On Normality of a Subgroup of Prime Index in a Group of Prime Power Order

Henry Wong

Abstract

In this paper we lay out the proof of this result in group theory using only elementary facts in group theory and the method of induction.

Theorem 1
Any subgroup of order p^{n-1} in a group G of order p^n, p a prime number, is normal in G.
Proof.
Let $P(n)$ be the statement that a subgroup of order p^{n-1} in a group of order p^n, p a prime number, is normal. Now assume $P(k-1)$ is true. Let G be a group of order p^k and H be subgroup of G of order p^{k-1}. By Lagrange's theorem, $|Z(G)| = p^m$ for some integer $0 \leqq m \leqq k$. Since $Z(G) \neq (e)$, p divides $|Z(G)|$ and so $Z(G)$ has an element a of order p. Let N be the subgroup of G generated by a. Then N is of order p. Since $a \in Z(G)$, N must be normal in G. Moreover, $|N \cap H|$ divides $|N|$. So $|N \cap H|$ divides p. Thus $|N \cap H| = 1$ or p. Suppose $|N \cap H| = 1$. Then

$$|NH| = \frac{|N||H|}{|N \cap H|} = p^k.$$

Since $NH \subset G$ and $|NH| = |G|$, so $NH = G$. Since $N \subset Z(G)$, every element of N commutes with every element of G. Let $g \in G$. So $g = nh$ for some $n \in N$ and $h \in H$. Let $x \in gH$. Thus $x = gh'$ for some $h' \in H$. Moreover, $x = gh' = (nh)h' = n(hh') = (hh')n \in Hn$. Hence $gH \subset Hn$. Since $gH \subset Hn$ and $|gH| = |Hn|$, so $gH = Hn$ and whence every left coset of H in G is a right coset of H in G. So H is normal in G and hence $P(k)$ is true. Now suppose $|N \cap H| = p$. Since $N \cap H \subset N$ and $|N \cap H| = |N|$, it follows that $N \cap H = N$ and hence $N \subset H$. Since G/N is a group of order p^{k-1} and H/N is a subgroup of G/N of order p^{k-2}, H/N must be normal in G/N by the induction hypothesis. Thus H is normal in G as well and hence $P(k)$ is true.

References

[1] I. N. Herstein, *Abstract Algebra*, Macmillan Publishing Company, New York, 1990.