfrac{1}{2}\tfrac{1}{2}-\tfrac{1}{2}+\tfrac{1}{2}-\tfrac{1}{2}-\tfrac{1}{2}\tfrac{1}{2}+-\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}-\tfrac{1}{2}+-\tfrac{1}{2}\tfrac{1}{2}-\tfrac{1}{2}\tfrac{1}{2}+-\tfrac{1}{2}-\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}+-\tfrac{1}{2}-\tfrac{1}{2}-\tfrac{1}{2}-\tfrac{1}{2}|Y)=1 \qquad \ldots (14)$$

Because the probabilities are equal to 1, we *expect* an odd number in the first case and we *expect* and even number in the second case.

Following the argument of Greenberger et al, we define a function $A(\beta_1)$ to be ± 1 according as $\sigma'_1 = \pm\frac{1}{2}$, i.e. according as the spinning motion of particle 1 actually possesses the property that its z component of spin in O_1 is $\pm\frac{1}{2}$. Similarly we define functions $B(\beta_2), C(\beta_3)$ and $D(\beta_4)$ in relation to the z components of spin σ'_2, σ'_3 and σ'_4. Then, from the definition of these functions, and on account of the results (13) and (14), we *expect*

$$A(\beta_1)B(\beta_2)C(\beta_3)D(\beta_4) = -1 \qquad (15)$$

whenever $\beta_1 + \beta_2 - \beta_3 - \beta_4 = 0$ and we *expect*

$$A(\beta_1)B(\beta_2)C(\beta_3)D(\beta_4) = 1 \qquad (16)$$

whenever $\beta_1 + \beta_2 - \beta_3 - \beta_4 = \pi$.

However, we cannot reason with relations (15) (and/or (16)) for more than one set of qualifying β values at a time. For such relations generally reflect expectations of incompatible attributes of the four particle system. Since Greenberger et al are not restricted in this regard they are able to derive contradictions. For example, if they set all β values equal to zero in (15) and set $\beta_1 = \pi$ and all other β values equal to zero in (16), and then multiply the two resulting equations together they can show that $A(0)A(\pi) = -1$; but by multiplying the four equations based on (15) with $(\beta_1, \beta_2, \beta_3, \beta_4) = (0,0,0,0)$, $(\frac{\pi}{2}, 0, \frac{\pi}{2}, 0)$, $(\frac{\pi}{2}, 0, 0, \frac{\pi}{2})$ and $(\pi, 0, \frac{\pi}{2}, \frac{\pi}{2})$, they get the contradictory result $A(0)A(\pi) = 1$.

We, however, are unable to reason in this way. Nor, of course, can expectations of (15) (or (16)) with more than one set of qualifying β values be generally confirmed by measurement because this would necessitate measurement of incompatible properties which is forbidden by the uncertainty principle.

However, the argument of Greenberger et al serves to show that we definitely cannot claim that expectations (15) and (16) (for beta values satisfying $\beta_1 + \beta_2 - \beta_3 - \beta_4 = 0$ and $\beta_1 + \beta_2 - \beta_3 - \beta_4 = \pi$ respectively) reflect *physical laws* controlling the numbers of positive and negative spin components. Although physical laws *can* be the reason for calculated probabilities of events being equal to 1, they evidently are not the reason in this case. Neither can either of the meanings (i) and (ii) of probability 1 (in section 5.5) apply. For, then, relations (15) and (16) would apply simultaneously to all qualifying β values, and contradictions would arise.

Since the attributes expressed by equations (15) and (16) (for different qualifying β values) cannot all be present, at least some must be absent (on any one occasion) so we might expect at least *occasionally* to find one of the attributes absent. We might expect, for example, to sometimes find an *even* number of negative spin components when $\beta_1 + \beta_2 - \beta_3 - \beta_4 = 0$. Why do we not? Well to account for why we do not we should calculate the probability for there being an even number of negative spin components. This we have already done using the quantum formalism as the way to calculate probabilities, and the result is zero. So we do not expect to find an even number of negative spin components when $\beta_1 + \beta_2 - \beta_3 - \beta_4 = 0$, and it seems we never do.[24] This may appear amazing but no contradiction between theory and experiment can actually be demonstrated. As we have noted before, the repeated fulfilment of our expectations (in many trials) should not lead us necessarily to think (by induction) that what we expect to be true must always be true.

6.5 The Kochen-Specker paradox

Accounts of the Kochen-Specker paradox are given in [15], [18] and [19]. The paradox arises in connection with the inferred properties (of section 5.4), when we imagine assigning values (at any one time t) to all the inferred properties of a system. Using a certain logical argument it can seem impossible to make such an assignment, while our realist approach clearly requires that it be possible.

Starting with one inferred property going with a particular complete set of orthogonal functions $\phi_j(x_i)$ ($j = 1,...N$), we can set up N orthogonal unit vectors v_j ($j = 1,...N$) from the origin of the function space of N dimensions. Then, with our quantum mechanical system in any particular condition, we can imagine each vector v_j ($j = 1,...N$) to be labelled 1 or 0 according as P is (at time t) quantified by that particular value of j or not. So only one of those vectors will carry the label 1, the rest being labelled 0. We can imagine appropriately labelling the vectors v'_j ($j = 1,...N$) similarly associated with property P' according as P' is in fact quantified by j or not, and similarly labelling the v''_j associated with property P'', …etc. The set of vectors v'_j or v''_j …etc. can be formed by rotating the original set v_j (as a whole) about the origin of the function space in an appropriate manner.

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[24] It is as if nature plays a trick on us - fulfilling our expectations whenever we test them but evidently not always fulfilling them when we choose not to! This is undoubtedly the strangest result of the present interpretation, but one which may come, in time, to be regarded as natural. (Zeno's four famous paradoxes once caused a crisis in Greek thought, but, in time, we got not to worry about Zeno's theoretical objections to the possibility of motion. We got to believe that an infinite number of steps *could* sometimes be completed in a finite time, and so on.)

Then whenever a set v'_j (formed by such a rotation) shares a common member with another set v''_j similarly formed, Kochen and Specker feel the logical need to assign the *same* label (1 or 0) to the shared vector; but, when applying that rule generally, consistent labelling of the vectors in the sets formed by all possible rotations of the set v_j is shown by Kochen and Specker to be impossible (at least when $N \geq 3$ which of course is very often the case).[25]

Now, there would indeed be a need to assign the same label (1 or 0) to a shared vector, say v'_j and v''_k (equal for a particular value of j and a particular value of k) if the properties P'_j and P''_k associated with the vectors where the same or where physically correlated with each other (so that one was present if and only if the other was). We need not, and do not, claim such sameness or correlation. We claim only that when we (fully) expect P'_j to be present we should also (fully) expect P''_k to be present, and vice versa, and therefore that

$$P(P'_j|P''_k) = P(P''_k|P'_j) = 1. \qquad (17)$$

-a result that in fact follows easily from (4) and (5) of section 5.4.

The Kochen-Specker paradox is therefore only telling us that we cannot interpret (17) as meaning properties P'_j and P''_k (for the particular values of j and k) are the same or are physically correlated. We therefore simply suppose that knowledge ((full) expectation) of one property entails (full) expectation of the other. We do not commit to viewing proposition P'_j as implying proposition P''_k or vice versa. Hence, for us, there is no paradox.

6.6 The paradox of particle reflection by a drop down in potential

Pedro, L. et al [20] have drawn attention to the fact that the one-dimensional Schrödinger equation of quantum mechanics predicts the possibility (non-zero probability) of reflection of a particle by a drop *down* in potential, i.e. by a region in space in which potential $V(x)$ falls monotonically from one constant value to another. They show how a one-dimensional particle wave packet approaching the drop down in potential is partially reflected and partially transmitted by it, and therefore that the particle momentum may be reversed.

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[25] For the purpose of the proof it is sufficient to consider only complete sets of orthogonal wave functions derived from the one set $\phi_j(x_i)$ by rotation (in function space) using *real-valued* rotation matrices, so that the orthogonal vector sets v_j, v'_j, v''_j,… can be pictured in a *Euclidian* space of N dimensions, and a proof of impossibility in the case $N = 3$ is effectively given in the argument on pp166-168 of [19].

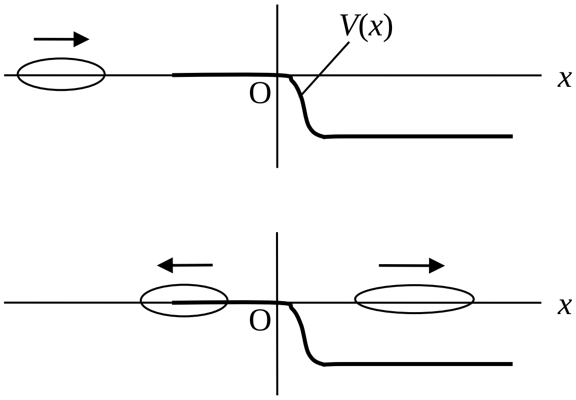

Figure 3. Wave packet (incident from the left) partially reflected by and partially transmitted through a drop down in potential.

This, of course, violates Newton's second law of motion according to which particle momentum (mass times velocity) must be *increased* by motion of the particle into a region of negative potential gradient. It also seems to pose a paradox for us, if we claim (as we do) that Newton's law applies to the rate of change of the particle's internal property of momentum as it drifts through a scalar potential field (see section 1). For, we could start off fully expecting the particle to have a certain definite momentum p_0 -as in a very long wave packet of uniform de Broglie wavelength approaching the potential drop (a packet that could extend as far as we like to the left). Then, at a later time, quantum mechanics gives two long (transmitted and reflected) wave packets and a non-zero probability for momentum $p = -p_0$. This seems to violate our law of momentum change.

Note, however, that the uncertainty principle tells us that the particle's initial momentum and the particle's final momentum (after impact with the potential drop) are *incompatible properties*. They cannot both be known ((fully) expected) from the start under any initial conditions. This has an effect on our logical reasoning.

If Y = 'the initial particle momentum $= p_0$' and X = 'the final particle momentum is greater than or equal to p_0', then, according to us, proposition Y implies proposition X (because the potential drop can only increase the momentum). However, because X and Y claim incompatible attributes it does not follow that the probability $P(X|Y) = 1$. Rule (ii) in section 3 prevents us from drawing this result. To find $P(X|Y)$ we must instead apply quantum mechanics, and this gives a positive value less than one. The probability $P(\overline{X}|Y)$ for the final momentum being less than p_0 is calculated to be greater than zero. These results can be confirmed by measuring the final momentum in many trials. If the final momentum in one trial is measured to be $-p_0$ no contradiction arises because (a) our inference regarding final momentum is one of expectation only, and (b) our initial knowledge (of momentum being

p_0) amounted again only to (full) expectation of the momentum value -not *certainty* with regard to its value. We thus avoid contradiction.[26]

We see in this example the importance of not claiming to be *sure* of the dynamical property on which pure knowledge is based, but only claiming (full) *expectation* of it inferred from measurements or system preparation. Our theorising (using the quantum formalism) is still useful however, because our (properly calculated) expectations seem always to be confirmed when we test for them; that is macroscopic effects (individual events or frequencies in repeated trials) can be reliably predicted.

6.7 Why do our expectations seem always to be fulfilled when we test them, while they are evidently not always fulfilled when we do not?

There is no logical need to answer this question, and in the interests of simplicity we are probably better *not* doing so. However if we should really feel the need to do so, it might be possible to proceed as follows.

We note first that when we test an expectation under a pure-state of knowledge, we need to set up measuring apparatus. The presence of this apparatus and the way it is set (to measure, for example, particular particle spin components rather than others) might, for all we know, be somehow ensuring that our expectation is actually brought about: be that an expectation of a particular system attribute (such as an odd number of negative spin components in section 6.4) or of a particular mean value over many trials (as in section 6.2). That is, the presence of the apparatus, set the way it is, might be partly determining (through physical laws we do not know) the motion of the quantum mechanical system under study in such a way that leads to our expectations being fulfilled.

Let us suppose this is the case.

Then to preserve the consistency of our interpretation, we need to claim that while this new knowledge (regarding the effect of settings of measuring apparatus) is of physical import, it is nonetheless *redundant* knowledge. That is, redundant as far as our pure-state quantum mechanical calculation of probabilities of system properties is concerned. The calculation of those probabilities need be based only on knowledge of the system dynamics acquired from the state of the apparatus used to *prepare* the system. That knowledge is of course necessarily limited on account of the uncertainty principle, and the possibility is left open that the existence and settings of measuring apparatus employed in the future are indeed contributing to the determination, *right from the start*, of those initial properties of the system

[26] In [5] we took knowledge of a physical property to reflect *certainty* of that property. On that assumption, our initial knowledge (of momentum being p_0) means that that particular momentum *certainly* applies initially, and when we attain (by measurement) knowledge that the final momentum is $-p_0$ this is *certainly* the final momentum. Hence the paradox of reflection *is* present under this interpretation of knowledge, and we cannot then maintain the claim (declared in the 'third law of potential action' in [5]) that momentum *always* increases at the rate it does in classical mechanics. We may, however, maintain that claim when we take (as we do in the present paper) knowledge to reflect only (full) expectation.

that we do not know. Since we have no knowledge of how this part-determination (by future measuring apparatus) of those unknown properties comes about, our (newly claimed) knowledge of the reality of this part-determination is naturally redundant; it is information that cannot contribute to our probability calculations.

While our new knowledge (of this part-determination) is redundant in the way described, it *does* serve in our assessment of the *meaning* of probabilities calculated to equal 1. So, for example, when in section 6.4, we calculate a probability equal to 1 for an odd number of negative spin components in a particular set of directions, we may now take this to mean that, *with future measuring apparatus set up to confirm this expectation* we can be *certain* that an odd number is present. While, with future measuring apparatus not set up in that way we just *expect* an odd number of those spin components to be present without being certain this follows from our initial knowledge of system dynamics.

We stress that this in no way implies that a change in the directional settings of the spin component measuring apparatus after system preparation in the pure state (11) must be changing spin components of the particles during their flight. Rather we can still assume (as a physical law) that all particle spin components (in the absence of a magnetic field) remain constant during particle flight (whatever the apparatus settings) but that those spin components are, *from the time of system preparation*, partly determined by the settings of the measuring apparatus present at the later time.

Similarly, when we try to confirm our expectation regarding the value of the correlation coefficient (6) between spin components of two particles in a singlet state, it is not that setting a spin measuring apparatus to measure a particular spin component of one particle sometimes instantly alters the spin component of the other particle which is about to be measured. It is rather that nature knows already which spin components are to be measured even if these are chosen at the last minute, and it sees to it that the spin components at the moment the particles are prepared (ejected from the source in a singlet state) are such as to fulfil our expectations when the spin components are measured. So there is no need for any 'spooky action at a distance'. Instead, as Bohr might have said, the whole experimental set up contributes to the determination of the motions of a quantum mechanical system under study.

Interestingly this rules out the possibility of free will on our part. That is, we cannot outdo nature by making a choice (of apparatus settings) that nature does not already know about! This should really not surprise us, for we have good reason to believe (from the space-time picture in relativity theory) that the state of all matter is fixed throughout time once and for all.

Another case in which (full) expectation might be taken to mean certainty occurs with regard to particle motion in the fine-tuned interferometer in section 6.1.

There (without null detection) the sum rule (1) gives, for the probability that the particle exits the one way, the result

$$\int \left|\Phi(\boldsymbol{r}|Y)\right|^2 d^3\boldsymbol{r} = 1,$$

where the integral is taken over the space occupied by the exiting wave packet, $d^3\boldsymbol{r}$ being a volume element of space, Y our pure-state of knowledge of the particle motion, and $\Phi(\boldsymbol{r}|Y)$ the wave function (in the position representation) at a time when the particle has left the interferometer. We are thus led to *expect* the particle to leave the interferometer the one way without being certain it must. Put that way, our view of the matter is the same as that given in section 6.1 of the present paper.

There is, however, an alternative view (coinciding with the main line taken in [5]) according to which exiting one way from the fine-tuned interferometer is a physical consequence of the experimental setup, i.e. that (full) expectation (that the particle exits one way) reflects *certainty* on account of physical law. In [5] it is suggested that when a particle moves from one place to another its motion can be affected not only by the potential fields it experiences in its vicinity as it travels but also in general on the potential fields along paths it *might* have taken but did not. (This is reminiscent of the principle of least action formulation of classical mechanics where, in effect, the whole potential field determines the path actually followed by a particle.) Then, in the fine-tuned interferometer of section 6.1, the potential associated with the mirror the particle does not hit might still contribute to determining the particle's motion through the final beam splitter ensuring it exits always one way, and when null detection is achieved (in the way discussed in section 6.1), the apparatus used for this null detection then effectively eliminates one possible path through the interferometer so removing any influence of the mirror in that path. That is, potentials associated with the null detection itself contribute to determining the path taken by the particle when it leaves the interferometer.[27]

It happens often in physics, that the detailed assumptions of, ways of looking at, or ways of formulating a theory are to some extent up to us. That is, a certain amount of personal choice is possible in this regard.[28] So whether or not, in quantum mechanics, a particle's motion is partly determined by the presence of apparatus in the future designed to measure a property of it, or whether or not a particle's future motion is taken to be dependent on potentials in paths the particle might have taken in the past but did not, may be a matter of personal choice. Only in the one case, when future apparatus and potentials outside the paths followed by particles are assumed to have no effect on particle motions and the question *why our expectations are seemingly always fulfilled when we test them* is considered not to be in need of answering, might our theory of the physical world qualify as a 'local' theory –a

[27] Only under this view of things can we adopt the claim made in [5] (see page 406 of [5]), that inferred properties are physically brought about by acquisition of pure knowledge that renders them (fully) expected. This is because that claim implies null detection must generally affect particle motion. To show this connection, we can take the two wave packets inside the interferometer together at one time to form one of the complete set of orthogonal wave functions $\phi_j(x_i)$ of section 5.4. The property value P_j associated with this wave function then certainly applies without null-detection but does not necessarily apply just after null-detection (when just one packet is present), and this is inconsistent with the null-detection having *no* effect on the particle's motion.

[28] We make personal choices, for example, in choosing which fields are fundamental in classical electromagnetism, or in choosing whether to base classical mechanics on Newton's laws or on the principle of least action.

theory in which the motion of particles is claimed to be influenced only by the fields they are experiencing in their immediate localities.

7. The modelling of measurements

In the usual interpretation of pure-state quantum mechanics, the modelling of measurements is not really possible. It is the one 'no go' area of the subject because the collapse of the wave function is not covered by the theory. For us, however, measurements *can* be modelled *within the theory*, and without having to employ photons or other matter not present in the (non-relativistic) theory itself.

Several examples of ideal measurement modelling are provided in Chapter XIII of [5] together with explanations for the (unpredictable) effects those measurements can have on properties incompatible with the property being measured. The fuller theory in [5] is not however needed for this purpose. We give, in sections 7.2 and 7.3, brief accounts of these models using only ideas presented in this paper, but first we pause to reflect on the form the theory of the electromagnetic field should take in quantum mechanics, because this is crucial in relation to our measurement modelling.

7.1 The electromagnetic field in quantum mechanics

Since our quantum mechanical theorising is all non-relativistic, it would be unfitting to employ the full Maxwell electromagnetic field theory which is really a *relativistic* theory. Instead, for consistency, we should employ a form of Maxwell's equations that is *Galilean* invariant (not Lorentz invariant). That means leaving out displacement currents and adopting only the quasi-static and quasi-stationary equations of the theory. Then, moving point charges do not generate magnetic fields but only electric fields, and the currents generating external magnetic fields cannot be modelled by moving charges but must be taken as sources in their own right distinct from charge sources.[29]

In employing this theory of electromagnetism, we should not consider it to be an approximation of the 'proper' theory, but as the actual theory going with quantum mechanics. That way there is no problem in understanding, for example, why electrons in an atom do not radiate electromagnetic waves (since there are no electromagnetic waves in the theory), and we can model apparatus suitable for measurements that *in principle* our *non-relativistic* theory allows. Only in relativistic quantum theory is there need for a fuller (Lorentz invariant) form of electromagnetism.

[29] See Appendix C of [5] for a fuller discussion of this.

7.2 Measurement of the position of a charged particle and the effect on its momentum

In modelling ideal measurements we may employ any apparatus built of matter allowed in quantum mechanics. We may use (in principle) particles of any mass and charge with any law of inter-particle potential we care to invent.[30]

By employing particles (material *points*) of very high mass, carrying charges and sensitive to appropriate inter-particle potentials of non-electromagnetic kind, we can 'construct' means for generating any specified time dependent electric field $\boldsymbol{E}(t)$ in a region R of space that may be as small as we please. We just have to imagine suitable particles assembled near the boundary of the region and to let the inter-particle potentials vary in time so as to move those particles about keeping them always close to the boundary. By using equal numbers of particles with a positive and with (an equal) negative charge, and by choosing that charge small enough and the number of particles large enough, we can make the particles form a layer with a charge density (charge per unit area) and a dipole density (normal dipole strength per unit area) variable over the boundary, and variable in time (subject to a zero net charge). In this way the jump in electric potential and the jump in the normal component of electric field across the boundary can be independently adjusted to produce any specified field inside region R and any field (that does not require a net boundary charge or sources at infinity) outside that region. Being of very large mass, the particles move according to the laws of classical mechanics (i.e. quantum theory is not needed for them). Our measuring apparatus is therefore purely classical.

The 'apparatus' described can be designed to produce a large pulsed uniform electric field in the x direction inside region R with no electric field outside that region at any time (see Figure 4).[31] This apparatus thus sits dormant producing no electric field till, at a certain time, it generates its pulsed uniform electric field within the region (which region, as we have said may be as small as we please), and then falls silent again.

[30] Of course, in non-relativistic atomic and molecular physics for example, the particles (electrons and nuclei) have charges that are multiples of the electronic charge and masses that have certain restricted values, and the inter-particle potentials are all of Coulomb type. This is not, however, a requirement of *quantum mechanics*. The only fundamental constant of the theory is Planck's constant, and there is no *theoretical* restriction on particle charges and masses or on inter-particle potentials.

[31] Strictly speaking, we should let the masses of the particles tend to infinity, their number tend to infinity and their charge tend to zero. The electric field they produce is then a collective effect and the field of any one particle is vanishingly small and has no effect on a charged particle whose position is to be measured (this is more fully explained in section 2.3 of Chapter XIII of [5]).

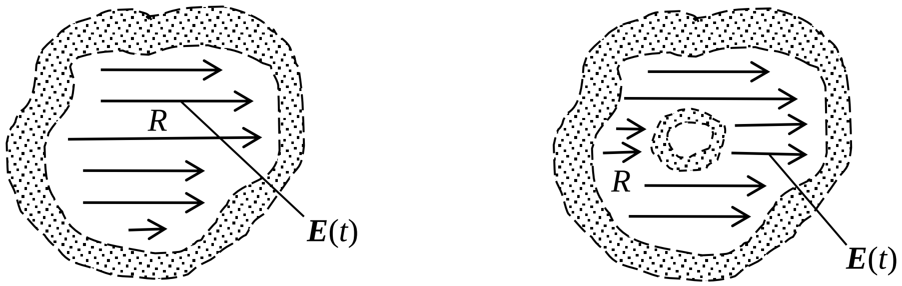

Figure 4. Swarm of charged particles in the boundary of region *R* producing a uniform electric field $\boldsymbol{E}(t)$ in the region. Left: Simply bounded region, Right: Region between two closed boundaries. (The boundaries are shown with exaggerated thickness.)

Now envisage a free quantum mechanical particle (for example an electron), our pure knowledge of its motion being represented by a wave function $\phi(\boldsymbol{r})$ occupying a volume W of space comparable to the size of our apparatus. Let $\overline{p}$ be the expected mean value of the particle's momentum and suppose our apparatus is placed in or over W so that an electric field pulse appears over only part of W (as, for example in Figure 5).

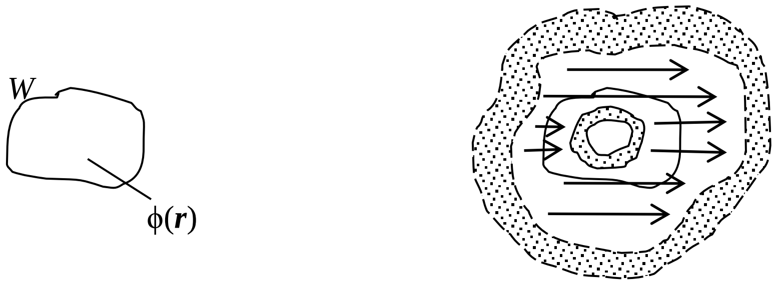

Figure 5. Left: Wave function confined to region *W*. Right: Apparatus placed over *W* so electric field will act over a part of *W* only.

Before and after the pulse, our apparatus has no effect on the particle (the charge and dipole densities in the boundary of region R being zero then and the electric potential being zero everywhere). The effect of the pulse can be judged using the Schrödinger equation in which, owing to its very large magnitude, the potential dominates in the Hamiltonian all be it for a short time only. The effect is to leave the part of the wave function outside R unchanged, and to multiply the part of the wave function inside R by $e^{ikx\delta/\hbar}$ where k is the (large) electric field strength times the particle's charge and δ the duration of the pulse. With $k\delta >> \overline{p}$ (as we suppose it is), this has the effect of setting the part of the wave function in R at the time (we call this the sub-packet) into uniform motion in the x direction at a high velocity (Figure 6).

Figure 6. Part of the wave function left behind (on the left) and the remaining part of wave function (on the right) set in fast motion by the electric field pulse and, by now, far away from its original position in *W*.

The remaining wave function (which we call the 'new' wave function $\phi'(\boldsymbol{r})$) now occupies a smaller region of space and it starts to evolve differently as a result. This evolution is relatively slow and unimportant, while the motion of the sub-packet is fast, and it is soon far away. The sub-packet is a wave packet containing many de Broglie wavelengths and therefore, on its own, would represent a particle expected to move closely to a classical path.

Just after the applied pulse, we expect the particle must have either remained behind in the region occupied by the 'new' wave function or have set off with the sub-packet. Supposing we fail to detect a particle in a particle detector placed in the path of the sub-packet. We then expect the particle to have stayed behind, and we may (in accordance with our rule in section 5.2) collapse our wave function to the (renormalized) 'new' wave function $\phi'(\boldsymbol{r})$.

Our expectation distribution of the particle's position is then sharper than before, and by shaping and positioning the region R and by employing an inner and outer boundary to R so that R covers all but a small part of W, we can, by chance, effectively get to know the particle's position to any required accuracy. We might get to know just its x coordinate as accurately as we please by applying our electric field pulse in two regions R one to the left and one to the right of two closely spaced planes perpendicular to the x axis and crossing W

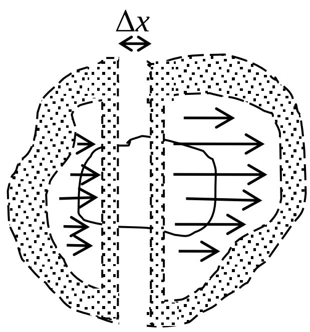

Figure 7. Apparatus used to improve knowledge only of the *x* coordinate of the particle

as in Figure 7.

We might thus measure the particle's position (or just its x coordinate), but in doing so, do we affect its momentum? Well we might do, because in the boundary (or boundaries) of R there are very large electric field strengths lying within the dipole layer(s) and these might change the particle's momentum greatly.

When narrowing our expectation of the particle's x coordinate, for example, the electric fields in the dipole layers (over the plane boundaries of the two R regions) point in the x direction and tend to infinity as the dipole layers are made infinitely thin. Taking the expected change in momentum proportional to the electric field (as in classical electrodynamics) the expected momentum imparted to the particle (when it is not detected in the sub-packet) stays the same. That is, it cannot be made smaller and smaller by diminishing the thickness of our dipole layers. An order of magnitude calculation (section 2.4 of Chapter XIII of [5]) shows that the expected (i.e. the mean) disturbance Δp_x to the x component of momentum is at least as great as the uncertainty relation $\Delta x \Delta p_x \approx \hbar$ would imply, Δx being here the spacing between the two plane boundaries into which the particle's wave function is collapsed.

Note that the relation $\Delta x \Delta p_x \approx \hbar$ above, is not a *general* formula for the expected increase Δp_x in uncertainty of the x component of momentum following a measurement in which our uncertainty of the particle's x coordinate falls to Δx. As we saw in section 6.1, when the wave function of a particle takes the form of two wave packets in separate regions of space, it is quite possible (by null detection in one of the regions) to acquire narrower expectation distributions over both the particle's position *and* the particle's momentum. We must be mindful of the difference between the likely physical effect of a measurement on a property incompatible with the one being measured, and the change in our expectations in relation to the value of that property following the measurement and the new knowledge the measurement gives us. These are generally quite different things.

By repeating the position measurement procedure in many trials of the same quantum mechanical process, and counting the relative number of successful measurements, we can confirm (as far as is possible) the probability for the particle being in any selected region of space as predicted by the quantum theory.

Also, if a successful position measurement in a particular trial is immediately repeated, the result will evidently be the same. That is, position measurement is reproducible, supporting the claim that such a measurement does not change the particle's position.

7.3 Measurement of the z component of spin of a particle and its effect on other spin components

In Chapter 5 of [17], Feynman et al describe Stern-Gerlach type filtration apparatus that might be used to determine, by chance, the component of spin of a (charge-free) particle in any chosen direction (say the z direction) while leaving that spin component unchanged. The particle moves near the classical limit in a well-defined path (along the y axis) through the apparatus. The magnetic field H_z has a strong gradient in the z direction causing the path to bend by a small amount depending on the component of spin in the z direction. By employing three Stern-Gerlach magnets in series each possible path can be bent back to the path the particle would have taken if no Stern-Gerlach magnets were present. By blocking all but one of the paths inside the central Stern-Gerlach magnet we can filter out all spin components except one.

In section 3 of Chapter XIII of [5] a somewhat different method for spin component measurement is suggested which effectively accomplishes the measurement instantly and returns the particle to the same position as it was before. We now briefly describe this method.

Supposing, as in section 7.2, that the spatial component of our particle wave function is confined at some time to a small volume W of space. At that time, a magnetic field H_z with a strong gradient in the x direction (rather than the z direction) is applied over W.[32] The field is pulsed, i.e. switched on and then off again. The particle experiences an effective potential gradient in the x direction, the potential being the potential of its magnetic moment in the magnetic field H_z. This potential gradient is proportional to its spin in the z direction. This causes the region W occupied by the wave function to start moving as a wave packet in the x direction at a velocity that depends on the component of the particle's spin in the z direction. The bigger that component, the faster the packet travels. Generally, however, the spin component will be unknown, with the particle's wave function containing the z component of spin as one of its variables. This wave function can be expressed as the superposition of wave functions each of which is a product of a function of position and a (delta) function of spin component indicating a definite value of that component. Under the magnetic impulse the wave packet therefore divides into as many parts as there are possible spin components. At some time after the magnetic impulse is applied, the parts of the wave packet are well separated in space and an impulse of opposite sign and twice as great as the first is applied to each one of the separate wave packets. This has the effect of reversing the velocity of those packets and returning them, in time, to the position of the original wave function. At the moment they arrive back a third impulse is applied, equal to the first, to bring the packets to rest again. All this can be accomplished quickly, i.e. in a time small compared to the time of natural evolution of the wave packets (aside from the motions induced in them by our apparatus). The reassembled wave packets brought back to rest add up (in the limit as the impulses are made greater and greater) to exactly the same wave function that we started with.[33]

If, during the process, we switch on particle detectors in the paths of all but *one* of the flying wave packets, and if no particle is detected, we expect the particle to be in the packet

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[32] This requires a uniform current density source to be present in the *y* direction filling the space between two planes (perpendicular to the *x* axis) one on either side of region *W*. This current need not interfere with the particle whose spin component is being measured. (See *Note on the relativistic modelling of macroscopic electromagnetic field sources* in Appendix C of [5].)

[33] Note that the argument used on pp 350-351 of [5] to confirm that the particle itself returns to the same position in the limit is too much of a stretch. It depends on taking two limits (of region W tending to zero and of the applied magnetic impulses tending to infinity). These limits are incompatible. Nonetheless, we may reasonably argue the particle *is* returned to the same place (in the limit of infinite magnetic impulses) just because its velocity *relative* to the wave packet motion during measurement, is expected to be finite, while the wave packet is returned instantly (in the limit).

that was not subjected to momentary detection, and therefore expect it to have the corresponding component of spin. The wave function is then collapsed to the part carrying that particular value of spin component.

The measurement is evidently repeatable supporting the view that a spin component is a real attribute of a particle that can be measured without changing that spin component.

By switching on particle detectors in all but a *number* of flying wave packets, we can determine by chance, through null-detection, that one or other of the corresponding spin component values is expected to apply just before and just after our measurement. This allows us to collapse the particle's wave function in a general manner with respect to its spin component variable.

Does a measurement of the spin component in one direction affect the components of spin in other directions? Well, yes it does, at least if we assume, as we do, that spin precession in a magnetic field is a real process. During the motion of each wave packet induced by the measurement, we have no way to deduce that the particle retains *exactly* the same position *inside* its wave packet. It may drift a little. Therefore the magnetic field it experiences, and the rate of precession of any one spin component, is likely quite different at the different stages of the measurement process due to the high gradient of the applied fields. So the net turning (by precession) of any spin component (other than the z component) is not zero.[34] If a spin component in another direction (say the y direction) is initially known, and we successfully measure the z component in the way described, then the y component will have undergone precession about the z direction, and after the measurement of the z component, the y component value will apply to a different axis in the xy plane, and the component value in the y direction will now be different, being the *unknown* component in the direction which precession caused to turn into the y direction. This makes it impossible for us to measure the particle's spin components in more than one direction at once.

7.4 Modelling the preparation of pure-states

Related to the modelling of measurements is the modelling of pure-state preparation processes. Both are possible under the realist interpretation of quantum mechanics we are proposing.

As an example of pure-state preparation, suppose, using classical apparatus, we have managed to establish that a single electron is present in an enclosure with perfectly reflecting walls and that its kinetic energy is less than or equal to a known amount T. Taking this to be all our information about the electron's dynamics, we are in a mixed state of knowledge of that dynamics. This mixed state can be represented (in the position representation) by an array made up of the wave functions of all the orthogonal stationary states of the electron in the enclosure (except those of energy greater than T) and associated positive weights (one

[34] This remains true even in the limit as the magnetic field gradients tend to infinity and the wave packets are instantly returned.

for each wave function). Using Jaynes's theory of probability in connection with mixed states[35], we can apply the principle of maximum entropy or the (extended) principle of indifference to deduce that the positive weights are all equal. If we switch off the potential field enclosing the electron and successfully perform an accurate measurement of the position of the particle following the method of section 7.2, each wave function of our array collapses to one-and-the-same delta function of position. Our array then represents a *pure-state* with the electron in a known position.[36] With the aid of suitable potential fields applied from this time onward, we can arrange for the wave function to evolve into any other form we wish. Any pure-state of knowledge of the electron's orbital motion can thereby be prepared.

8. Concluding remarks

By passing from a frequency interpretation to a kind of *rational* Bayesian interpretation of probability, it seems possible to construct a *realist* interpretation of quantum mechanics. In this realist interpretation, what were previously viewed as only 'potential properties' (like particle positions and particle momenta) become *actual* properties possessed by a quantum mechanical system, and probabilities refer not just to future 'personal experiences' of an individual (as the QBists think with their subjective Bayesian approach to probability) but to actual physical properties supposed real.

In proposing this realist view we have to relax the interpretation of probability equal to 1, and take it to imply (full) expectation rather than certainty, just as the QBists (and subjective Bayesians generally) might. We thus take (full) expectation to be a state of mind in which, by calculation, we fully expect the event in question to occur but do not generally claim it is *certain* to occur.

This change, from certainty to expectation, seems generally necessary anyway in order to resolve a contradiction that arguably arises when applying the principle of indifference to deduce certain (rational Bayesian) probabilities in ordinary life or in classical mechanics (as in the biased die rolling example in Appendix A). It also usefully allows the possibility of recalculating conditional probabilities based on a proposition claiming an event whose probability (before conditioning) is calculated to be zero.

In our proposed Bayesian probability theory for quantum mechanics, the idea that a calculated probability equal to 1 implies 'expectation' (rather than certainty), seems to be an essential requirement. For it appears that, while we might sometimes calculate a probability equal to 1 for *each* of two (or more) incompatible system properties (properties incompatible under the uncertainty principle), and while our expectations of those individual properties seem always to be borne out by measurement, we should not expect the compounded property of the two (or more) of them together[37], for then contradictions will arise. This

[35] See sections 7 and 10 of [12], or sections 5 and 6 of Chapter XIV in [5].

[36] See section 12.1 of Chapter XIV of [5] for more detailed explanation of the realisation of this pure-state.

[37] Neither, of course can we *test* whether or not the two (or more) properties are present together, because the uncertainty principle prevents this.

change in our way of reasoning, accompanying the uncertainty principle, may seem strange, but it is forced on us. It is connected to our claim that joint probabilities of incompatible properties are non-existent, and is thus arising from a change in probability theory rather than a change in the axioms of propositional calculus. (Even though we view (with Jaynes) probability as an *extension* of logic, there is evidently no connection between our work and the work trying to establish a new *quantum propositional logic*, as discussed, for example, in Section 7.4 of [15].)

It is surely not so strange that probability theory should change as science progresses. We have seen already (in relativity theory) how our notions of geometry and of absolute time (both once thought to be unalterable) must change, and change in a way not easy to comprehend at first, and there is likely no limit to the subtle changes we will need to make in our ways of thinking according as new experiences may demand.

In our realist interpretation of quantum mechanics we have adopted the physical picture of quantum mechanical particles performing both irregular motions in space over infinitesimal distances in infinitesimal times, and irregular drifting motions through space over ordinary distances in ordinary (non-infinitesimal) times. We have supposed that a particle's spin and linear momentum components are properties of its infinitesimal motion, and that all the inferred properties of a system of particles (properties represented by complete sets of observables in the quantum formalism) are momentary properties generally dependent on the infinitesimal motions of its particles, and are properties generally dependent (both qualitatively and quantitatively) on the Cartesian coordinate system to which they are referred. It may be that (without invalidating the uncertainty principle or the probabilistic theory of quantum mechanics presented here) more detailed mathematical laws of motion governing the irregular infinitesimal and drifting particle motions (under the action of external and inter-particle potentials) will one day be postulated. This would provide a much deeper insight into the nature of the physical world at the quantum level (even if this nature could never be substantiated by direct observation nor employed to make new predictions).

Now the rational Bayesian probability theory for quantum mechanics presented in this paper relies on the formalism of pure-state quantum mechanics (with the additions we have noted in sections 5.1-5.3). This formalism serves (in this paper) as the set of rules for calculating our probabilities under any pure-state of knowledge we might hold of a system's dynamical properties. As we have said already, we view this set of rules as resulting from a mix of probabilistic rules and physical laws. As Jaynes has remarked, it is necessary to separate out this mix.[38] This is the task tackled in [5], where the simplest possible rules of *complex-valued* probability and the simplest possible physical laws are sought that lead, through Bayesian reasoning, to the same predictions as those of the quantum formalism. Because of this equivalence with the usual formalism, there are no new predictions in [5], but by unscrambling the 'mix', a new probability theory emerges, and some light is cast on the nature of the physical world (at the quantum level).

[38] See footnote 14.

There is no space in this paper to provide a full account of the theory in [5]. In Appendix B, however, we provide a flavour of that theory by (i) explaining the role of phases (of complex-valued probabilities) in relation to what (full) expectation might mean, (ii) giving some of the assumed physical laws, and some of the *deduced* physical laws, and (iii) giving an example of the use of the principle of indifference and of the method of transformation groups as formulated in the theory of complex-valued probability.

The task of constructing a Bayesian theory of complex-valued probabilities for application to quantum mechanics has been tackled by other researches.

In particular Saul Youssef has written several papers ([21]-[25]) suggesting how this task might be accomplished, and he has maintained a list of papers [26] by others pursuing the same end. Youssef argues that a complex-valued probability calculus for quantum mechanics can be provided using Cox's logical derivation (of the Bayesian probability calculus) in [27] (or Jaynes' similar derivation in Chapter 2 of [2]) but dropping the assumption made by Cox (and Jaynes) that probabilities must be represented by real numbers. Youssef derives a complex-valued form of the probability calculus in this way. A difference between Youssef's calculus and ours (in [5]) lies in the sum rule for mutually exclusive propositions. For two such propositions (A and B) claiming compatible properties, the sum rule is (for us) as in (18) of Appendix B with $\Phi(AB|Y)$ put equal to zero, but for Youssef it is

$$\Phi(A+B|Y) = \Phi(A|Y) + \Phi(B|Y) .$$

$\Phi(A|Y)$ denoting the complex-valued probability of A under knowledge Y , …etc. This is in line with Feynman's theory of probability amplitudes, but it raises problems for normalisation, and the idea that the probability of a disjunction of propositions claiming compatible attributes can be zero while the probabilities of the component propositions are different from zero seems hard to reconcile with probabilities reflecting degrees of belief or degrees of expectation under given knowledge.

QBists have tended not to develop a theory of complex valued probabilities as such. Instead, they stick to ordinary positive-valued probabilities (of an individual's future measurement experiences) and try to explain how the quantum formalism used for calculating such probabilities comes about through one or other supposed rational principle for gambling – a principle from which the Born rule follows as the coherent way of formulating probabilities. They are adopting De Finetti's approach to probability and are trying to account for *why* a basic entity (wave function or density matrix) in the quantum formalism serves (through the Born rule) as a compendia of rational probabilities. In particular Fuchs and Schack [28] and Benavoli, Facchini and Zaffalon [29] are working on this kind of theory.

However, it is unclear to us how the quantum formalism can be regarded as (or reinterpreted as) *pure* probability theory, as these researchers seem to think. The reason is that the formalism contains at least one dimensional quantity – the unit of action, or Planck's constant which appears in some of the commutation rules. Physical constants (for example a particle's mass) also appear in the Schrödinger equation. Calculated probabilities based on knowledge of properties (or of past experiences) are generally dependent on the values of these constants. This seems to us to mean that the calculation of probabilities using the formalism is in part a reflection of physical laws. In other words, the formalism should surely be conceived as arising from physical laws *as well as* from laws (or principles) of probability.

By denying physical laws play a part in accounting for the formalism, these researchers seem to us to be missing something, and their work cannot cast light on what is most important for physicists, namely the nature of the physical world. Whereas the approach we are putting forward (particularly as developed in [5]) can do that.

Appendix A: The resolution of some problems in rational Bayesian probability theory

Jaynes [2] has done much to counter criticism of Bayesian methods used to calculate probabilities in ordinary life and in classical physics.

In particular he has indicated (in section 12.3 of [2]) how measure theory can be employed to avoid ambiguity in the way the principle of maximum entropy is applied when probability distribution functions are *continuous*. The point here seems to be that, in applying the principle of maximum entropy to a continuous distribution function (or, indeed, the principle of indifference to a continuous distribution function), one should employ the *natural* measure associated with the variable space in question. For example, if the distribution required is one over the continuous variable x denoting the position of a classical particle along a straight line, the natural measure is that of distance along the line. So if we know only that the particle lies between two points A and B on the line, then, on applying the principle of maximum entropy (or the principle of indifference), we should partition the line between these two points into *equal* intervals of distance, and set the probability for the particle lying in any one interval the same. Our prior probability distribution (probability density) is therefore a constant between A and B and zero elsewhere.

In section 9.2 of [2] entitled 'The poorly informed robot', Jaynes draws attention to a paradox concerning the principle of indifference. Supposing, he says, we roll a biased die many times, say n times, and denote the result as $r_1 r_2 ... r_n$, where integers r_i are in the range $1 \le r_i \le 6$ (all i). Then suppose we give a robot (programmed to calculate probabilities according to Bayesian rules) limited information about the process, namely that a process (of 'order n') has been run and this process has 6^n possible outcomes symbolically labelled as $r_1 r_2 ... r_n$, where $1 \le r_i \le 6$ (all i). We will denote this information by I. The robot is indifferent between the outcomes and assigns probability $\frac{1}{6^n}$ to each. Using the sum rule of probability it accordingly assigns a probability $\frac{1}{6}$ for r_1 being 3, and $\frac{1}{6^2}$ for r_1 being 3 *and* r_2 being 3, etc. If we were to tell the robot the values r_1 and r_2, it would use the product rule to deduce, for the probability of any value of r_3, the result

$$P(r_3|r_1r_2I) = \frac{P(r_3r_1r_2|I)}{P(r_1r_2|I)} = \frac{1/216}{1/36} = \frac{1}{6} = P(r_3|I)\,.$$

In this way the robot calculates that the probability of *any* one value of any one term r_i is independent of given values of the others. Of course the robot is quite mislead by not being told that $r_1r_2...r_n$ denotes an outcome of a set of biased die rolls, and so it is not in a position to learn anything from the information given it about the actual values of some of the r_i. The robot calculates that when n is large, the number three (or any particular number from 1 and 6) is likely to occur in $r_1r_2...r_n$ close to $\frac{n}{6}$ times, i.e. close to the fraction $\frac{1}{6}$ of the time. More precisely the robot calculates, using the sum rule and simple counting, that the probability for just m threes occurring in $r_1r_2...r_n$ is given by the binomial distribution

$${}_nC_m\left(\frac{1}{6}\right)^m\left(\frac{5}{6}\right)^{n-m}$$

and therefore that the probability for 3 occurring in $r_1r_2...r_n$ a fraction between $\frac{1}{6}-\varepsilon$ and $\frac{1}{6}+\varepsilon$ of the time tends to 1 as $n \to \infty$ for any positive value of ε however small. The robot appears to be certain about this limiting relative frequency of occurrence. In other words the robot seems to logically deduce something to be true that isn't true. Jaynes talks a little about this kind of problem on p. 337 of [2] in connection with coin tossing, and it seems he would have argued that the limit $n \to \infty$ is not physically possible (something we did not tell the robot!). Nonetheless there seems to be a contradiction here *in principle*, and, as noted in section 2, this is one of the motivations for us regarding probability 1 as expressing (full) expectation rather than certainty.

If we replace the robot by a person, all the last paragraph still of course applies (so long as our person reasons in a proper Bayesian manner). Now, if we go on to tell this person the *actual* limiting relative frequencies (in $r_1r_2...r_n$) of each of the values 1 to 6 (as judged by us secretly rolling our biased die very many times) then this person must rethink their calculations. They must condition on a proposition whose probability they had determined (under information I) to be zero, but because we take zero probability to indicate only *expectation* of non-occurrence rather than *certainty* of non-occurrence, the person is happy to condition on the new information provided. They might use their imagination (in a way that a robot might currently not be able to do) and come up with the theory that (i) occurrence $r_1r_2...r_n$ is the result of n trials of a random process each with 6 possible outcomes labelled 1 to 6, (ii) the probabilities of the outcomes in one trial are equal to certain values (taken equal to the limiting relative frequencies given to them) and (iii) the outcomes of each trial are logically independent. Then they would have satisfactorily taken into account their new knowledge.

Appendix B: Some details of the complex-valued probability theory

B.1 Complex-valued probabilities

In [5] probabilities take complex values whose squared moduli should be termed degrees of expectation ranging from 0 to 1, and whose arguments should be termed 'phases of expectation' ranging from 0 to 2π. Accordingly, the complex numbers representing probabilities lie within or on the unit circle in the complex plane.

Wave functions in the quantum formalism are (in [5]) complex-valued probability distributions over the possible values of the dynamical property employed in the representation. These complex-valued probability distributions (and the complex-valued probabilities of disjunctions of the basic propositions of a representation) are deducible using a certain complex-valued probability calculus and assumed physical laws together with rules for assigning prior probabilities (generalisations of the rules formulated by Jaynes for assigning real-valued prior probabilities).

Under pure knowledge Y of the dynamical properties of a system, (full) expectation of an event arises when the probability (denoted $\Phi(A|Y)$), of the proposition A claiming the event, is calculated to be of the form $e^{i\alpha}$ where α is the real-valued phase of the probability (defined modulo 2π). That is, an event is *expected* when its probability lies on the unit circle in the complex plane.

In [5], much as in section 5.5 of the present paper, an event that is *expected*, may (i) be a pure-logical consequence of the proposition Y expressing our knowledge, (ii) be a consequence of Y by physical law (independently of our knowledge of truth of Y) or (iii) be brought about (or ensured) physically by the acquisition of our knowledge Y. Or it may be (iv) that none of these apply and we can say only that the event is expected to occur without being *certain* it does.

In [5] we postulate that when the phase α is determinate (has a definite calculated value) then (i) or (ii) above must be true, but when α is indeterminate which of (i)-(iv) applies remains an open question.

Indetermination of the phase of some probabilities arises in [5] through the sum rule for complex-valued probabilities which we state here in the form

$$\Phi(A+B|Y) = e^{i\beta}\sqrt{|\Phi(A|Y)|^2 + |\Phi(B|Y)|^2 - |\Phi(AB|Y)|^2} \qquad (18)$$

where β is a phase which is generally indeterminate, and (both in (18) and (19)) A and B are propositions in the sample space whose atomistic propositions (like the x_i in section 5.1) are claiming basic properties of any one representation. (A and B are thus general disjunctions of those atomistic propositions.)

The product rule (in the same sample space) takes the form

$$\Phi(AB|Y) = \Phi(A|Y)\Phi(B|AY)e^{i(k+\varepsilon)} \qquad (19)$$

where AY constitutes a pure-state of knowledge, and k and ε are the phases characteristic of knowledge Y and of knowledge A respectively and are generally indeterminate.

The rule for wave function collapse (i.e. equation (3) in the present paper) is a consequence of the product rule (19).

B.2 Physical Laws

One of the *assumed* physical laws in [5] (as in this paper) is that any quantum mechanical particle moves continuously but not smoothly through space. So its motion is irregular even at the smallest scales. A particle therefore moves rather like a pollen particle in Brownian motion. Indeed, the derivation of the Schrodinger equation in [5] using the complex-valued probability calculus and Bayesian rules for assigning prior complex-valued probabilities, has much in common with a similar (classical Bayesian probability) derivation of the diffusion equation for Brownian motion of a classical particle (see Appendix H of [5]).

To mention some of the other assumed physical laws, we claim in [5] that (i) particle momentum is an *internal* property of a particle just as particle spin angular momentum is, (ii) spin components stay constant over short times but (in a magnetic field) may jump occasionally from one of their possible discrete values to another, (iii) any particle system in motion can move in exactly the same way when displaced or rotated as a whole in fixed space or when displaced in time, (iv) for every possible motion of a particle system in an external field, the time-reversed motion in the time-reversed field is also a possible motion, (v) potentials are more fundamental than fields, affecting particle motion even when they are uniform throughout space, and (vi) that *two* full rotations are required to bring a coordinate system[39] (or any rigid body) back to occupying space in the same way.

Certain physical laws are *derived* in [5] as consequences of certain calculated probabilities taking the form $e^{i\alpha}$ where α is a *known* (determinate) phase. Of these derived laws, we have (i) that the momentum of a particle stays constant during free motion, (ii) that in a potential of uniform gradient the momentum of a particle increases at a constant rate the same as in Newtonian mechanics, (iii) that the spin components of a particle stay constant when there is no magnetic field and (iv) undergo precession about a magnetic field at a certain definite rate equal to that calculated in textbooks on quantum mechanics.

The physical laws *assumed* in [5] may be adopted as assumed physical laws in the simpler theory presented in the main part of this paper, and the physical laws *derived* in [5] can be

[39] A coordinate system can be modelled in quantum mechanics as a structure of very many particles of very high mass held together by inter-particle potentials. It can include synchronised clocks at every point of the space it marks out. It is a classical system, at rest, or moving in a specified way, and not in interaction with the quantum mechanical particle system under study.

adopted as well, by claiming *certainty* applies to particular (real-valued) probabilities calculated to equal 1. We have done this already to some extent, and in this way we have provided (as we have in [5]) a partial description of physical dynamics at the quantum level.

B.3 The complex-valued probability calculus and laws of prior probability assignment

Along with the sum rule (18) and product rule (19) we claim, in [5], various other rules of the complex-valued probability calculus. In particular, we adopt the well-known rule

$$\Phi(x_i|Y) = \sum_{j=1}^{N} \Phi(x_i|y_j)\Phi(y_j|Y) \tag{20}$$

connecting wave functions in any two representations (x and y), *as a rule of the complex-valued probability calculus.* It is a rule that determines the extent to which the distribution of our degrees of expectation over property x values become less sharp as the distribution of our degrees of expectation over property y values becomes sharper. In adopting (20) as a rule of the complex-valued probability calculus, we thus incorporate the uncertainty principle into probability theory itself. Rule (20) leads to the ‘interference of probabilities’, and the transformation functions $\Phi(x_i|y_j)$ in it (themselves wave functions) are derivable using physical laws and laws of prior probability assignment (claimed in [5]).

One law of prior probability assignment is the principle of indifference which we here state as follows.

Principle of indifference

> If, on the basis of knowledge Y, we are indifferent between the mutually exclusive propositions x_i ($i = 1,...m$ where $m \le N$) claiming basic dynamical properties of a system in one representation, then we should set our degrees of expectation equal, i.e. we should set
>
> $$|\Phi(x_1|Y)|^2 = |\Phi(x_2|Y)|^2 = ... = |\Phi(x_m|Y)|^2 .$$
>
> If, in addition, the differences between the properties claimed by the x_i cannot be expressed *absolutely* using natural measures, natural orders, natural directions or other such concepts, then we are *absolutely indifferent* between the alternative propositions and we should set our *phases* of expectation equal also, and so set
>
> $$\Phi(x_1|Y) = \Phi(x_2|Y) = ... = \Phi(x_m|Y) .$$

We claim no converse to the principle of indifference. (So the above relations being true do not imply indifference.) We do however make the following claim.

Principle of indifference (cont.)[40]

> If we are indifferent but not absolutely indifferent between the propositions x_i ($i = 1,...m$) we should *by default* set the phases of their probabilities (under knowledge Y) unequal, unless of course we have reason (other than indifference) to do otherwise.

To illustrate the use of this principle of indifference, we consider the problem of finding the position/momentum transformation function $\Phi(d^3\boldsymbol{r}|d^3\boldsymbol{p})$ for a free particle. This is the complex-valued probability for the particle being in volume element $d^3\boldsymbol{r}$ at $\boldsymbol{r}$, knowing its momentum is in element $d^3\boldsymbol{p}$ (of momentum space) at $\boldsymbol{p}$.

Taking the volume elements to be equal cubical elements filling all space (i.e. elements of equal natural measure), and noting that (knowing only its momentum) we are indifferent between the particle occupying any one or any other of these elements of space, we should assign probabilities whose squared moduli are all equal, i.e. we should put

$$\Phi(d^3\boldsymbol{r}|d^3\boldsymbol{p}) = \phi_{\boldsymbol{p}}(\boldsymbol{r})\sqrt{d^3\boldsymbol{r}} \quad \text{with} \quad \phi_{\boldsymbol{p}}(\boldsymbol{r}) = ke^{if(\boldsymbol{r},\boldsymbol{p})}$$

where k is independent of $\boldsymbol{r}$, and $f(\boldsymbol{r},\boldsymbol{p})$ is a (non-dimensional) real valued function. The square root over the volume element $d^3\boldsymbol{r}$ is evidently necessary on account of the form of the sum rule (20) according to which our degree of expectation for the particle lying in $d^3\boldsymbol{r}$, i.e. $\left|\Phi(d^3\boldsymbol{r}|d^3\boldsymbol{p})\right|^2$, should be proportional to $d^3\boldsymbol{r}$.

Since there is only one fundamental constant in quantum mechanics (the fundamental unit of action, $\hbar$), we can form only one absolute vector of displacement, namely $\boldsymbol{p}/\hbar$. In any plane perpendicular to this vector we are *absolutely* indifferent between the particle occupying one or other of the points on this plane, because we have no natural way of distinguishing them. Therefore f can only be a function of $\boldsymbol{r}.\boldsymbol{p}/\hbar$. So we see how we have got so far toward calculating the required transformation function using our principle of indifference.

We can get further by applying Jaynes' method of transformation groups generalised to complex-valued probabilities. Consider a coordinate system O' displaced from our coordinate system O by $\boldsymbol{\Delta}$. Under the same knowledge $\boldsymbol{p}$, the problem of finding $\phi'_{\boldsymbol{p}}(\boldsymbol{r}')$ in

[40] The continuation of the principle of indifference differs here from the form given it in [5]. The continuation as stated in [5] was too strong, leading to inconsistency. However, as restated above it serves the desired purpose well enough, (the purpose being to fix (on p.249 of [5]) the anti-symmetry of the wave function for two or more identical fermions).

O' is similar to that of finding $\phi_{\boldsymbol{p}}(\boldsymbol{r})$ in O. Therefore these probability distributions can differ only in a phase factor independent of $\boldsymbol{r}$ (but possibly a function of $\boldsymbol{\Delta}$ and $\boldsymbol{p}$). So we have

$$k'e^{if'(\boldsymbol{r}'.\boldsymbol{p}/\hbar)} = ke^{if(\boldsymbol{r}.\boldsymbol{p}/\hbar)}e^{i\alpha}$$

when $\boldsymbol{r}' = \boldsymbol{r}$, the LHS representing $\phi'_{\boldsymbol{p}}(\boldsymbol{r}')$ and the RHS being $\phi_{\boldsymbol{p}}(\boldsymbol{r})$ times a phase factor independent of $\boldsymbol{r}$. On the other hand, the probabilities for the same event must be the same, i.e.

$$k'e^{if'(\boldsymbol{r}'.\boldsymbol{p}/\hbar)} = ke^{if((\boldsymbol{r}'+\boldsymbol{\Delta}).\boldsymbol{p}/\hbar)}.$$

Using both these relations we have

$$f((\boldsymbol{r}'+\boldsymbol{\Delta}).\boldsymbol{p}/\hbar) = f(\boldsymbol{r}'.\boldsymbol{p}/\hbar) + \alpha$$

for all $\boldsymbol{r}'$. So $f(\boldsymbol{r}.\boldsymbol{p}/\hbar)$ can only be a linear function of its argument, and we find $\Phi(\boldsymbol{r}|\boldsymbol{p})$ has the form

$$\phi_{\boldsymbol{p}}(\boldsymbol{r}) = ke^{i\beta\boldsymbol{r}.\boldsymbol{p}/\hbar}$$

where k and β are independent of $\boldsymbol{r}$, and β is real.

See section 3 Chapter 6 of [5] for an alternative derivation of the position/momentum transformation function in which β and $|k|$ are fully evaluated. See also [5] for the derivation of many other transformation functions of the quantum formalism.

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