Standing on the shoulders of giants: Derivations of Einstein's $E = mc^2$ from Newtonian Laws of Motion

Zion Elani[a)]

(Dated: 16 July 2019)

This report presents a simple derivation of Einstein's famous equation, $E = mc^2$. Through the use of elementary physical quantities of Newtonian mechanics such as distance, force, momentum, velocity, and energy, our approach resembles a 'handling units' method. Further, two other proofs premising on the notion of mass and its dependence on velocity are discussed. These methods prove to be simple and physically intuitive, thus stimulating the amateur enthusiasts to a better understanding of various complex and difficult-to-derive formulas which are otherwise understood at a sophisticated academic level. Pedagogic significance of these methods is further discussed.

PACS numbers: 01.40.E 01.40.gb 01.40.Ha
Keywords: Science education, Physics education, Chemistry education, Energy conceptions

I. INTRODUCTION

It's no secret that lecturers, instructors, and educators face tremendous difficulty in explaining the difficult-to-derive formulas and concepts of modern physics, as well as their origins in history, to chemistry students in the college setting - especially freshmen students who are preparing to pursue studies in chemistry, biology, and engineering. The following report is inspired by these challenges faced by instructors and it is the hopes of the researchers involved that it serves as a valuable pedagogical resource for teachers, students, and amateur physics enthusiasts in their pursuit of knowledge.

Many instructors find themselves struggling to derive Einstein's $E = mc^2$ equation[1] without first looking to calculus. However, calculus always tends to be an area that causes much stress, intimidation, and apprehension among most college and pre-college students, as well as high school students and beyond.[2,3] And yet, the relevance and mysteriousness of the origins of $E = mc^2$ tend to keep students and enthusiasts alike in awe, forcing them to seek its origin asking 'how the heck this formula came up'? However, even for the freshman chemistry student, Einstein's equation is still notoriously difficult to derive.

Still, probably the most glorious "proof" that anyone knows of is an A-bomb. An A-bomb is built on one principle; that mass can be transformed into energy, and the formula that exactly predicts this conversion between energy and mass is $E = mc^2$. What has this celebrated formula has to do with Special Relativity? The answer is that $E = mc^2$ is derived directly from special relativity. But this poses another obstacle as the enthusiast amateur is not aware of the link between $E = mc^2$ and the special theory of relativity, and in fact, know nothing about relativity theory and its principles except the well-known fact that 'nothing can exceed the speed of light'.

It is not often realized by chemistry students during their first year, that the special theory of relativity is behind several aspects of (quantum) chemistry like electron spin for example. The predictions of molecular properties by non-relativistic (quantum) chemistry deviate significantly from the experimental results as the properties of various elements are influenced by relativistic effects. These include the liquidity of Mercury[4], the color of Gold and other chemical properties of heavy metals such as dipole moments, biological activity

[a)]zion.elani@biu.ac.il; Institute for Nanotechnology and Advanced Materials, Bar-Ilan University, Ramat-Gan, 5290002 Israel

and force constants. It is worth mentioning that relativistic treatments and corrections are consistently implemented in standard quantum chemical software packages, without the knowledge of advanced students who use them.[5] For a student to understand the importance of relativistic effects in chemistry, the yellow color of Gold offers a good example. According to special relativity, an object with a velocity close to the speed of light exhibits time dilation and length contraction. This is exactly the case for electrons in Gold with their velocities close to the speed of light, thus exhibiting length contraction. This relativistic effect causes the wavelength of light absorption shift to blue and to reflect the opposite color, i.e., yellow.[6]

After an extensive and comprehensive literature review[2] that included related pedagogical papers,[7–9] the researchers of this report were shocked to understand that a simple derivation for $E = mc^2$ does not yet exist or at the very least, one that would allow amateur enthusiasts and beginning chemistry students to grasp, understand, and appreciate without the prior knowledge of calculus and special relativity.

Moreover, after corresponding with experienced teachers and colleagues throughout the last number of years, we have determined that presenting the derivation in its simplest relation as "handling units" allows students and amateurs a chance to approach the concept from a more convincing and easily memorizable standpoint.

Here, we bring a simple eye-catching and easily comprehensible proof of $E = mc^2$ with no intention to present the nomenclature of the formula nor its interpretation and justification. Thus, we avoid the need to delve into the mass-energy equivalence which states that anything having mass has an equivalent amount of energy and vice versa.[10] In Treptow[11], the reader can find a pedagogic discussion on to what extent is mass conserved in the reactions of physics and chemistry. A mass-energy equivalence in this context is anything having energy that exhibits a corresponding mass m given by its energy E divided by the speed of light squared c^2. As already mentioned above, beside this mass-energy equivalence, what laymen usually associate with the $E = mc^2$ equation is that nothing can exceed the speed of light c, without even relating it to the special relativity principles and to Lorentz transformation. Following those "knowledge", we present, with a method similar to that of 'handling units', a simple and picturesque derivation of the formula using nothing but Newtonian laws of motions, that do not go beyond mere definitions, together with well-known elementary physical quantities as: distance, velocity, force, momentum and energy. In Leary and Ingham[12] the reader can find a pedagogic derivation of $E = mc^2$, which can supplement our approach.

The next section presents our derivation which is based on a method which resemble a handling unit method. In the section following the next, we present another derivation of $E = mc^2$ that is based on Einsteins own thought experiment (1906).[13,14] This proof is followed by a pedagogical explanation on the subtle notion of mass of a photon. The third section presents yet another proof that relies on elementary calculus as taught in high school though it uses only the mere definitions of differentials and derivatives. The derivation goes beyond proving Einstein's mass-energy relation by presenting the path toward the understanding of how the (relativistic) mass of an object depends on its velocity while keeping in mind that nothing can exceed the speed of light. A subsequent summary and outlook on the pedagogic merit of these presentations is given.

II. DERIVING THE FORMULA - A TREATMENT RESEMBLING A 'HANDLING UNITS' METHOD

As explained in the introduction, the derivation that we provide uses a method which resembles that of 'handling units', as well elementary knowledge of Newtonian laws of motion that do not extend beyond understanding the definitions of elementary quantities. *Let us consider a body that moves with a velocity v that is very close to that of c, the speed of light*, $c =$ [*speed of light*]. All we know about the 'speed of light' is that nothing can exceed it, no material body can reach the speed c or go beyond it. Let us recall that by *momentum* we mean [*mass*] $\times$ [*velocity*] and that a *Force* acting on a body is [*mass*] $\times$ [*acceleration*].

From mechanics, we know and understand that an impulse is the product of the average net force acting on an object and its duration, [*impulse*] = [*Force*] × [*time duration*]. Impulse applied to an object produces an equivalent change in its momentum, [*impulse*] = [*change in momentum*]. If a constant force applied for a *time interval* Δt on a body, the momentum of the body *changes by an amount* [*Momentum*] = [*Force*] × [*time interval*]. Now, what would have happened if a constant force F will act upon our body (that travel very close to the speed of light)? From elementary mechanics, we know that a *Work* is done on a body when an applied force moves it through a distance, [*Work*] = [*Force*] × [*distance*]. Since our body travels at a speed close to c, the constant force that we assume acting on it does not really change (i.e., increase) the velocity of the body. So, what changes due to the force that acts on our body? Our body has a momentum ([*mass*] × [*velocity*]), and since the velocity is not changing (it is already close to c, and cannot increase further), we are forced to conclude that it is the mass of the body that changes. This means that the momentum of our body is now [*mass*] × [c].

Let us now consider the *Energy* of the system. *Energy* is the ability to do work. The *increase of energy* of the body due to the constant force which acts on it along the distance that the body travels is, [*Energy*] = [*Force*] × [*distance the body travel*]. Now, what exactly is the distance that the body travels? Recall that [*distance*] = [*velocity*] × [*time*]. So, in *unit time*, the distance the body travels is just c, the speed of light (since in our case distance ∼ [*speed of light*] × [*time*]). Remarkably, we can now cast this relation between energy and force as

$$\big[Energy\big] = \big[Force\big] \times \big[c\big] \tag{1}$$

We recall that $\Delta P = F\Delta t$, the momentum of the body is equal to the force acting on the body multiply by the time interval, and in *unit time* it is simply equal to [*momentum*] = [*Force*]. But what exactly is the momentum, ΔP, gained by the body? It is merely [*mass*] × c, i.e., $\Delta m \times c$. The reader should be keen to take note that even though we are dealing with *unit time*, in this case, we choose to express the mass and energy as difference quantities for clarity. Now, the expression for the force acting on the body is

$$\big[\Delta F\big] = \big[\Delta m\big] \times \big[c\big] \tag{2}$$

Lets us now plug in expression (2) into expression (1) and we get, *in unit time*, the relation:

$$\big[\Delta E\big] = \big[\Delta F\big] \times \big[c\big] = \Big(\big[\Delta m\big] \times \big[c\big]\Big) \times \big[c\big] \tag{3}$$

Simplifying the above equation, one finally gets

$$E = mc^2. \tag{4}$$

The above simple derivation brings us Eq.(4). In plain English, this equations tells us that *the energy E gained by the body, in unit time, due to the action of the force is equal to the quantity mc^2, where m is the mass that the body gained.* We should remark though that such a hand-waving derivation is for this special case, where we consider a body that travels at a speed close to c, and *can not explain the generic case between energy and mass.*[7,8,10]

III. PHOTON ENERGY - EINSTEIN'S THOUGHT EXPERIMENT

In this section, we present another form of proof based on a thought experiment which was invented by Einstein himself in 1906.[13,14] The treatment shown here differs somehow from his original article, making the presentation more easy to grasp. Our objective in this thought experiment is to put forward the idea that energy must have a certain inertial mass equivalent that is associated with it (by inertial mass, we are referring to the ratio of linear momentum to velocity, namely $m = p/v$).

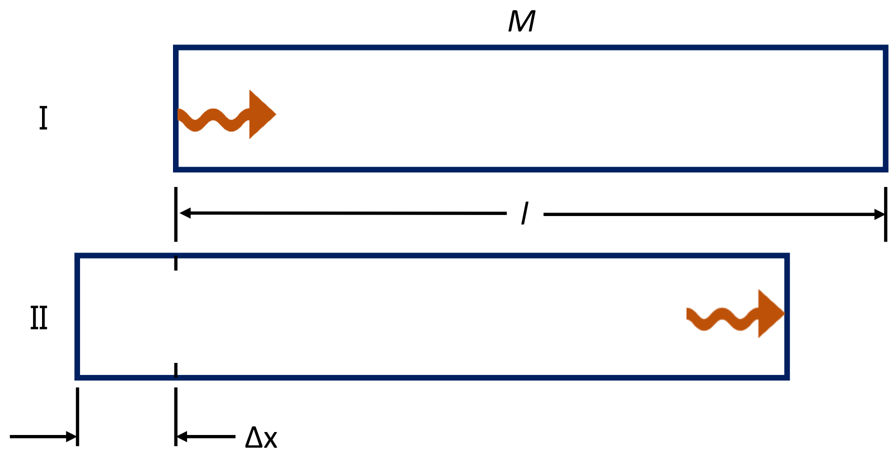

FIG. 1. Einstein's box experiment: An isolated box recoils from its initial position (I) to a final position (II) as a result of a photon traveling from one end of the box to the other.

Let us imagine a stationary isolated box of length l, floating in outer space (free from any external forces including gravitation). In addition, let us consider a finite amount of radiation (photons) of total energy E, that is being emitted from the left side of the box towards its right end (see fig.1). For simplicity we consider this radiation as a single photon (a 'particle of light') of the same energy. The energy of an amount of radiation has a momentum defined by $E = pc$. This phenomenon is known as radiation pressure. An interesting property that the reader should bear in mind is that photons are massless particles. A photon does indeed possess a momentum, but it does not posses mass, and this is confirmed by experiments conducted within strict limits. We can interpret the relation $E = pc$ by saying that a photon's momentum is a function of its energy, but it is not proportional to the velocity, which is always c. The relation between energy and momentum of radiation (photon) is outside the scope of Newtonian mechanics, but that shouldn't distract the reader from understanding the treatment presented here. In fact, this so called 'radiation pressure' will be derived in the next section (see Eq.(26)) and will be based solely on Newtonian mechanics and the mass-energy equivalence. From Newtonian mechanics, we know that momentum is a product of mass and velocity ($p = mv$). So, how do photons have momentum yet have no mass? In his great perception, Einstein considered that the energy of a photon must be equivalent to a quantity of mass and hence could be related to the momentum.[13,15] From Newton's laws of motion, we know that for an isolated system the momentum of the system must be conserved, i.e., $p_{\text{box}} = p_{\text{photon}}$. Due to this fact, we also know that the box must recoil to the left as the photon is emitted. At some time later, Δt, the photon collides with the other side of the box, transferring all of its momentum to the box. Since the total momentum of the system (box+photon) is conserved, the impact of the photon causes the box to stop moving.

No external forces are acting on this system, and as such, its center of mass is required to remain in the same location. However, the box has moved so how can the movement of the box be explained with the system's center of mass remaining fixed? Einstein resolved this evident contradiction by proposing that there must be a 'mass equivalent' to the energy of the photon. That is to say, the energy of the photon must be equivalent to a mass moving from left to right in the box; a mass that is large enough to cause the system center to remains stationary. Let us now spell out our finding mathematically, using only Newtonian mechanics.

If the energy of the photon is E then the momentum of the photon is given by

$$p_{\text{photon}} = \frac{E}{c} \tag{5}$$

The box of mass M, will recoil slowly in the opposite direction to the photon with speed v and box momentum that is given by

$$p_{\text{box}} = Mv \tag{6}$$

After a very short time, Δt, the photon will reach the other side of the box. During this time, the box will have moved a small distance, Δx (see fig.1). We can now write for the speed of the box as

$$v = \frac{\Delta x}{\Delta t} \tag{7}$$

Due to the fact that momentum is conserved, $p_{\text{box}} = p_{\text{photon}}$, we have

$$M\frac{\Delta x}{\Delta t} = \frac{E}{c} \tag{8}$$

Now, the time it takes for the photon to reach the other side of the box is

$$\Delta t = \frac{l}{c} \tag{9}$$

By substituting this into the conservation of momentum equation (8) and rearranging the terms we arrive at the expression

$$M\Delta x = \frac{El}{c^2} \tag{10}$$

Now, let us suppose for a moment that the photon has some mass, which we denote by m (remember that photons are massless particles! See discussion below). By doing that, we can calculate the center of mass of the system, box + photon. Let us now donate the positions of the box and the photon as x_1 and x_2 respectively. As such, the center of mass of our system (box + photon), by definition, given us

$$x_{cm} = \frac{Mx_1 + mx_2}{M + m} \tag{11}$$

As we argue above, and because the system is isolated, Newtonian mechanics tells us that its center of mass should not change. Namely, the center of mass at the beginning of our thought experiment must be the same as the end of the experiment. Mathematically, it is represented by

$$\frac{Mx_1 + mx_2}{M + m} = \frac{M(x_1 - \Delta x) + ml}{M + m} \tag{12}$$

Now, since the photon starts at the left of the box, we have $x_2 = 0$ for its position. Plugging this into the above equation and rearranging the terms we obtain the simplified expression

$$ml = M\Delta x \tag{13}$$

Let us now substitute eq. (10) into (13). This will give us

$$ml = \frac{El}{c^2} \tag{14}$$

Finally, by rearranging we obtain the mass-energy equivalence, Eq. (4), namely,

$$E = mc^2 \tag{15}$$

For a further discussion on this thought experiment and to answer some subtleties which have been omitted here, we recommend exploring and referencing the article by Antippa.[16]

IV. A GYMNASIUM CALCULUS DERIVATION

Here, we propound yet another form of proof for $E = mc^2$ based on Newtonian mechanics. And although it may not be as simple in that it will require a minimal knowledge of calculus, this method doesn't go beyond simple high school calculus with preliminary notions of differentials and derivatives. Let us consider a body of mass m that moves at a speed v very close to the speed of light. We also assume that a force F acts on the body and, as a result, the energy E of the body increases. As we know, nothing can exceed the speed of light and so the speed of the body v cannot exceed c and, as the force continues to act, the speed v approaches c asymptotically. From Newtonian mechanics we know that for a time interval Δt, the momentum of the particle changes by an amount $\Delta p = F\Delta t$, i.e., $F = dp/dt$ (force equal the rate of change of momentum). From Newton's second law, we also know that work, W, is equal to the change in kinetic energy E_k of the linear velocity of the body, $W = \Delta E_k$.

Mathematically, the relation between the net force and the acceleration is given by Newton's second law, $F = ma$, and the body displacement x can be expressed by the equation,

$$x = \frac{v_2^2 - v_1^2}{2a} \tag{16}$$

which follows from $v_2^2 = v_1^2 + 2ax$. Now, the work of the net force is calculated as the product of its magnitude and the particle displacement. Substituting the above equations, we obtain:

$$W = Fx = ma\left(\frac{v_2^2 - v_1^2}{2a}\right) = \frac{mv_2^2}{2} - \frac{mv_1^2}{2} = \Delta E_k \tag{17}$$

Using the fact that $F = dp/dt$, we understand the dependence of E as a function of time by taking the derivative of E with respect to t. By doing so we get for dE/dt,

$$\frac{dE}{dt} = Fv = \frac{d(mv)}{dt}v = v^2\frac{dm}{dt} + mv\frac{dv}{dt} \tag{18}$$

In this last equation, the time t is the independent variable. This equation shows how energy $E(t)$, mass $m(t)$, and speed $v(t)$ change as a function of time t. Our aim here is to determine exactly how the mass depends on the energy. In order to achieve that, the independent variable in the above expression should be the energy E such that the time, mass and velocity will all depend on E. In order to make the time $t(E)$, mass $m(E)$ and speed $v(E)$ to be functions of E, we will simply multiply Eq.(18) by $\frac{dt(E)}{dE}$ and use the chain rule to derive at,

$$1 = v^2\frac{dm}{dt}\frac{dt}{dE} + mv\frac{dv}{dt}\frac{dt}{dE} = v^2\frac{dm}{dE} + mv\frac{dv}{dE} \tag{19}$$

By rearranging the above expression, we arrive at

$$\frac{dm}{dE} = \frac{1}{v^2} - \frac{m}{v}\frac{dv}{dE} \tag{20}$$

The expression in Eq.(20) tells us how the mass depends on the energy E. Now, how will this expression appear when the energy E grows very large? Since nothing exceeds the speed of light, we know that in that limit, v approaches c asymptotically and the term $\frac{dv}{dE}$ approaches zero:

$$\lim_{E\to\infty}\frac{dm}{dE} = \frac{1}{c^2} \tag{21}$$

This equation shows us that for the addition of each unit of energy E, the mass m of a body grows incrementally by a factor of $\frac{1}{c^2}$. In plain English, Eq.(21) gives us the mass-energy

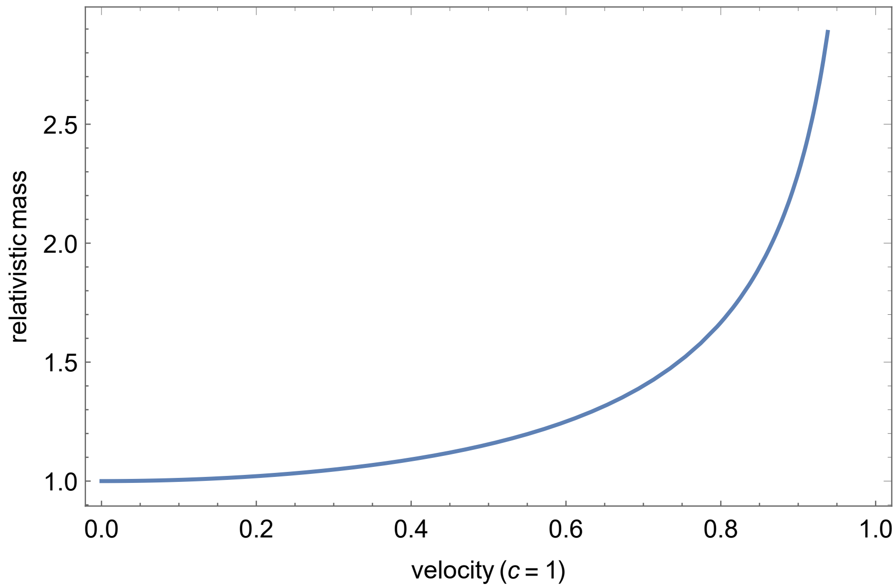

FIG. 2. Dependency between the mass of a body and its velocity. The graph shows how the mass approaches infinity as velocity approaches the speed of light. Thus, it is not possible for an object with mass to reach the speed of light.

equivalent relation, $E = mc^2$ (Eq. (4)). We should stress though, that while this result holds generally, this simple hand-waving calculation is valid only in the special case in which the energy E of the body has grown so large that its speed v is close to c.

Let us now try to put together some of the results we have discussed.[17] Let us combine Eq.(4), $E = mc^2$, and the definition of momentum, $m = p/v$, into a single expression,

$$E = \frac{c^2 p}{v} \tag{22}$$

Newtonian mechanics tells us that the change in the kinetic energy of a particle corresponds to the work done by external forces. By combining this with the relation between force and momentum, $F = dp/dt$ we have

$$dE = Fdx = \frac{dp}{dt}dx \tag{23}$$

and from this we get,

$$dE = vdp \tag{24}$$

Let us now combine Eqs.(22) and (24) by multiplying together the left and right sides of these two equations, $EdE = c^2 pdp$. By integrating them, we obtain the relation

$$E^2 = c^2p^2 + E_0^2 \tag{25}$$

where E_0^2 is a constant of integration, written explicitly as the square of some constant of energy. Eq.(25) is a relationship between energy E and momentum p for a particle, a relation that can now be proposed as a general one. If the body is a massless particle, $m_0 = 0$, then Eq.(25) reduces to Eq.(5):

$$E = pc \tag{26}$$

For a photon, this is the relation between radiant momentum (causing radiation pressure) and radiant energy (see previous section). Let us now substitute in Eq.(25) the relation

$cp = Ev/c$ that we got from Eq.(22). This will give us the *relation between energy and velocity*

$$E(v) = \frac{E_0}{(1 - v^2/c^2)^{1/2}} \tag{27}$$

For particles that travel in a speed very small compared to the speed of light, $v << c$, we can approximate this exact resulting, Eq.(27), by neglecting terms of higher order than v^2/c^2:

$$E(v) = E_0 + \frac{1}{2}\left(\frac{E_0}{c^2}\right) v^2 \tag{28}$$

Equation (28) should be similar to Newtonian mechanics at low velocities, and so we identify E_0/c^2 with the classical mass of a particle which we call the 'inertial mass' and denote by m_0. With this realization one can see that Eqs.(4) and (27) together lead to an explicit *relationship between mass and velocity* (see Fig.2):

$$m(v) = \frac{m_0}{(1 - v^2/c^2)^{1/2}} \tag{29}$$

We call this mass, which depends on v a 'relativistic mass'. The quantity m_0, which in Newtonian mechanics would be the 'inertial mass' of a body, now assumes a new role as the 'rest mass' of the body for $v = 0$ and the quantity E_0 $(= m_0c^2)$ is called the 'rest energy'.

Generally, 'relativistic mass' is a measure of the energy E of a particle, which changes with velocity (see eq. (29), and Fig.2). The reader should be aware that we commonly don't designate relativistic mass as the 'mass of a particle' (and in the case of a photon, it will be awry to say that the photon has mass in this way). By the mass of an object we usually mean its 'invariant mass' define as $m = \sqrt{E^2/c^4 - p^2/c^2}$ (see eq. (25)), and for a photon this quantity is zero. The reason that nowadays we define "mass" as the invariant mass of an object is mainly because the invariant mass is more useful and practical when doing any kind of calculation. However, with this definition in mind, mass is not to be considered a conserved quantity and furthermore, mass is not simply the sum of the masses of an objects parts. In the modern view "mass" is not equivalent to energy. The mass is just that part of the energy of a body which is not kinetic energy. Energy depends on velocity whereas mass does not depends on velocity. In contrast to this, relativistic mass is a quantity that is equivalent to energy, which is the prime reason for why relativistic mass is not a commonly used term today. For a more detailed discussion on the notion of mass in relativity, pedagogical discussions by Sandin[18], Hecht[19], Okun[20] and Roche[21] are recommended.

V. SUMMARY AND OUTLOOK

To introduce the properties and behavior of an equation or a formula to mathematically unequipped students, most of the instructors make use of simple, memory-less and physically intuitive methods such as that of "handling units". Here, we considered this line of approach to derive $E = mc^2$. Firstly, a simple derivation of this equation using handing units like approach was presented, followed by yet another proof based on Einstein's thought experiment. We further presented another proof relying solely on elementary Newtonian mechanics using basic high school calculus that doesn't go beyond the mere definitions of differential and derivative. Using terms from Newtonian mechanics we introduce to the reader the dependence of mass on velocity - a concept that is otherwise appreciated only while delving into the special theory of relativity. In addition to that, we elaborate on the notion of mass avoiding the need of introducing to the student the nomenclature of the concepts involved and their technical jargon. From experience, our approach will enable a student to understand such complex-to-derive equations when supplied with just a set of definitions of physical quantities like work, energy and force. The student can thus be taught to look upon them as shorthand statements of physical laws that are valid, no matter in what units one may choose to express any of the quantities involved. We believe that the treatment presented here has a great pedagogic merit as it will spark a better understanding of various mysterious and strenuous-to-derive equations in undergrad students or even amateur enthusiasts.

[1] R. Baierlein, "Teaching $E = mc^2$," American Journal of Physics **57**, 391–392 (1989), https://doi.org/10.1119/1.16034.
[2] D. Halliday, R. Resnick, and J. Walker, *Fundamentals of Physics* (John Wiley & Sons, 2010).
[3] R. Feynman, R. Leighton, and M. Sands, *The Feynman Lectures on Physics: The New Millennium Edition: Mainly Mechanics, Radiation, and Heat*, v. 1 (Basic Books, 2015).
[4] L. J. Norrby, "Why is mercury liquid? or, why do relativistic effects not get into chemistry textbooks?" Journal of Chemical Education **68**, 110 (1991).
[5] W.-K. Li and S. M. Blinder, "Introducing relativity into quantum chemistry," Journal of Chemical Education **88**, 71–73 (2011).
[6] D. R. McKelvey, "Relativistic effects on chemical properties," Journal of Chemical Education **60**, 112 (1983).
[7] F. Rohrlich, "An elementary derivation of $E = mc^2$," American Journal of Physics **58**, 348–349 (1990), https://doi.org/10.1119/1.16168.
[8] M. J. Feigenbaum and N. D. Mermin, "$E = mc^2$," American Journal of Physics **56**, 18–21 (1988), https://doi.org/10.1119/1.15422.
[9] E. Hecht, "How einstein confirmed $E = mc^2$," American Journal of Physics **79**, 591–600 (2011), https://doi.org/10.1119/1.3549223.
[10] E. Taylor and J. Wheeler, *Spacetime Physics* (W. H. Freeman, 1992).
[11] R. S. Treptow, "$E = mc^2$ for the chemist: When is mass conserved?" Journal of Chemical Education **82**, 1636 (2005).
[12] J. J. Leary and W. H. Ingham, "$E = mc^2$: An intuitive derivation," Journal of Chemical Education **84**, 1651 (2007).
[13] A. Einstein, "Das prinzip von der erhaltung der schwerpunktsbewegung und die trgheit der energie," Annalen der Physik **325**, 627–633 (1906), https://onlinelibrary.wiley.com/doi/pdf/10.1002/andp.19063250814.
[14] A. Einstein, A. Beck, and P. Havas, *The Collected Papers of Albert Einstein*, COLLECTED PAPERS OF ALBERT EINSTEIN No. ' 2 (Princeton University Press, 1989).
[15] G. Farmelo, *It Must be Beautiful: Great Equations of Modern Science* (Granta, 2003).
[16] A. F. Antippa, "Inertia of energy and the liberated photon," American Journal of Physics **44**, 841–844 (1976).
[17] A. French, *Special Relativity* (Taylor & Francis, 1968).
[18] T. R. Sandin, "In defense of relativistic mass," American Journal of Physics **59**, 1032–1036 (1991).
[19] E. Hecht, "There is no really good definition of mass," The Physics Teacher **44**, 40–45 (2006).
[20] L. B. Okun, "The concept of mass," Physics Today **42**, 31–36 (1989).
[21] J. Roche, "What is mass?" European Journal of Physics **26**, 225–242 (2005).