The Goldbach Theorem

The proof of the Goldbach's conjeture

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Abstract

The proof of Goldbach's strong conjecture is presented, built on the foundations of the *theory of gap*, which, when combined with certain criteria about the existence of prime numbers in successions, gives us the evidence cited. In reality, We have proof a more general statement in relation to that attributed to Goldbach. As result, it is proved how a even number is the sum of two odd primes, of infinite ways and as a corollary, the conjecture about of the *twin primes* is also proof.

1 Introduction

When someone claims that he has the proof of a legendary and famous conjecture, the answer he gets is disbelief. And it's not for less, considering the centuries of passed history and the amount of geniuses that failed when trying to solve it. Skepticism is well received, but must give way to analysis as compelling events are revealed.

In my opinion, if one can explain why a phenomenon occurs, then one has a proof for that phenomenon. Goldbach's strong conjecture had not been proven simply because nobody could explain why it happens. This document is a technical explanation of why this phenomenon exists. The theory used is not new to the author, I have called it the *theory of gap* and I used it in my research work on Fermat's Last Theorem, even before Andrew Wiles presented his laborious proof. A few days ago it occurred to me to dust off those ideas and apply them to Goldbach's conjecture to see what came out and the result is the content of this document.

The proof, finally, has been relatively easy to achieve. They do not require more than the four basic operations of arithmetic and a bit of fundamentals about primality theory. This can "turn on alarms"among those who believe that the most important open problems can only be solved through very elaborate and complex theories! A very elaborate "proof", usually can only be understood by three or four specialists in the world which makes it terribly unpopular. This is because only these people have, not intelligence, but rather, the necessary experience in the issues involved and hence the review of a work of these can take years. However, nowhere is it written that difficult problems must be solved in the same way. On the contrary, the simpler a proof for a difficult problem, more aroused admiration, since, in this case, there is more merit in discovering the facts that for centuries specialists have ignored that, in continuing to pay a certain cult to the difficulty and moving away more and more the pure mathematics of the “mortal currents“. This proof is specially built so that anyone with basic education can understand it, I hope, without major difficulties.

Finally, I dedicate this proof to the memory of my deceased parents: Néstor and Sofía.

2 The Goldbach Conjeture

The Goldbach conjecture is one of the oldest open problems in mathematics.[1] is defined as:[2]

Definition 1. *Any even number greater than 2 can be written as the sum of two prime numbers.*

The conjecture is easy to see for small pairs, as illustrated in table 1:

No	$p+q$	p	q
1	4	2	2
2	6	3	3
3	8	3	5
4	10	5	5
5	12	5	7
6	14	7	7
7	16	5	11
8	18	5	13
9	20	7	13
10	22	11	11

Table 1: First 10 pairs in the Goldbach conjecture.

The conjeture:[3]

[1] G.H. Hardy in 1921, in his famous speech at the Copenhagen Mathematical Society, commented that probably the Goldbach conjecture is not only one of the most difficult unresolved problems of number theory, but of all mathematics (see [7]).

[2] Christian Goldbach (1690-1764), Prussian mathematician.

[3] See [7]

it has been checked by computers for all even numbers less than 10^{18}. **Most mathematicians believe that the conjecture is true, and rely mostly on statistical considerations about the probabilistic distribution of prime numbers in the set of natural numbers**: the larger the even integer, the makes more "likely"that can be written as the sum of two prime numbers.

Goldbach made two related conjectures about the sum of prime numbers:

- The strong conjeture and,
- The "weak"conjeture of Goldbach.

This research is about Goldbach's strong conjecture, which is often mentioned simply as **conjeture of Goldbach**.

The weak conjeture of Goldbach state that (see [8]):

> "All odd number greater than 5 can be expressed as the sum of three prime numbers."

This conjecture was demonstrated by Harald Andrés Helfgott [4], and receives the name of "weak "because the strong Goldbach conjecture, automatically implies the weak Goldbach conjecture. This is feasible because if every even number greater than 4 is the sum of two odd primes, you can add three to even numbers greater than 4 to produce odd numbers greater than 7, so sometimes it is usually stated as follows:

> "All odd number greater than 7 can be expressed as the sum of three odd prime numbers."

The following results summarize the advances obtained in this matter in the last two centuries:[5]

1. In 1923, **Hardy and Littlewood** showed that, assuming a certain generalization of the **Riemann hypothesis**, the weak Goldbach conjecture is true for all sufficiently large odd numbers.

[4]Two works published in the years 2012 and 2013 by the Peruvian mathematician Harald Andrés Helfgott, who claim the improvement of the estimates of the major and minor arcs, are considered sufficient to unconditionally demonstrate the conjecture weak of Goldbach.

[5]See [8].

2. In 1937, the Russian mathematician **Iván Matvéyevich Vinográdov** [6] It was able to eliminate the dependence of the Riemann hypothesis and directly showed that all sufficiently large odd numbers can be written as a sum of three primes.

3. Chen Jing-run showed that each sufficiently large number is the sum of a prime number with a number that has no more than two prime divisors.

4. Olivier Ramaré showed in 1995 that every even number greater than four ($n \geq 4$) is in fact the sum of, at most, six primes, so it follows that each odd number $n \geq 5$ is the sum of at most, seven primes.

5. Leszek Kaniecki showed that every odd integer is the sum of at most, five primes, under the condition of the Riemann hypothesis. In 2012, Terence Tao demonstrated this without need of using the Riemann hypothesis.

6. Perhaps the most important result has been achieved by the Peru mathematic Harald Andrés Helfgott, who in January of 2014 presented the document [2] with the aforementioned demonstration.

On the result of Helfgott it is commented on [5]:

> "The proof is based on the advances made in the early twentieth century by Hardy, Littlewood and Vinogradov. In 1937, Vinogradov proved that the conjecture is true for all odd numbers greater than some constant C. (Hardy and Littlewood had shown the same under the assumption that the generalized Riemann Hypothesis was true, we will discuss this later.) Since then, the constant C has been specified and gradually improved, but the best value (this is, the smallest) of C that was available was $C = e^{3100} > 10^{1346}$ (Liu-Wang), which was by far too large. Even

[6]Vinográdov could not determine what "was sufficiently large "exactly, his student K. Borodzin showed that $3^{14,348,907}$ is a upper bound for the concept of "large enough". This number has more than six million digits, so checking the conjecture in each number below this level would be impossible. Fortunately, in 1989 Wang and Chen reduced this level to 10^{43000}. This means that if each of the odd numbers less than 10^{43000} turns out to be the sum of three prime numbers, then the weak Goldbach conjecture will be proven. However, this level must still be greatly reduced before each number can be checked below it.

> $C = 10^{100}$ would be too much: as 10^{100} is larger than the product of the estimated number of subatomic particles in the universe by the number of seconds since the Big Bang, there was no hope of checking each case up to 10^{100} per computer (even assuming that one was an alien dictator using the entire universe as a highly highly parallel computer). "

For the purposes of this investigation, the following definition is considered for Goldbach's strong conjecture:

Definition 2. *All even number greater than 4 is the sum of two odd primes.*

For convenience, we exclude the case $4 = 2 + 2$ and focus on the odd primes.

3 Gap Theory

3.1 Gap Successions of Goldbach

Consider the succession:

$$
\begin{array}{llll}
0 \cdot 16 & = 0 & & \\
 & & 17 & \\
1 \cdot 17 & = 17 & & 2 \\
 & & 19 & \\
2 \cdot 18 & = 36 & & 2 \\
 & & 21 & \\
3 \cdot 19 & = 57 & & 2 \\
 & & 23 & \\
4 \cdot 20 & = 80 & & 2 \\
 & & 25 & \\
5 \cdot 21 & = 105 & & 2 \\
 & & 27 & \\
6 \cdot 22 & = 132 & & 2 \\
 & & 29 & \\
7 \cdot 23 & = 161 & & 2 \\
 & & 31 & \\
8 \cdot 24 & = 192 & & 2 \\
 & & 33 & \\
9 \cdot 25 & = 225 & & 2 \\
 & & 35 & \\
10 \cdot 26 & = 260 & & 2 \\
 & & 37 & \\
11 \cdot 27 & = 297 & & 2 \\
 & & 39 & \\
12 \cdot 28 & = 336 & & 2 \\
 & & 41 & \\
13 \cdot 29 & = 377 & & 2 \\
 & & 43 & \\
14 \cdot 30 & = 420 & & 2 \\
 & & 45 & \vdots \\
15 \cdot 31 & = 465 & & \\
 & \vdots & &
\end{array}
$$

Named as *Gap Succession of Goldbach* (GSG). In the succession GSG, by inspection, we can make the following observations:

1. Each element of the sequence is obtained by the difference of the two consecutive elements that precede it in the column on the left.

2. All the elements of the first column [7], with the exception, perhaps, of the element that immediately follows zero (17 in this case), are numbers compounds, that is, that can be expressed as the product of two nontrivial factors.

3. Each element in the sequence can be obtained from the elements that precede it by means of sums. For example, $17 = 17+0$, y $36 = 17+19 = 17+17+2$, equally $57 = 17+19+21 = 17+17+2+17+4$, etc.

4. Since the last column is constant and equal to $2! = 2$, each new element is obtained by adding 2 to the element that precedes it in the middle column, a result that is added in a similar way to the first column to get a new element.

5. The first column is then composed of "compound numbers", the second column are all odd numbers and the third column are constant values.

6. Note that **the factors involved in the first column all have the same difference**, that is, $31-15 = 16$, $30-14 = 16$, $29-13 = 16$, $\ldots$, $17-1 = 16$, y $16-0 = 16$. In addition, 16 serves as a generator of the entire sequence since the first element is obtained by doing $17 = 0+(16+1) = 0+17$, and successively are obtained $36 = 19+17 = (17+2)+17$, etc.

7. Note also that, the sum of the factors involved in the first column corresponds to the difference that appears in the second column decreased by one unit, that is, $31+15 = 46 = 45+1$, $14+30 = 44 = 43+1$, $13+29 = 42 = 41+1$, etc.

On the other hand, in a similar way, we can define the sequence:

[7] It can be seen as "gaps "of zero order.

$$
\begin{array}{llll}
0\cdot 42 & = 0 & & \\
 & & 41 & \\
1\cdot 41 & = 41 & & \text{-2} \\
 & & 39 & \\
2\cdot 40 & = 80 & & \text{-2} \\
 & & 37 & \\
3\cdot 39 & = 117 & & \text{-2} \\
 & & 35 & \\
4\cdot 38 & = 152 & & \text{-2} \\
 & & 33 & \\
5\cdot 37 & = 185 & & \text{-2} \\
 & & 31 & \\
6\cdot 36 & = 216 & & \text{-2} \\
 & & 29 & \\
7\cdot 35 & = 245 & & \text{-2} \\
 & & 27 & \\
8\cdot 34 & = 272 & & \text{-2} \\
 & & 25 & \\
9\cdot 33 & = 297 & & \text{-2} \\
 & & 23 & \\
10\cdot 32 & = 320 & & \text{-2} \\
 & & 21 & \\
11\cdot 31 & = 341 & & \text{-2} \\
 & & 19 & \\
12\cdot 30 & = 360 & & \text{-2} \\
 & & 17 & \\
13\cdot 29 & = 377 & & \text{-2} \\
 & & 15 & \\
14\cdot 28 & = 392 & & \text{-2} \\
 & & 13 & \vdots \\
15\cdot 27 & = 405 & & \\
 & \vdots & &
\end{array}
$$

In this last sequence we can also make some similar observations:

1. The factors involved in the elements of the first column add up all the same, that is, $27 + 15 = 42$, $14 + 28 = 42$, $13 + 29 = 42$, etc.

2. The difference of the factors involved in the first column now corre-

sponds to the elements of the second column decreased in the unit. This is, $27 - 15 = 12 = 13 - 1$, $28 - 14 = 14 = 15 - 1$, etc.

3. The elements of the last column, the constant column, are now negative.

4. All the elements of the first column, with the exception perhaps of the element immediately following to zero (41 in this case), are compound numbers, that is, they can be expressed as the product of two nontrivial factors.

In both cases, we have two even numbers, namely, the difference and the sum of the factors involved in the first column. **These even numbers are, by nature, associated with an infinite series of pairs of prime numbers**, which leads to the direct proof of the Goldbach conjecture, as will be shown more clearly in the section 4.1.

3.2 Gap Successions of Fermat

The successions seen above are intended to provide proof of the Goldbach conjecture, however, they are not the only ones. We can define similar sequences to attack other types of problems. For example, the following sequence, calculated for the first time in 1991, can be used to directly study Fermat's Last Theorem:

$$\begin{array}{llccc}
17^3 - (16^3 + 11^3) & = -514 & & & \\
 & & 433 & & \\
18^3 - (17^3 + 10^3) & = -81 & & \text{-54} & \\
 & & 379 & & 6 \\
19^3 - (18^3 + 9^3) & = 298 & & \text{-48} & \\
 & & 331 & & 6 \\
20^3 - (19^3 + 8^3) & = 629 & & \text{-42} & \\
 & & 289 & & \vdots \\
21^3 - (20^3 + 7^3) & = 916 & & & \\
 & \vdots & & &
\end{array}$$

3.3 The Successions Theorem

Given a generating numerical sequence $g = (g_i)_{i\in\mathbb{Z}}$, the infinite matrix is built

$$E(g) = [e_{ij}]_{i\in\mathbb{Z}, j\in\mathbb{N}}$$

of the following way

- For each $i \in \mathbb{Z}$: $\quad e_{i0} = g_i$
- For each $i \in \mathbb{Z}, j \in \mathbb{N}$: $\quad e_{i,j+1} = e_{i+1,j} - e_{ij}$

schematically you have
Theorem 1[8]. For each $i \in \mathbb{Z} \quad j \in \mathbb{N}$, we have

$$e_{ij} = \sum_{k=0}^{j} \binom{j}{k} (-1)^{j-k} g_{i+k} \tag{1}$$

Proof. (Induction on j)

1. $j = 0$ By definition, for any $i \in \mathbb{Z}$ we have $e_i = g_i$ and trivially

$$g_i = \sum_{k=0}^{0} \binom{0}{k} (-1)^{0-k} g_{i+k} \tag{2}$$

2. Suppose that the formula is valid for j (and any $i \in \mathbb{Z}$) by definition $e_i^{j+1} = e_{i+1,j} - e_{ij}$; For both $e_{i+1,j}$ and e_{ij} the formula is valid, then it can be replaced:

$$e_{ij} = \sum_{l=0}^{j} \binom{j}{l} (-1)^{j-l} g_{i+1+l} - \sum_{k=0}^{j} \binom{j}{k} (-1)^{j-k} g_{i+k} \tag{3}$$

$$\tag{4}$$

[8]Oostra-Useche, 1991.

(changing index: $k = l + 1$)

$$= \sum_{k=1}^{j+1} \binom{j}{k-1} (-1)^{j-k+1} g_{i+k} + \sum_{k=0}^{j} \binom{j}{k} (-1)^{j-k+1} g_{i+k} \tag{5}$$

$$= (\sum_{k=1}^{j} \binom{j}{k-1} (-1)^{j-k+1} g_{i+k} + g_{i+j+1} + (-1)^{j+1} g_i + \tag{6}$$

$$\sum_{k=1}^{j} \binom{j}{k} (-1)^{j-k+1} g_{i+k}) \tag{7}$$

$$= (-1)^{j+1} g_i + \sum_{k=1}^{j} (\binom{j}{k-1} + \binom{j}{k}) (-1)^{j-k+1} g_{i+k} + g_{i+j+1} \tag{8}$$

$$\tag{9}$$

(by a property of the binomial coefficients)

$$= \binom{0}{j+1} (-1)^{j+1-0} g_{i+0} + \sum_{k=1}^{j} \binom{j+1}{k} (-1)^{j+1-k} g_{i+k} + \tag{10}$$

$$\binom{j+1}{j+1} (-1)^{j+1-(j+1)} g_{i+(j+1)} \tag{11}$$

$$= \sum_{k=0}^{j+1} \binom{j+1}{k} (-1)^{j+1-k} g_{i+k} \tag{12}$$

and this is the formula for $j + 1$ $\square$

3.3.1 Special cases

1. If b is a non-zero real number, for each $i \in \mathbb{Z}$ let $g_i = b^i$. Then, according to the affirmation 1, it is received

$$e_{ij} = \sum_{k=0}^{j} \binom{j}{k} (-1)^{j-k} b^{i+k} \tag{13}$$

$$= b^i \sum_{k=0}^{j} \binom{j}{k} b^k (-1)^{j-k} \tag{14}$$

$$= b^i (b-1)^j \tag{15}$$

2. If n is positive integer, for each $i \in \mathbb{Z}$ let $g_i = i^n$

The following section is dedicated to proof that in this case for each $i \in \mathbb{Z}$ we have

$$e_{in} = n! \tag{16}$$

combining this fact with the proven statement, we obtain the following "surprising expressions "of the factorial

1. For each whole number $i \in \mathbb{Z}$,

$$n! = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (i+k)^n \tag{17}$$

2. In special,

$$n! = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} k^n \tag{18}$$

3.

$$n! = \sum_{k=0}^{n} \binom{n}{k} k^k (-k)^{n-k} \tag{19}$$

4.

$$n! = \sum_{k=0}^{n} \binom{n}{k} (-1)^k (i-k)^n \tag{20}$$

4 Hard Number

The "hard numbers "are integers of the form $n = p \cdot q$, where $p, q \in \mathbb{P}$. That is, they are integers in whose canonical decomposition only two nontrivial prime factors appear. For example $77 = 11 \cdot 7$ is a hard number that, however, does not honor its name simply because its magnitude is small. These numbers, having only two prime factors are very difficult to factoring when their size exceeds a certain dimension. However, in this paper, the factorization of these numbers does not represent a problem, as will be appreciated immediately. These numbers play a central role in demonstrating the strong Goldbach conjecture and other results.

The GSG defined in 3.2 enclose some mathematical details that allow us to know its nature. In effect, consider the GSG, written as shown below:

$$
\begin{array}{llcc}
0\cdot 16 & = 0 & & \\
 & & 17 & \\
1\cdot 17 & = 17 = (\frac{17+1}{2})^2 - (\frac{17-1}{2})^2 = 9^2 - 8^2 & & 2 \\
 & & 19 & \\
2\cdot 18 & = 36 = (\frac{19+1}{2})^2 - (\frac{17-1}{2})^2 = 10^2 - 8^2 & & 2 \\
 & & 21 & \\
3\cdot 19 & = 57 = (\frac{21+1}{2})^2 - (\frac{17-1}{2})^2 = 11^2 - 8^2 & & 2 \\
 & & 23 & \\
4\cdot 20 & = 80 = (\frac{23+1}{2})^2 - (\frac{17-1}{2})^2 = 12^2 - 8^2 & & 2 \\
 & & 25 & \\
5\cdot 21 & = 105 = (\frac{25+1}{2})^2 - (\frac{17-1}{2})^2 = 13^2 - 8^2 & & 2 \\
 & & 27 & \\
6\cdot 22 & = 132 = (\frac{27+1}{2})^2 - (\frac{17-1}{2})^2 = 14^2 - 8^2 & & 2 \\
 & & 29 & \\
7\cdot 23 & = 161 = (\frac{29+1}{2})^2 - (\frac{17-1}{2})^2 = 15^2 - 8^2 & & 2 \\
 & & 31 & \\
8\cdot 24 & = 192 = (\frac{31+1}{2})^2 - (\frac{17-1}{2})^2 = 16^2 - 8^2 & & 2 \\
 & & 33 & \\
9\cdot 25 & = 225 = (\frac{33+1}{2})^2 - (\frac{17-1}{2})^2 = 17^2 - 8^2 & & 2 \\
 & & 35 & \\
10\cdot 26 & = 260 = (\frac{35+1}{2})^2 - (\frac{17-1}{2})^2 = 18^2 - 8^2 & & 2 \\
 & & 37 & \\
11\cdot 27 & = 297 = (\frac{37+1}{2})^2 - (\frac{17-1}{2})^2 = 19^2 - 8^2 & & 2 \\
 & & 39 & \\
12\cdot 28 & = 336 = (\frac{39+1}{2})^2 - (\frac{17-1}{2})^2 = 20^2 - 8^2 & & 2 \\
 & & 41 & \\
13\cdot 29 & = 377 = (\frac{41+1}{2})^2 - (\frac{17-1}{2})^2 = 21^2 - 8^2 & & 2 \\
 & & 43 & \\
14\cdot 30 & = 420 = (\frac{43+1}{2})^2 - (\frac{17-1}{2})^2 = 22^2 - 8^2 & & 2 \\
 & & 45 & \vdots \\
15\cdot 31 & = 465 = (\frac{45+1}{2})^2 - (\frac{17-1}{2})^2 = 23^2 - 8^2 & & \\
 & \vdots & &
\end{array}
$$

A remarkable fact appears in this presentation, namely, that each gap of zero order in a GSG can be written as the difference of two squares, namely two squares of the form:

$$(\frac{a+1}{2})^2 - (\frac{b-1}{2})^2$$

where a **represents the sum of the two factors decreased in the unit**, and b **represents the difference between the same factors**. This implies that, being n a gap of zero order of the GSG, then we can write its decomposition as:

$$
\begin{aligned}
n &= (\frac{a+1}{2})^2 - (\frac{b-1}{2})^2 && (21)\\
&= (a')^2 - (b')^2 && (22)\\
&= (a'+b')\cdot(a'-b') && (23)\\
&= p \cdot q && (24)
\end{aligned}
$$

whore $a' = \frac{a+1}{2}$ y $b' = \frac{b+1}{2}$. Also, $p = a' + b'$ y $q = a' - b'$.

We define the numbers $n = p \cdot q$, where, both p and q are odd primes, like the **"hard numbers"** of the sequence and we represent them by $\hbar$. That is, wherever $\hbar$ appears, it will be understood that we are talking about a compound "hard "(hard number), a number with only two non-trivial prime factors. Some authors call these numbers as *false primes* or *pseudoprimes*.

For example, the hard numbers of the example given in the section 3.1 are:

$$
\begin{aligned}
3 \cdot 19 = 57 = 11^2 - 8^2 &= (11-8)\cdot(11+8) && (25)\\
7 \cdot 23 = 161 = 15^2 - 8^2 &= (15-8)\cdot(15+8) && (26)\\
13 \cdot 29 = 377 = 21^2 - 8^2 &= (21-8)\cdot(21+8) && (27)
\end{aligned}
$$

Of course, only for the interval shown.

4.1 Existence

The existence of hard numbers ($\hbar$) allows associating an even number with two odd primes, so that this pair is just the sum of such primes. In practice there is more than one way to do that, however, in order to prove the strong Goldbach conjecture, in theory, it is only necessary to guarantee the existence of at least one hard number in each GSG.

Essentially, we look for numbers of the form $x \pm y$, where $x + y =$ a prime and $x - y =$ other prime, which are part of the first column of a GSG. By the Dirichlet theorem[9] on arithmetic progressions, it is known that there are

[9] See [9]

linear functions $f(x) = ax + b$ that produce infinite prime numbers as long as a and b are relative primes, that is, $(a, b) = 1$. There are many results that involve primes generated by linear functions. However, in this study, we will follow a different strategy to guarantee the existence of infinite $\hbar$ in any GSG.

Let's go back to the GSG of the example given in the section 3.1. If we detail a little the elements of the first column, that is,

$$\begin{array}{llcc}
0 \cdot 16 & = 0 & & \\
 & & 17 & \\
1 \cdot 17 & = 17 & & 2 \\
 & & 19 & \\
2 \cdot 18 & = 36 & & 2 \\
 & & 21 & \\
3 \cdot 19 & = 57 & & 2 \\
 & & & \vdots \\
 & \vdots & &
\end{array}$$

The elements, $0, 17, 36, 57$, etc., all them with exception of 17, are compound numbers that assume the form $n = p \cdot q$, with $p, q \in \mathbb{N}$, and in some special cases (hard numbers) we have $p, q \in \mathbb{P}$.

We can imagine the elements of the GSG as being part of a "strip "double of numbers, which have an offset equal to the difference of $q - p$. The first strip, that is, the values $0, 1, 2, 3, \ldots$, naturally goes over all the integers and therefore runs through the complete set of prime numbers $2, 3, 5, \ldots$. What will be the probability that one of these integers turns out to be a number prime ? considering a given interval. To calculate this value, we first use the prime counting function, that is,

$$\pi(x) \sim \frac{x}{\ln x} \tag{28}$$

N	$\pi(x)$	$P(x)$
10	4	0.4
100	21	0.21
1000	144	0.144
10000	1085	0.1085
100000	8685	0.08685
1000000	72382	0.072382
10000000	620420	0.0620420
100000000	5428681	0.0542868
1000000000	48254942	0.0482549
10000000000	434294481	0.04343

Table 2: First values for $\pi(x)$, with $x \leq 10^{10}$.

The table 2 shows the values of $\pi(x)$ for different 10-powers with $x \leq 10^9$. The value of $P(x)$ refers to the probability that an integer is prime for different intervals. For example, the first line indicates that there are four primes less than 10 (2, 3, 5, and 7). Therefore, the probability that any of those integers in the interval $(0, 10)$ turns out prime, is equal to $P(4) = 4/10 = 0.4$. In general we can express this probability as:

$$P(x) = \pi(x)/x = \frac{1}{\ln x} \tag{29}$$

On the other hand, for be a hard number, there must be two odd primes. This leads us to ask ourselves the question: What is the probability that the two factors in the first column of a GSG are prime numbers ? Since the two "strips"considered above are essentially the same, the probability that both factors are prime numbers is a composite probability and in our case it is:

$$P(x, y) = P(x) \cdot P(y) = (\frac{1}{\ln x})^2 \tag{30}$$

The equation in 30 gives us the probability of having a hard number ($\hbar$) in a GSG.

4.1.1 Improving $\pi(x)$

The estimate for $\pi(x)$ given in the table 2 corresponds to the classical expression for $\pi(x)$. However, we can improve the precision of this value a bit more by computing $\pi(x)$ with the expression:

$$\pi(x) \sim \frac{x}{\ln x - 1.08366} \tag{31}$$

which was introduced by Legendre, 25 years after Gauss discovered the approach (see [10]).

N	$\pi(x)$	$P(x)$
10	8	0.43429
100	28	0.21715
1000	172	0.14476
10000	1231	0.10857
100000	9588	0.08686
1000000	78543	0.07238
10000000	665140	0.06204
100000000	5768004	0.05429
1000000000	50917519	0.04825
10000000000	455743004	0.04343

Table 3: Improving values for $\pi(x)$, with $x \leq 10^{10}$.

The table 3, slightly improves the estimate of $\pi(x)$. However, the estimate for $P(x)$ does not vary substantially and for this reason this modification will not affect the statistical calculations.

4.2 Prediction

In the section 4.1, method and formula (Ec. 30) is given to calculate the probability that any element of the GSG sequence will be a **hard number**. Based on the Ec. 30 we can obtain the amount of hard numbers ($N(\hbar)$) expected in the GSG sequence. The expression for this value is:

$$N(\hbar) = x \cdot P(x, y) = x(\frac{1}{\ln x})^2 \tag{32}$$

Where, $N(\hbar)$ denotes the theoretical hard numbers in GSG, and x defines the sample size.

The expression given in Ec. 32 allows us to know **how many hard numbers can be expected in a GSG before a certain value**, as illustrated in the table 4.

x	$\ln x$	$(1/\ln x)^2$	$\pi(x)$	$N(\hbar)$
10	2.30259	0.1886117	4	2
10^2	4.60517	0.0471529	22	5
10^3	6.90776	0.0209569	145	21
10^4	9.21034	0.0117882	1086	118
10^5	11.51293	0.0075445	8686	754
10^6	13.81551	0.0052392	72382	5239
10^7	16.11810	0.0038492	620421	38492
10^8	18.42068	0.0029471	5428681	294706
10^9	20.72327	0.0023285	48254942	2328539
10^{10}	23.02585	0.0018861	434294482	18861170
10^{11}	25.32844	0.0015588	3948131654	155877436
10^{12}	27.63102	0.0013098	36191206825	1309803451
10^{13}	29.93361	0.0011160	334072678387	11160455444
10^{14}	32.23619	0.0009623	3102103442166	96230457659
10^{15}	34.53878	0.0008383	28952965460217	838274208941
10^{16}	36.84136	0.0007368	271434051189532	7367644414516
10^{17}	39.14395	0.0006526	2554673422960305	65263562979797
10^{18}	41.44653	0.0005821	24127471216847324	582134867319796
10^{19}	43.74912	0.0005225	228576043106974624	5224700748244154
10^{20}	46.05170	0.0004715	2171472409516259072	47152924252903480

Table 4: Expected number of hard numbers ($\hbar$) before a certain value x, compare with the value of $\pi(x)$.

With the help of this table, we collect the following facts:

1. for $x = 4$, the equation for $N(\hbar)$ yields:

$$N(4) = 4(\frac{1}{\ln 4})^2 = 4 \cdot (0.72134752)^2 = 4 \cdot 0.52034225 \approx 2$$

   We also find, $N(6) = 2, N(8) = 2, ..., N(50) = 3$

   Therefore it is clear that $N(\hbar) > 0$ for all $x \in \mathbb{N}$. This is important because the condition $N(\hbar) > 0$ guarantees that in each GSG, there is at least one **hard number**, that is, a n such that $n = p \cdot q$ with $p, q \in \mathbb{P}$ and n forming part of GSG.

2. The table shows that the number of primes before a certain value x, that is, $\pi(x)$, is always greater than the number of hard numbers $N(\hbar)$,

[10] in the same interval, however, its magnitudes are comparable, having a close order, so that if $\pi(x) \to \infty$, then, likewise $N(\hbar) \to \infty$. This results in the presence of infinite $\hbar$ for each GSG.

3. This behavior is easy to see in the expression:

$$\lim_{x\to\infty} x(\frac{1}{\ln x})^2 = \lim_{x\to\infty} x \cdot \lim_{x\to\infty} (\frac{1}{\ln x})^2 = \infty$$

Since $\lim_{x\to\infty} x$ is ∞, the full expression it evaluates to ∞.

The equation 32 is a consequence of the symmetry of the problem, and has a singularity in $x = 1$, since $\frac{1}{\ln 1} =$ indeterminate. However, this condition does not affect the calculation of $N(\hbar)$ in any case.

We can now formulate the following:

Theorem 2: Every GSG contains infinite values $\hbar$ in its elements of zero order.

Proof. It is a consequence of the arguments presented in numerals 1 to 3 of this section $\square$.

4.3 $n-$Primes

Theorem 2 leads us to define the primes in special series, so that in each GSG, $s = p \pm q =$constant, for all elements of zero order in the GSG. That is, here s represents the sum of the factors p, q in the broad sense. To see it more directly, consider the following examples.

4.3.1 $n = 2$ (Twin primes)

Indeed, we have an infinite series of $\hbar$, associated with an even number and, in this order of ideas, we can generate all the possible SGS, starting with the first pair, $n = 2$, to obtain something like

[10]Which is very obvious, since its definition.

$$
\begin{array}{llcc}
0 \cdot 2 & = 0 & & \\
 & & 3 & \\
1 \cdot 3 & = 3 & & 2 \\
 & & 5 & \\
2 \cdot 4 & = 8 & & 2 \\
 & & 7 & \\
3 \cdot 5 & = 15 & & \\
 & \vdots & &
\end{array}
$$

Succession in which all so-called “twin primes”appear (see section 5.2), that is, 3 and 5, 11 and 13, 17 and 19, etc.

But this situation is not exclusive to even 2.

4.3.2 $n = 4$ (4-primes)

Another GSG with $n = 4$ will be:

$$
\begin{array}{llcc}
0 \cdot 4 & = 0 & & \\
 & & 5 & \\
1 \cdot 5 & = 5 & & 2 \\
 & & 7 & \\
2 \cdot 6 & = 12 & & 2 \\
 & & 9 & \\
3 \cdot 7 & = 21 & & \\
 & \vdots & &
\end{array}
$$

Where other primes are obtained, which can no be called “twins ”because they are separated by 4 units, then we can tell them the “4-primes”, of which the first is $3 \cdot 7 = 21$, and following:

$$\begin{aligned}
7 - 3 &= 4 \\
11 - 7 &= 4 \\
17 - 13 &= 4 \\
23 - 19 &= 4 \\
41 - 37 &= 4 \\
47 - 43 &= 4 \\
71 - 67 &= 4 \\
83 - 79 &= 4 \\
101 - 97 &= 4 \\
&\vdots
\end{aligned}$$

And then come “6-primes”, and so on for every pair.

4.3.3 $n = 16$, (16-primes)

For a moment, let's go back to the example in the section 3.1 and suppose you want to know, in this case, the value of $N(\hbar)$ for the given sequence and for different sample sizes. First, consider a sample of 10 elements that according to 32 will produce:

$$N(\hbar) = 10 \cdot (\frac{1}{\ln 10})^2 = 10 \cdot (0.43429448)^2 = 10 \cdot 0.1886 = 1.886 \approx 2$$

hard number. When reviewing the first 10 elements of the sequence one by one, there are actually two hard numbers (57 and 161). Repeating this same exercise achieves the table 5. The table shows that the theoretical prediction for $N(\hbar_t)$ is always less than the real value, and the relationship between these two quantities (column 5) “seems”converge to a certain value around 0.6.

Sample	$P(x)^2$	$N(\hbar)$ **Real**	$N(\hbar)$ **Theoretical**	$N(\hbar_t)/N(\hbar_r)$
10	0.189	2	1	0.5
100	0.047	9	4	0.44
1000	0.021	39	20	0.51282
10000	0.012	200	117	0.585
100000	0.008	1233	754	0.61152
1000000	0.005	8210	5239	0.63812
10000000	0.004	58606	38490	0.65679
100000000	0.003	441055	294705	0.66818

Table 5: $N(\hbar)$ in GSG for even number 16 and a sample size that varies from 10 to 10^8 items in succession.

Actually, it is difficult to know the value of this relationship for large values of the sample size in GSG, because $N(\hbar_r)$, must be calculated by "brute force ", that is, verifying one by one the hard numbers existing in the sequence. So, for very large sample sizes, this calculation is impractical.

4.3.4 $n = 98$, (98-primes)

If we change the reference par value from 16 to 98, then the relation

$$\frac{N(\hbar_t)}{N(\hbar_r)}$$

suffers, of course, some changes as shown in table 6.

Muestra	$P(x)^2$	$N(\hbar)$ **Real**	$N(\hbar)$ **Teórico**	$N(\hbar_t)/N(\hbar_r)$
10	0.189	1	1	1
100	0.047	7	4	0.57143
1000	0.021	40	20	0.50000
10000	0.012	231	117	0.50649
100000	0.008	1485	754	0.50774
1000000	0.005	9815	5239	0.53377
10000000	0.004	70139	38492	0.54880
100000000	0.003	528631	294705	0.55749

Table 6: $N(\hbar)$ in GSG for even number 98 and a sample size that varies from 10 to 10^8.

The situation is essentially the same, noting that the observed relation, now, has varied from 0.6 to 0.5, but the fact that the theoretical values are lower than the real is maintained, excluding the case $N(\hbar(10))$, where the comparison is a bit abusive.

4.3.5 Cases $6 \leq x < 50$

The table 7 shows the result of evaluating $N(\hbar)$ for even numbers $6 \leq x < 50$. It is observed how the values of $N(\hbar_t)$ are shown less than $N(\hbar_r)$, the latter, always growing to the right in the box.

Therefore, regardless the size of the reference pair used, the expression used in Ec. 32 guarantees that there will always be an infinite number of $\hbar$ for any GSG sequence. A fact that was expected due to the way "combine the strips"considered in GSG, that is, the whole factors of the zero-order gaps.

Even/N	10	10^2	10^3	10^4	10^5	10^6
Theor value.	1	4	20	117	754	5239
6	2	16	74	411	2447	16386
8	1	8	37	207	1259	8241
10	2	11	51	270	1624	10934
12	2	15	70	404	2421	16378
14	1	9	47	244	1487	9877
16	2	9	39	200	1233	8210
18	1	14	74	417	2477	16451
20	0	9	47	268	1644	10971
22	1	7	41	226	1351	9171
24	1	14	78	403	2474	16342
26	1	8	41	239	1347	8927
28	1	8	41	248	1468	9784
30	1	18	99	536	3329	21990
32	0	5	36	195	1203	8195
34	1	8	43	214	1305	8745
36	2	14	76	404	2463	16441
38	1	7	38	212	1291	8699
40	2	10	48	267	1638	10967
42	1	15	87	489	2931	19839
44	0	6	40	226	1408	9209
46	1	6	39	201	1292	8599
48	0	12	71	408	2482	16500

Table 7: $N(\hbar)$ in GSG for $4 < N < 50$ and 6 sample sizes $\leq 10^6$. The first line corresponds to the theoretical values that do not depend on the even number.

5 New statement

Now that we have proven results, it is convenient to review the classic statement of the strong conjecture to try to discover the meaning of the expression "every even number is the sum of two prime numbers", a statement that is associated with the strong Goldbach conjecture.

However, some discrepancy arises when the expression "sum" is given a meaning different from that used by Goldbach towards the first years of the

18th century. In fact, to see it better, consider how the conjecture materializes around the pair $n = 8$. So, it is easy to verify that $8 = 5 + 3$ would be the only solution in the sense used by Goldbach. Such solutions are obtained from the partitions for par 8 as shown below:

$$\begin{aligned} 8 &= 1 + 7 \\ &= 2 + 6 \\ &= 3 + 5 \\ &= 4 + 4 \end{aligned}$$

Since 3, 5 are the only pair of odd primes and therefore the only value $\hbar$ associated with par 8 under the concept of Goldbach sum, which implies a restrictive sense of the sum that does not take into account the signature of the addends. However, we can establish a relation of "sum", in a more generalized additive sense, as shown below:

$$\begin{aligned} 8 &= 11 + (-3) = 11 - 3 \\ 8 &= 12 + (-4) = 12 - 4 \\ 8 &= 13 + (-5) = 13 - 5 \\ 8 &= 14 + (-6) = 14 - 6 \\ 8 &= 15 + (-7) = 15 - 7 \\ 8 &= 16 + (-8) = 16 - 8 \\ 8 &= 17 + (-9) = 17 - 9 \\ 8 &= 18 + (-10) = 18 - 10 \\ 8 &= 19 + (-11) = 19 - 11 \\ 8 &= 20 + (-12) = 20 - 12 \\ 8 &= 21 + (-13) = 21 - 13 \\ 8 &= 22 + (-14) = 22 - 14 \\ 8 &= 23 + (-15) = 23 - 15 \\ &\vdots \end{aligned}$$

That is, we look for all combinations $x \pm y =$ constant pair. By doing this, the $\hbar$ values are achieved

$$8 = 11 - 3$$
$$8 = 13 - 5$$
$$8 = 19 - 11$$
$$\vdots$$

that according to the results obtained in the section 4.2 yields infinite values $\hbar$, which correspond to all the ways of writing par 8 as the sum of two odd primes, including the signature, as indicated above :

```
11 - 3 = 8
13 - 5 = 8
19 - 11 = 8
31 - 23 = 8
37 - 29 = 8
61 - 53 = 8
67 - 59 = 8
79 - 71 = 8
97 - 89 = 8
109 - 101 = 8
139 - 131 = 8
157 - 149 = 8
181 - 173 = 8
199 - 191 = 8
241 - 233 = 8
271 - 263 = 8
277 - 269 = 8
367 - 359 = 8
397 - 389 = 8
409 - 401 = 8
439 - 431 = 8
457 - 449 = 8
487 - 479 = 8
499 - 491 = 8
571 - 563 = 8
577 - 569 = 8
```

5.1 The proposed statement

All these values of $\hbar$ are part of the GSG sequence associated with the par 8 and when it is evaluated for a sample size of one million, 8241 hard numbers ($\hbar$) are obtained, from where we have taken the sample previous to illustrate the case par $= 8$. We have chosen 8 for this example because in the restricted sense of the sum, only one value is obtained for $8 = 5 + 3$. But as we have just seen, by changing the meaning of the sum and accepting a more general concept as it is done now, infinite solutions appear for $8 = p_1 \pm p_2$, where $p_1, p_2 \in \mathbb{P}$. This approach leads us to the next

Theorem 3: Every even number is the sum[11] of two odd prime in infinite ways.
Proof. Observe that since the *Theorem 1* the existence of a sequence is guaranteed in such a way that the difference (or sum) $q-p$ of all its elements is always the same, that is, it is constant for all the numbers $n_i = p_i \cdot q_i$ that are gaps of zero order, that is, $d = q_i - p_i =$ cte. This allows an arbitrary number pair ($q - p$ or $q + p$) be associated with the product of two factors, $n = p \cdot q$, in infinite forms, as shown in the sections "Gap Theory"and "Gap Succession Goldbach". By combining this result with *Theorem 2*, the desired result is achieved $\square$ In the section "Hard Number in GSG"it was shown and proved that, statistically speaking, the zero-order gaps of the resulting GSG contain, effectively, infinite hard numbers.

As a corollary we can say that Goldbach's strong conjecture, as we know it and have been stated here, is a special case of *Theorem 3*, which, being more general, include the concept of addition in the restricted sense and allows use a broader concept that involves no longer a pair, but infinite pairs of prime numbers, all them being part of the same "family "of numbers as zero-order gaps and all them constructed from the repetitive sum of the same quantity, an even number, as illustrated above in the section 3.1.

5.2 The twin primes

Another very legendary conjecture states that:

> "There is an infinite number of primes p such that $p + 2$ is also prime."

[11]Notice that here we are using the term sum in its broadest sense, as illustrated above.

For example, 3 and 5, are twin primes, also 11 and 13, 17 and 19, etc. All of them are built with:

$$
\begin{array}{llll}
0 \cdot 2 & = 0 & & \\
 & & 3 & \\
1 \cdot 3 & = 3 & & 2 \\
 & & 5 & \\
2 \cdot 4 & = 8 & & 2 \\
 & & 7 & \\
3 \cdot 5 & = 15 & & 2 \\
 & & 9 & \\
4 \cdot 6 & = 24 & & 2 \\
 & & 11 & \\
5 \cdot 7 & = 35 & & 2 \\
 & & 13 & \\
6 \cdot 8 & = 48 & & 2 \\
 & & 15 & \\
7 \cdot 9 & = 63 & & 2 \\
 & & 17 & \\
8 \cdot 10 & = 80 & & 2 \\
 & & 19 & \\
9 \cdot 11 & = 99 & & 2 \\
 & & 21 & \\
10 \cdot 12 & = 120 & & 2 \\
 & & 23 & \\
11 \cdot 13 & = 143 & & 2 \\
 & & 25 & \\
12 \cdot 14 & = 168 & & 2 \\
 & & 27 & \\
13 \cdot 15 & = 195 & & 2 \\
 & & 29 & \\
14 \cdot 16 & = 224 & & 2 \\
 & & 31 & \vdots \\
15 \cdot 17 & = 255 & & \\
 & \vdots & &
\end{array}
$$

in where, we separate the values $\hbar$ and we obtain the set of all the twin primes. Now, admitting the Theorem 3, it follows then that there are infinite twin primes in the GSG given above. In this way, the conjecture of the twin primes becomes the twin primes theorem, in the following:

Theorem 4. *There are infinite twin primes.*
Proof. It follows from Theorem 3.

Although we have stated this theorem separately, in reality, it has been a corollary of Theorem 3.

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