Pi , Three Series

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ABSTRACT. We give three series for Pi.

1. Introduction. Euler numbers

The Euler numbers are defined by

$$\operatorname{sech} x = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} E_n x^{2n} \quad , |x| < \pi / 2 \tag{1}$$

$$\{E_n : n \geq 0\} = \{1,1,5,61,1385,50521,2702765,...\} \tag{2}$$

$$E_0 = 1 \quad , E_n = (-1)^n \sum_{k=1}^{n} \left(\frac{1}{2}\right)^{k-1} \sum_{m=1}^{k} (-1)^m \binom{2k}{k-m} m^{2n} \quad , n \geq 1 \tag{3}$$

2. Three series for Pi

Recall that

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - ...\right) = 3.1415926535... \tag{4}$$

Entry 1. If $c_n = \sum_{k=0}^{[n/2]} \frac{(-1)^k \, 2^{2k} E_{n-2k}}{(4k+4)!(2n-4k)!}$, $n \geq 0$, then

$$\pi = 6\sin\left(\frac{\ln 3}{2}\right) + 24\sum_{n=0}^{\infty}(-1)^n \frac{1}{2n+5}\left(\frac{\ln 3}{2}\right)^{2n+5} c_n \tag{5}$$

$$\pi = 4\sin\left(\ln\left(1+\sqrt{2}\right)\right) + 16\sum_{n=0}^{\infty}(-1)^n \frac{1}{2n+5}\left(\ln\left(1+\sqrt{2}\right)\right)^{2n+5} c_n \tag{6}$$

$$\pi = 3\sin\left(\ln\left(2+\sqrt{3}\right)\right) + 12\sum_{n=0}^{\infty}(-1)^n \frac{1}{2n+5}\left(\ln\left(2+\sqrt{3}\right)\right)^{2n+5} c_n \tag{7}$$

Remark:

$$\{c_n : n \geq 0\} = \left\{\frac{1}{24}, \frac{1}{48}, \frac{173}{20160}, \frac{421}{120960}, \frac{675691}{479001600}, \frac{4057}{7096320}, \ldots\right\} \tag{8}$$

- $[x]$ is the integer part of x .

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