Elementary Sums

Edgar Valdebenito

JUNE 20 , 2019

ABSTRACT. We recall some elementary sums for Pi

Introduction

Recall that

$$\pi = 4\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-...\right)=3.1415... \tag{1}$$

$$\phi = \frac{1+\sqrt{5}}{2} \tag{2}$$

Entry 1.

$$\tan^{-1}\left(\phi-\sqrt{\phi}\right)=\frac{\pi}{4}-\frac{1}{2}+\frac{1}{2}\sum_{n=1}^{\infty}\binom{2n}{n}\frac{(-1)^{[(n-1)/2]}\,2^{-3n}}{2n+1} \tag{3}$$

$$\tan^{-1}\left(\phi-\sqrt{\phi}\right)=\frac{\pi}{4}-\frac{1}{2}+\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\,2^{-6n}\binom{4n}{2n}\left(\frac{8n}{(4n-1)^2}+\frac{1}{4n+1}\right) \tag{4}$$

$$\tan^{-1}\left(\phi-\sqrt{\phi}\right)=\sum_{n=0}^{\infty}\left(\sqrt{\phi}-1\right)^{n+1}\sum_{k=0}^{[n/2]}\frac{(-1)^k}{2k+1}\binom{2k+1}{n-2k}a(k,n-2k) \tag{5}$$

where

$$a(n,k)=\begin{cases}1 & k\le 2n+1\\ 0 & k>2n+1\end{cases} \tag{6}$$

Remark: $[x]$ is the integer part of x .

Elementary Sums for Pi

Entry 2.

$$\pi = \frac{4}{\sqrt[4]{2}} - \frac{4}{\sqrt[4]{2}} \sum_{n=1}^{\infty} \binom{2n}{n} \frac{(-1)^{[(n-1)/2]}\, 2^{-3n}}{2n+1} \left(\frac{1}{\sqrt{2}}\right)^{n} \tag{7}$$

$$\pi = \frac{6}{\sqrt[4]{12}} - \frac{6}{\sqrt[4]{12}} \sum_{n=1}^{\infty} \binom{2n}{n} \frac{(-1)^{[(n-1)/2]}\, 2^{-4n}}{2n+1} \left(\frac{1}{\sqrt{3}}\right)^{n} \tag{8}$$

$$\pi = \frac{10}{\sqrt[4]{50+22\sqrt{5}}} - \frac{10}{\sqrt[4]{50+22\sqrt{5}}} \sum_{n=1}^{\infty} \binom{2n}{n} \frac{(-1)^{[(n-1)/2]}\, 2^{-3n}}{2n+1} \left(\frac{1}{\sqrt{50+22\sqrt{5}}}\right)^{n} \tag{9}$$

$$\pi = \frac{12}{\sqrt{3\sqrt{6}+5\sqrt{2}}} - \frac{12}{\sqrt{3\sqrt{6}+5\sqrt{2}}} \sum_{n=1}^{\infty} \binom{2n}{n} \frac{(-1)^{[(n-1)/2]}\, 2^{-3n}}{2n+1} \left(\frac{1}{\sqrt{3\sqrt{6}+5\sqrt{2}}}\right)^{2n} \tag{10}$$

Entry 3.

$$\pi = \frac{4}{\sqrt[4]{2}} - \frac{4}{\sqrt[4]{2}} \sum_{n=1}^{\infty} (-1)^{n-1}\, 2^{-7n} \binom{4n}{2n} \left(\frac{8\sqrt{2}\, n}{(4n-1)^2} + \frac{1}{4n+1} \right) \tag{11}$$

$$\pi = \frac{6}{\sqrt[4]{12}} - \frac{6}{\sqrt[4]{12}} \sum_{n=1}^{\infty} (-1)^{n-1}\, 2^{-8n}\, 3^{-n} \binom{4n}{2n} \left(\frac{16\sqrt{3}\, n}{(4n-1)^2} + \frac{1}{4n+1} \right) \tag{12}$$

$$\pi = \frac{10}{\sqrt[4]{50+22\sqrt{5}}} - \frac{10}{\sqrt[4]{50+22\sqrt{5}}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, 2^{-6n}}{\left(50+22\sqrt{5}\right)^{n}} \binom{4n}{2n} \left(\frac{8\sqrt{50+22\sqrt{5}}\, n}{(4n-1)^2} + \frac{1}{4n+1} \right) \tag{13}$$

$$\pi = \frac{12}{\sqrt{3\sqrt{6}+5\sqrt{2}}} - \frac{12}{\sqrt{3\sqrt{6}+5\sqrt{2}}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, 2^{-6n}}{\left(3\sqrt{6}+5\sqrt{2}\right)^{2n}} \binom{4n}{2n} \left(\frac{8\left(3\sqrt{6}+5\sqrt{2}\right) n}{(4n-1)^2} + \frac{1}{4n+1} \right) \tag{14}$$

References

1. G. Almkvist, C. Krattenthaler, and J. Petersson: Some new formulas for π . Experiment. Math. 12, 2003 , 441-456.
2. D.H. Bailey, J.M. Borwein, P.B. Borwein, and S. Plouffe: The quest for pi. Math. Intelligencer, 19, 1997, 50-57.
3. F. Bellard: A new formula to compute the nth binary digit of pi, 1999. See reference in http://mathworld.wolfram.com/P/Pi.html.
4. B. Berndt: Ramanujans Notebooks, I (1985), II (1991), III (1991), IV (1994), Springer-Verlag, New York.
5. D.V. Chudnovsky and G.V. Chudnovsky: Classification of hypergeometric identities for π and other logarithms of algebraic numbers, Proc. Natl. Acad. Sci. USA, 1998, 2744-2749.
6. M. Hirschhorn: A new formula for π , Aust. Math. Soc. Gazette 26, 1999, 82-83.
7. J. Guillera: Some binomial series obtained by the WZ-method, Advances in Applied Mathematics, vol. 29, 2002, 599-603.