"Collection of Three Monographs pertaining to Quaternionic Analysis."

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I. ABSTRACT.

This particular submission contains (inter alia) copies of three (3) monographs, whose purpose is to further elaborate upon various topics having been enunciated in the author's previous set of submissions, namely -

(a) ***"A Supplementary Discourse on the Classification and Calculus of Quaternion Hypercomplex Functions - PARTS 1/10 to 10/10."***;

(b) ***"Supplementary Notes pertaining to a Specific Quaternion Analogue of the Cauchy-Goursat Theorem."***,

which have been published under the **'VIXRA' Mathematics subheading:-** ***'Functions and Analysis'***.

II. COPY OF AUTHOR'S ORIGINAL MONOGRAPH No.1.

(1) Title of Monograph:-

"Correlation of Specific Results having been enunciated in Various Expository Articles and Papers."

Re:- Mathematical Paper, thus entitled – "A Supplementary Discourse on the Classification and Calculus of Quaternion Hypercomplex Functions."

(2) Date of Completion:- 14th June 2018.

(3) Total number of A4 pages to follow = 22.

(4) Referee's Assessment:-

For further details, the reader should accordingly refer to the contents of **Appendix A1**.

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Correlation of Specific Results having been enunciated in Various Expository Articles and Papers.

Re:- Mathematical Paper, thus entitled - "A Supplementary Discourse on the Classification and Calculus of Quaternion Hypercomplex Functions."

I. Copies and Sources of Selected Extracts.

1. Extract No. E1/12.

[1] Copy of Extract:-

Center

The center of a noncommutative ring is the subring of elements c such that $cx = xc$ for every x. The center of the quaternion algebra is the subfield of real quaternions. In fact, it is a part of the definition that the real quaternions belong to the center. Conversely, if $q = a + bi + cj + dk$ belongs to the center, then

$$0 = iq - qi = 2cij + 2dik = 2ck - 2dj,$$

and $c = d = 0$. A similar computation with j instead of i shows that one has also $b = 0$. Thus $q = a$ is a real quaternion.

The noncommutativity of multiplication has some unexpected consequences, among them that polynomial equations over the quaternions can have more distinct solutions than the degree of the polynomial. For example, the equation $z^2 + 1 = 0$, has infinitely many quaternion solutions, which are the quaternions $z = bi + cj + dk$ such that $b^2 + c^2 + d^2 = 1$. Thus these "roots of −1" form a unit sphere in the three-dimensional space of purely imaginary quaternions.

[2] Source of Extract:- Page 4/13 of web page article [1].

2. Extract No. E2/12.

[1] Copy of Extract:-

Square roots of −1

In the complex numbers, **C**, there are just two numbers, i and $-i$, whose square is -1. In **H** there are infinitely many square roots of minus one: the quaternion solution for the square root of -1 is the unit sphere in $\mathbf{R}^3$. To see this, let $q = a + bi + cj + dk$ be a quaternion, and assume that its square is -1. In terms of a, b, c, and d, this means

$$a^2 - b^2 - c^2 - d^2 = -1,$$
$$2ab = 0,$$
$$2ac = 0,$$
$$2ad = 0.$$

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To satisfy the last three equations, either $a = 0$ or b, c, and d are all 0. The latter is impossible because a is a real number and the first equation would imply that $a^2 = -1$. Therefore, $a = 0$ and $b^2 + c^2 + d^2 = 1$. In other words: a quaternion squares to -1 if and only if it is a vector (that is, pure imaginary) with norm 1. By definition, the set of all such vectors forms the unit sphere.

Only negative real quaternions have infinitely many square roots. All others have just two (or one in the case of 0).

[2] Source of Extract:- Pages 7/13 & 8/13 of web page article [1].

3. Extract No. E3/12.

[1] Copy of Extract:-

Functions of a quaternion variable

Like functions of a complex variable, functions of a quaternion variable suggest useful physical models. For example, the original electric and magnetic fields described by Maxwell were functions of a quaternion variable.

Exponential, logarithm, and power

Given a quaternion,

$$q = a + bi + cj + dk = a + \mathbf{v}$$

the exponential is computed as

$$\exp(q) = \sum_{n=0}^{\infty} \frac{q^n}{n!} = e^a \left(\cos \|\mathbf{v}\| + \frac{\mathbf{v}}{\|\mathbf{v}\|} \sin \|\mathbf{v}\| \right)$$

$$\ln(q) = \ln \|q\| + \frac{\mathbf{v}}{\|\mathbf{v}\|} \arccos \frac{a}{\|q\|}.$$ [27]

It follows that the polar decomposition of a quaternion may be written

$$q = \|q\| e^{\hat{n}\theta} = \|q\| \left(\cos(\theta) + \hat{n} \sin(\theta) \right),$$

where the angle θ and the unit vector $\hat{n}$ are defined by:

$$a = \|q\| \cos(\theta)$$

and

$$\mathbf{v} = \hat{n} \|\mathbf{v}\| = \hat{n} \|q\| \sin(\theta).$$

Any unit quaternion may be expressed in polar form as $e^{\hat{n}\theta}$.

The power of a quaternion raised to an arbitrary (real) exponent α is given by:

$$q^\alpha = \|q\|^\alpha e^{\hat{n}\alpha\theta} = \|q\|^\alpha \left(\cos(\alpha\theta) + \hat{n} \sin(\alpha\theta) \right).$$

[2] Source of Extract:- Pages 8/13 & 9/13 of web page article [1].

4. Extract No. E4/12.

[1] Copy of Extract:-

2 The notion of an $\mathbb{H}$-derivative

We begin by the following

Definition 2.1. *A quaternion function* $f(z)$, $z = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3$, *defined on some neighborhood* $G \subset \mathbb{H}$ *of a point* $z^0 = x_0^0 + x_1^0 i_1 + x_2^0 i_2 + x_3^0 i_3$, *is called* $\mathbb{H}$-differentiable *at* z^0 *if there exist two sequences of quaternions* $A_k(z^0)$ *and* $B_k(z^0)$ *such that* $\sum_k A_k(z^0)B_k(z^0)$ *is finite and that the increment* $f(z^0+h) - f(z^0)$ *of the function* $f(z)$ *can be represented as*

$$f(z^0+h) - f(z^0) = \sum_k A_k(z^0) \cdot h \cdot B_k(z^0) + \omega(z^0, h), \qquad (2.1)$$

where

$$\lim_{h \to 0} \frac{|\omega(z^0, h)|}{|h|} = 0 \qquad (2.2)$$

and $z^0 + h \in G$. *In this case, the quaternion* $\sum_k A_k(z^0)B_k(z^0)$ *is called the* $\mathbb{H}$-derivative of the function f at the point z^0 *and is denoted* $f'(z^0)$. *Thus*

$$f'(z^0) = \sum_k A_k(z^0)B_k(z^0). \qquad (2.3)$$

The uniqueness of the $\mathbb{H}$-derivative follows from the fat that the right-hand part of (2.3), if it exists, is just the partial derivative $f'_{x_0}(z^0)$ of $f(z)$ at z^0 with respect to its real variable.

In the sequel, the symbol $o(h)$ denotes any function $\omega(z^0, h)$ satisfying (2.2).

[2] Source of Extract:- Page 4 of Dzagnidze [2].

5. Extract No. E5/12.

[1] Copy of Extract:-

Quaternionic differential forms. When it is necessary to avoid confusion with other senses of differentiability which we will consider, we will say that a function $f : \mathbb{H} \to \mathbb{H}$ is *real-differentiable* if it is differentiable in the usual sense. Its differential at a point $q \in \mathbb{H}$ is then an $\mathbb{R}$-linear map $df_q : \mathbb{H} \to \mathbb{H}$. By identifying the tangent space at each point of $\mathbb{H}$ with $\mathbb{H}$ itself, we can regard the differential as a quaternion-valued 1-form

$$df = \frac{\partial f}{\partial t}\,dt + \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz. \qquad (2.20)$$

[2] Source of Extract:- Page 5 of Sudbery [3].

6. Extract No. E6/12.

[1] Copy of Extract:-

3 Rules for calculating $\mathbb{H}$-derivatives

The rules for calculating $\mathbb{H}$-derivatives are identical to those derived in a standard calculus course.

Proposition 3.1. *Let f and φ be two functions defined on a neighborhood of $z^0 \in \mathbb{H}$. If both f and φ are $\mathbb{H}$-differentiable at z^0, then*

1. *both cf and fc are $\mathbb{H}$-differentiable at z^0 for all $c \in \mathbb{H}$ and $(cf)'(z^0) = cf'(z^0)$ and $(fc)'(z^0) = f'(z^0)c$;*
2. *$f + \varphi$ is $\mathbb{H}$-differentiable at z^0 and $(f+\varphi)'(z^0) = f'(z^0) + \varphi'(z^0)$;*
3. *$f\varphi$ is $\mathbb{H}$-differentiable at z^0 and $(f\varphi)'(z^0) = f'(z^0)\varphi(z^0) + f(z)\varphi'(z^0)$.*

[2] Source of Extract:- Page 9 of Dzagnidze [2].

7. Extract No. E7/12.

[1] Copy of Extract:-

Proposition 3.4. *If a function φ is $\mathbb{H}$-differentiable at a point z^0 and if $\varphi \neq 0$ in a neighborhood of z^0, then[3] $\frac{1}{\varphi}$ is also $\mathbb{H}$-differentiable at z^0 and we have:*

$$\left(\frac{1}{\varphi}\right)'(z^0) = -\frac{1}{\varphi(z^0)} \cdot \varphi'(z^0) \cdot \frac{1}{\varphi(z^0)}.$$

[2] Source of Extract:- Page 10 of Dzagnidze [2].

8. Extract No. E8/12.

[1] Copy of Extract:-

The following functions introduced by Hamilton

$$z^n, \quad n = 0, 1, 2, \dots,$$
$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots,$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots,$$
$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

are the *basic elementary quaternion functions of one quaternionic variable z.*

Let us now show that the basic elementary functions are $\mathbb{H}$-differentiable.

Proposition 2.3. $(z^n)' = nz^{n-1}$ for $n = 0, 1, 2, \ldots,$ and for $z \in \mathbb{H}$.

[2] Source of Extract:- Page 5 of Dzagnidze [2].

9. Extract No. E9/12.

[1] Copy of Extract:-

Corollary 3.5. *For $z \neq 0$ we have:*

$$(z^m)' = mz^{m-1}, \quad m = -1, -2, \ldots.$$

[2] Source of Extract:- Page 11 of Dzagnidze [2].

10. Extract No. E10/12.

[1] Copy of Extract:-

Proposition 2.5. *We have the equality*

$$(e^z)' = e^z.$$

[2] Source of Extract:- Page 7 of Dzagnidze [2].

11. Extract No. E11/12.

[1] Copy of Extract:-

Proposition 2.6. *The equality*

$$(\sin z)' = \cos z$$

is valid.

Proposition 2.7. *The equality*

$$(\cos z)' = -\sin z$$

is valid.

[2] Source of Extract:- Page 8 of Dzagnidze [2].

12. Extract No. E12/12.

[1] Copy of Extract:-

4 The $\mathbb{H}$-derivative of the quaternion logarithm function

A quaternion w is called the *logarithm* of a finite quaternion $z \neq 0$ if $z = e^w$, in which case we write $w = \ln z$.

In order to define the $\mathbb{H}$-derivative $w' = (\ln z)'$, we first note that the $\mathbb{H}$-derivative of the left-hand side of the identity $z = e^{\ln z}$ exits and is 1 by Proposition 2.3. Applying now Proposition 3.8 to the right-hand side and taking into account (2.5), we get

$$\begin{aligned} 1 = &\left(1 + \frac{w}{2!} + \frac{w^2}{3!} + \cdots\right) \cdot w' + \left(\frac{1}{2!} + \frac{w}{3!} + \frac{w^2}{4!} + \cdots\right) \cdot w' \cdot w \qquad (4.1) \\ &+ \left(\frac{1}{3!} + \frac{w}{4!} + \frac{w^2}{5!} + \cdots\right) \cdot w' \cdot w^2 + \cdots . \end{aligned}$$

Thus, the $\mathbb{H}$-derivative $w' = (\ln z)'$ satisfies Equality (4.1).

Remark 4.1. If ww' and $w'w$ were equal, then we could write $w \cdot w', w^2 \cdot w', \dots$ instead of $w' \cdot w, w' \cdot w^2, \dots$, and then Equality (4.1) would take the form

$$\begin{aligned} 1 &= \left(1 + \frac{w}{2!} + \frac{w^2}{3!} + \cdots\right) \cdot w' + \left(\frac{w}{2!} + \frac{w^2}{3!} + \cdots\right) \cdot w' \\ &\quad + \left(\frac{w^2}{3!} + \frac{w^3}{4!} + \frac{w^4}{5!} + \cdots\right) \cdot w' + \cdots \\ &= \left(1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \cdots\right) \cdot w' = e^w \cdot w' = e^{\ln z} \cdot (\ln z)' = z \cdot (\ln z)'. \end{aligned}$$

So, we would obtain the classical formula

$$(\ln z)' = \frac{1}{z},$$

that is well known in the case of a complex variable z.

[2] Source of Extract:- Page 13 of Dzagnidze [2].

II. Supplementary Notes.

1. Correlation of Analogous Results having been enunciated in Section I and the Undersigned Author's Paper respectively.

(1) With reference to Extract Nos. E1/12 & E2/12, the statements underlined in RED are logically consistent with the contents of Theorem TI-1; Theorem TI-2 & Definition DI-8 having been enunciated on pages 16-22 of the aforesaid paper.

(2) With reference to Extract No. E3/12, the formulae underlined in RED are logically consistent with the contents of Theorem TI-3; Definition DI-9; Theorem TI-4; Definition DI-14; Theorem TI-20; Definition DI-15; Theorem TI-21 & Definition DI-17 having been enunciated on pages 22-29; 31-33; 89-95 & 113-114 of the aforesaid paper, whereupon a derivation of the formula for q^2 is likewise provided in Appendix A1.

(3) With reference to Extract No. E4/12, the formula and its concomitant statement underlined in RED are logically consistent with the contents of Definition DII-5 & Theorem TII-2 having been enunciated on pages 152-158 of the aforesaid paper.

(4) With reference to Extract No. E5/12, the formula underlined in RED is logically consistent with the contents of Theorem TII-3 having been enunciated on pages 159-161 of the aforesaid paper, bearing in mind that the independent variable, $q = t + ix + jy + kz$, as specified on page 2 of Sudbery [3], and hence the function, $f = f(q)$.

(5) With reference to Extract No. E6/12, Statements 1, 2 & 3 underlined in RED are logically consistent with the contents of Theorem TII-4 having been enunciated on pages 162-168 of the aforesaid paper.

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(6) With reference to Extract No. E7/12, the formula underlined in RED is logically consistent with the contents of Theorem TII-4 having been enunciated on pages 162-168 of the aforesaid paper, whereupon a derivation of the formula for $(\frac{1}{\phi})'(q)$ is likewise provided in Appendix A2.

(7) With reference to Extract No. E8/12, the formula underlined in RED is logically consistent with the contents of Theorem TII-5 having been enunciated on pages 168-171 of the aforesaid paper.

(8) With reference to Extract No. E9/12, the formula underlined in RED is logically consistent with the contents of Theorem TII-5 having been enunciated on pages 168-171 of the aforesaid paper.

(9) With reference to Extract No. E10/12, the formula underlined in RED is logically consistent with the contents of Theorem TII-6 having been enunciated on pages 171-172 of the aforesaid paper.

(10) With reference to Extract No. E11/12, the formulae underlined in RED are logically consistent with the contents of Theorem TII-7 having been enunciated on pages 172-175 of the aforesaid paper.

(11) With reference to Extract No. E12/12, the formula underlined in RED is logically consistent with the contents of Theorem TII-9 having been enunciated on pages 177-179 of the aforesaid paper.

2. Correlation of Specific Formulae pertaining to the Definite Integration of Quaternion Hypercomplex Functions.

(1) With reference to the following extract from page 11 of Sudbery [3], namely-

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4. *Cauchy's theorem and the integral formula*. The integral theorems for regular quaternionic functions have as wide a range of validity as those for regular complex functions, which is considerably wider than that of the integral theorems for harmonic functions. Cauchy's theorem holds for any rectifiable contour of integration; the integral formula, which is similar to Poisson's formula in that it gives the values of a function in the interior of a region in terms of its values on the boundary, holds for a general rectifiable boundary, and thus constitutes an explicit solution to the general Dirichlet problem.

The algebraic basis of these theorems is the equation

$$\begin{aligned} d(g\,Dq\,f) &= dg \wedge Dq\,f - g\,Dq \wedge df \\ &= \{(\bar{\partial}_r g)f + g(\bar{\partial}_l f)\}v, \end{aligned} \tag{4.1}$$

which holds for any differentiable functions f and g. Taking $g = 1$ and using Proposition 3, we have:

PROPOSITION 4. *A differentiable function f is regular at q if and only if*

$$Dq \wedge df_q = 0.$$

From this, together with Stokes's theorem, it follows that if f is regular and continuously differentiable in a domain D with differentiable boundary, then

$$\int_{\partial D} Dq\,f = 0.$$

it should be noted that the definite integral underlined in RED is a non-commutative analogue of the definite integral,

$$\int_C f(q)\,dq = 0 \quad \left[\underline{\text{N.B.}}\ f(q) = f(x + Q^*\xi) = U(x,\xi) + Q^* V(x,\xi), \text{ where } Q^* = \frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}} \ \&\ \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}. \right],$$

whose existence is enunciated via Theorem T II-25 on pages 235 – 241 of the undersigned author's paper.

(2) Similarly, it should be noted that the following differential equation, namely –

$$\bar{\partial}_l f = \frac{\partial f}{\partial t} + i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} = 0 \quad \left[\text{viz. PROPOSITION 3 – The Cauchy-Riemann-Fueter Equations.} \right]$$

[N.B. $f = f(q)$, where $q = t + ix + jy + kz$.], which has been enunciated on page 10 of Sudbery [3], likewise guarantees the existence of the aforesaid definite integral underlined in RED and, furthermore, is a non-commutative analogue of the differential equation,

$$\frac{\partial}{\partial x}(f(q)) + Q^*\frac{\partial}{\partial \xi}(f(q)) = 0 \left[\underline{\text{N.B.}}\ f(q) = f(x+Q^*\xi) = \mathcal{U}(x,\xi) + Q^*\mathcal{V}(x,\xi).\right].$$

(3) As is evident from the contents of Appendix A3, this particular differential equation generates specific quaternion analogues of the Cauchy-Riemann equations, whose satisfaction subsequently guarantees the existence of the definite integral,

$$\int_C f(q)\,dq = 0,$$

by virtue of the aforesaid Theorem TII-25.

3. Correlation of Specific Formulae pertaining to the Series Expansions of Quaternion Hypercomplex Functions, restricted to Smooth Arcs embedded in q-Space, about a Non-singular Point.

(1) As is evident from the contents of Appendix A4, the series expansion,

$$f(q(t)) = \sum_{n=0}^{\infty}\left[\frac{d^n}{dt^n}[f(q(t))]\right]_{t=t_0}(t-t_0)^n/n! \quad [\underline{\text{N.B.}}\ t\in(a,b)\subseteq\mathbb{R}.],$$

whose existence is enunciated via Theorem TIII-5 on pages 294 - 304 of the aforesaid paper, may in certain circumstances be reduced to its more familiar real valued analogue (i.e. Taylor series expansion), namely -

$$f(x(t)) = \sum_{n=0}^{\infty}\left[\frac{d^n}{dt^n}[f(x(t))]\right]_{t=t_0}(t-t_0)^n/n! \quad [\underline{\text{N.B.}}\ t\in(a,b)\subseteq\mathbb{R}.].$$

(2) Furthermore, from the contents of this same appendix, we likewise note the application of the formula,

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$$\frac{d}{dt}(f(q(t))) = \frac{\partial}{\partial x}(f(q))\frac{dx}{dt} + \frac{\partial}{\partial y}(f(q))\frac{dy}{dt} + \frac{\partial}{\partial \hat{x}}(f(q))\frac{d\hat{x}}{dt} + \frac{\partial}{\partial \hat{y}}(f(q))\frac{d\hat{y}}{dt}$$

$$\left[\begin{array}{l} \underline{\text{N.B.}}\ q = q(t);\ x = x(t);\ y = y(t);\ \hat{x} = \hat{x}(t);\ \hat{y} = \hat{y}(t)\ \& \\ \quad f(q) = f(x + iy + j\hat{x} + k\hat{y}) = u_1(x,y,\hat{x},\hat{y}) + i v_1(x,y,\hat{x},\hat{y}) + \\ \qquad\qquad j u_2(x,y,\hat{x},\hat{y}) + k v_2(x,y,\hat{x},\hat{y}) \end{array}\right],$$

which is logically consistent with the formula underlined in RED, as previously depicted in Extract No. E5/12 from Section I.

III. APPENDICES.

A1. Derivation of Formula for q^α via Extract No. E3/12 from Section I.

Theorem T(A1)−1.

Let there exist a constant, $\alpha \in \mathbb{R}$. Hence, it may be proven that, if the independent variable, $q = a + ib + jc + kd$ $(a, b, c, d \in \mathbb{R})$, then the function,

$$q^\alpha = \|q\|^\alpha(\cos(\alpha\theta) + \hat{n}\sin(\alpha\theta)),$$

where the quaternion, $v = ib + jc + kd$; the moduli, $\|v\| = \sqrt{b^2 + c^2 + d^2}$ & $\|q\| = \sqrt{a^2 + b^2 + c^2 + d^2}$; the quaternion, $\hat{n} = v/\|v\|$ and the angle, $\theta = \cos^{-1}(a/\|q\|)$.

* * * * *

PROOF:-

From the relevant definitions and theorems enunciated in Pearson [4] & [5], we recall that the function,

$$q^{\alpha} = \begin{cases} e^{\alpha \log(q)} \\ e^{\log(q)\alpha} \end{cases} \quad (A1-1).$$

However, since the constant, $\alpha \in \mathbb{R}$, it immediately follows that the product, $\alpha \log(q) = \log(q)\alpha$, and hence Eq. (A1-1) may be rewritten as

$$q^{\alpha} = e^{\alpha \log(q)} \quad (A1-2).$$

Similarly, we recall from the aforesaid definitions and theorems that the logarithmic function,

$$\log(q) = \log(a + ib + jc + kd)$$

$$= \log(\|q\|) + \hat{n}\cos^{-1}(a/\|q\|)$$

$$= \log(\|q\|) + \hat{n}\theta \quad (A1-3).$$

N.B. $\|q\| = \sqrt{a^2+b^2+c^2+d^2}$; $v = ib + jc + kd \Rightarrow \|v\| = \sqrt{b^2+c^2+d^2}$; $\hat{n} = v/\|v\|$ & $\theta = \cos^{-1}(a/\|q\|)$, as previously stated in the preamble.

After making the appropriate algebraic substitutions, it likewise follows from Eqs. (A1-2) & (A1-3) and the aforesaid definitions and theorems that the function,

$$\begin{aligned} q^{\alpha} &= e^{\alpha \log(q)} \\ &= \exp[\alpha \log(q)] \\ &= \exp[\alpha(\log(\|q\|) + \hat{n}\theta)] \\ &= \exp[\alpha \log(\|q\|) + \hat{n}\alpha\theta] \\ &= \exp[\alpha \log(\|q\|)]\exp[\hat{n}\alpha\theta] = \|q\|^{\alpha}(\cos(\alpha\theta) + \hat{n}\sin(\alpha\theta)), \end{aligned}$$

as required. Q.E.D.

..../13

A2. Derivation of Formula for $\left(\frac{1}{\phi}\right)'(q)$ via Statement (6) in Subsection 1 of Section II.

Theorem T(A2–1).

Let there exist a quaternion function, $\phi(q)$, where the independent variable, $q \in \mathcal{D} \subseteq \mathbb{H}$. Hence, it may proven that, if $\phi(q) \neq 0$, then the resultant first derivative of $1/\phi(q)$ with respect to x,

$$\left(\frac{1}{\phi}\right)'(q) = \frac{\partial}{\partial x}\left[\frac{1}{\phi(q)}\right] = -\frac{1}{\phi(q)}\frac{\partial}{\partial x}\left[\phi(q)\right]\frac{1}{\phi(q)} = \frac{\partial}{\partial x}\left[\overline{\phi(q)}/|\phi(q)|^2\right]$$

$$= \frac{|\phi(q)|^2 \frac{\partial}{\partial x}\left(\overline{\phi(q)}\right) - \frac{\partial}{\partial x}\left(|\phi(q)|^2\right)\overline{\phi(q)}}{|\phi(q)|^4}$$

* * * * *

PROOF:–

From the relevant definitions and theorems enunciated in Pearson [4] & [5], we recall that the first derivative of $\frac{1}{\phi(q)}$ with respect to x,

$$\frac{\partial}{\partial x}\left[\frac{1}{\phi(q)}\right] = \lim_{\delta x \to 0}\left[\frac{\frac{1}{\phi(q+\delta x)} - \frac{1}{\phi(q)}}{\delta x}\right],$$

whereupon it immediately follows that the difference quotient,

$$\frac{\frac{1}{\phi(q+\delta x)} - \frac{1}{\phi(q)}}{\delta x} = \frac{1}{\delta x}\left(\frac{1}{\phi(q+\delta x)} - \frac{1}{\phi(q)}\right)$$

$$= \frac{1}{\delta x\,\phi(q+\delta x)} - \frac{1}{\delta x\,\phi(q)}$$

$$= \frac{1}{\phi(q+\delta x)\,\delta x} - \frac{1}{\delta x\,\phi(q)}$$

$$= \frac{1}{\phi(q+\delta x)\delta x}\left[\frac{\phi(q)}{\phi(q)}\right] - \left[\frac{\phi(q+\delta x)}{\phi(q+\delta x)}\right]\frac{1}{\delta x\,\phi(q)}$$

$$= \frac{\phi(q)}{\phi(q+\delta x)\,\delta x\,\phi(q)} - \frac{\phi(q+\delta x)}{\phi(q+\delta x)\,\delta x\,\phi(q)}$$

$$= \frac{1}{\phi(q+\delta x)}\left[\frac{\phi(q)}{\delta x\,\phi(q)} - \frac{\phi(q+\delta x)}{\delta x\,\phi(q)}\right]$$

$$= \frac{1}{\phi(q+\delta x)}\left[\frac{\phi(q)}{\delta x} - \frac{\phi(q+\delta x)}{\delta x}\right]\frac{1}{\phi(q)}$$

$$= -\frac{1}{\phi(q+\delta x)}\left[\frac{\phi(q+\delta x)-\phi(q)}{\delta x}\right]\frac{1}{\phi(q)}\,.$$

Furthermore, in view of the aforesaid definitions and theorems, since the function,

$$\frac{1}{\phi(q)} = \frac{\overline{\phi(q)}}{|\phi(q)|^2} \Longrightarrow \frac{1}{\phi(q+\delta x)} = \frac{\overline{\phi(q+\delta x)}}{|\phi(q+\delta x)|^2}\,,$$

it likewise follows that the difference quotient,

$$\frac{\frac{1}{\phi(q+\delta x)} - \frac{1}{\phi(q)}}{\delta x} = \frac{\frac{\overline{\phi(q+\delta x)}}{|\phi(q+\delta x)|^2} - \frac{\overline{\phi(q)}}{|\phi(q)|^2}}{\delta x}$$

$$= \frac{1}{\delta x}\left[\frac{\overline{\phi(q+\delta x)}}{|\phi(q+\delta x)|^2}\cdot\frac{|\phi(q)|^2}{|\phi(q)|^2} - \frac{|\phi(q+\delta x)|^2}{|\phi(q+\delta x)|^2}\cdot\frac{\overline{\phi(q)}}{|\phi(q)|^2}\right]$$

$$= \frac{1}{\delta x}\left[\frac{\overline{\phi(q+\delta x)}\,|\phi(q)|^2 - |\phi(q+\delta x)|^2\,\overline{\phi(q)}}{|\phi(q+\delta x)|^2\,|\phi(q)|^2}\right]$$

$$= \frac{1}{|\phi(q+\delta x)|^2 |\phi(q)|^2} \left[\frac{\overline{\phi(q+\delta x)}\,|\phi(q)|^2 - |\phi(q+\delta x)|^2\,\overline{\phi(q)}}{\delta x} \right]$$

$$= \frac{1}{|\phi(q+\delta x)|^2 |\phi(q)|^2} \left[\frac{\overline{\phi(q+\delta x)}\,|\phi(q)|^2 - \overline{\phi(q)}\,|\phi(q)|^2 + \overline{\phi(q)}\,|\phi(q)|^2 - |\phi(q+\delta x)|^2\,\overline{\phi(q)}}{\delta x} \right]$$

$$= \frac{1}{|\phi(q+\delta x)|^2 |\phi(q)|^2} \left[\frac{\left(\overline{\phi(q+\delta x)} - \overline{\phi(q)}\right)|\phi(q)|^2 + |\phi(q)|^2\,\overline{\phi(q)} - |\phi(q+\delta x)|^2\,\overline{\phi(q)}}{\delta x} \right]$$

$$= \frac{1}{|\phi(q+\delta x)|^2 |\phi(q)|^2} \left[\frac{\left(\overline{\phi(q+\delta x)} - \overline{\phi(q)}\right)|\phi(q)|^2 + \left(|\phi(q)|^2 - |\phi(q+\delta x)|^2\right)\overline{\phi(q)}}{\delta x} \right]$$

$$= \frac{1}{|\phi(q+\delta x)|^2 |\phi(q)|^2} \left[\frac{\left(\overline{\phi(q+\delta x)} - \overline{\phi(q)}\right)|\phi(q)|^2 - \left(|\phi(q+\delta x)|^2 - |\phi(q)|^2\right)\overline{\phi(q)}}{\delta x} \right]$$

$$= \frac{1}{|\phi(q+\delta x)|^2 |\phi(q)|^2} \left[\frac{|\phi(q)|^2\left(\overline{\phi(q+\delta x)} - \overline{\phi(q)}\right) - \left(|\phi(q+\delta x)|^2 - |\phi(q)|^2\right)\overline{\phi(q)}}{\delta x} \right]$$

$$= \frac{1}{|\phi(q+\delta x)|^2 |\phi(q)|^2} \left[|\phi(q)|^2 \left(\frac{\overline{\phi(q+\delta x)} - \overline{\phi(q)}}{\delta x} \right) - \left(\frac{|\phi(q+\delta x)|^2 - |\phi(q)|^2}{\delta x} \right) \overline{\phi(q)} \right].$$

Subsequently, we may now write

$$\frac{\frac{1}{\phi(q+\delta x)} - \frac{1}{\phi(q)}}{\delta x} = -\frac{1}{\phi(q+\delta x)} \left(\frac{\phi(q+\delta x) - \phi(q)}{\delta x} \right) \frac{1}{\phi(q)}$$

$$= \frac{1}{|\phi(q+\delta x)|^2 |\phi(q)|^2} \left[|\phi(q)|^2 \left(\frac{\overline{\phi(q+\delta x)} - \overline{\phi(q)}}{\delta x} \right) - \left(\frac{|\phi(q+\delta x)|^2 - |\phi(q)|^2}{\delta x} \right) \overline{\phi(q)} \right].$$

Once again, by virtue of the aforesaid definitions and theorems, we recall that the limits,

$$\lim_{\delta x\to 0}\left(\frac{1}{\phi(q+\delta x)}\right)=\frac{1}{\phi(q)}\,;\ \lim_{\delta x\to 0}\left(\frac{1}{\phi(q)}\right)=\frac{1}{\phi(q)}$$

$$\lim_{\delta x\to 0}\left(\frac{1}{|\phi(q+\delta x)|^2|\phi(q)|^2}\right)=\frac{1}{|\phi(q)|^4}\,;\ \lim_{\delta x\to 0}\left(|\phi(q)|\right)^2=|\phi(q)|^2\,;$$

$$\lim_{\delta x\to 0}\left(\overline{\phi(q)}\right)=\overline{\phi(q)}\,;\ \lim_{\delta x\to 0}\left(\frac{\phi(q+\delta x)-\phi(q)}{\delta x}\right)=\frac{\partial}{\partial x}[\phi(q)]\,;$$

$$\lim_{\delta x\to 0}\left(\frac{\overline{\phi(q+\delta x)}-\overline{\phi(q)}}{\delta x}\right)=\frac{\partial}{\partial x}[\overline{\phi(q)}]\,;\ \lim_{\delta x\to 0}\left(\frac{|\phi(q+\delta x)|^2-|\phi(q)|^2}{\delta x}\right)=\frac{\partial}{\partial x}(|\phi(q)|^2),$$

likewise imply the existence of the limit,

$$\lim_{\delta x\to 0}\left[\frac{\frac{1}{\phi(q+\delta x)}-\frac{1}{\phi(q)}}{\delta x}\right]=-\frac{1}{\phi(q)}\frac{\partial}{\partial x}(\phi(q))\frac{1}{\phi(q)}$$

$$=\frac{|\phi(q)|^2\frac{\partial}{\partial x}(\overline{\phi(q)})-\frac{\partial}{\partial x}(|\phi(q)|^2)\,\overline{\phi(q)}}{|\phi(q)|^4}.$$

Finally, by setting

$$\frac{1}{\phi}'(q)=\frac{\partial}{\partial x}\left[\frac{1}{\phi(q)}\right]=\frac{\partial}{\partial x}\left[\frac{\overline{\phi(q)}}{|\phi(q)|^2}\right],$$

we immediately deduce from the preceding statements that

$$\frac{1}{\phi}'(q)=\frac{\partial}{\partial x}\left[\frac{1}{\phi(q)}\right]=-\frac{1}{\phi(q)}\frac{\partial}{\partial x}(\phi(q))\frac{1}{\phi(q)}=\frac{\partial}{\partial x}\left[\frac{\overline{\phi(q)}}{|\phi(q)|^2}\right]$$

$$=\frac{|\phi(q)|^2\frac{\partial}{\partial x}(\overline{\phi(q)})-\frac{\partial}{\partial x}(|\phi(q)|^2)\,\overline{\phi(q)}}{|\phi(q)|^4}$$

, as required. Q.E.D.

..../17

A3. Generation of Specific Quaternion Analogues of the Cauchy – Riemann Equations.

Theorem T(A3) – 1.

Let us define the following differential equation, namely –

$$\frac{\partial}{\partial x}(f(q)) + Q^* \frac{\partial}{\partial \xi}(f(q)) = 0,$$

where

(a) the quaternion constant,

$$Q^* = \frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}, \text{ such that } \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} \ \& \ [Q^*]^2 = -1;$$

(b) the quaternion function,

$$f(q) = f(x + Q^*\xi) = \mathcal{U}(x,\xi) + Q^* \mathcal{V}(x,\xi),$$

such that the functions, $\mathcal{U}(x,\xi), \mathcal{V}(x,\xi) \in \mathbb{R}$.

Hence, it may be proven that this particular differential equation likewise generates the following quaternion analogues of the Cauchy – Riemann Equations, namely –

$$\frac{\partial}{\partial x}[\mathcal{U}(x,\xi)] = \frac{\partial}{\partial \xi}[\mathcal{V}(x,\xi)] \ \& \ \frac{\partial}{\partial x}[\mathcal{V}(x,\xi)] = -\frac{\partial}{\partial \xi}[\mathcal{U}(x,\xi)].$$

* * * * *

PROOF:–

..../18

Let us assume that the differential equation,

$$\frac{\partial}{\partial x}(f(q)) + Q^*\frac{\partial}{\partial \xi}(f(q)) = 0,$$

exists, as previously indicated in the preamble. Hence, after making the appropriate algebraic substitutions in accordance with the relevant definitions and theorems enunciated in Pearson [4] & [5], we subsequently obtain

$$\frac{\partial}{\partial x}[\mathcal{U}(x,\xi) + Q^*\mathcal{V}(x,\xi)] + Q^*\frac{\partial}{\partial \xi}[\mathcal{U}(x,\xi) + Q^*\mathcal{V}(x,\xi)] = 0$$

$$\therefore \frac{\partial}{\partial x}[\mathcal{U}(x,\xi)] + Q^*\frac{\partial}{\partial x}[\mathcal{V}(x,\xi)] + Q^*\left(\frac{\partial}{\partial \xi}[\mathcal{U}(x,\xi)] + Q^*\frac{\partial}{\partial \xi}[\mathcal{V}(x,\xi)]\right) = 0$$

$$\therefore \frac{\partial}{\partial x}[\mathcal{U}(x,\xi)] + Q^*\frac{\partial}{\partial x}[\mathcal{V}(x,\xi)] + Q^*\frac{\partial}{\partial \xi}[\mathcal{U}(x,\xi)] - \frac{\partial}{\partial \xi}[\mathcal{V}(x,\xi)] = 0$$

$$\therefore \frac{\partial}{\partial x}[\mathcal{U}(x,\xi)] - \frac{\partial}{\partial \xi}[\mathcal{V}(x,\xi)] + Q^*\left(\frac{\partial}{\partial x}[\mathcal{V}(x,\xi)] + \frac{\partial}{\partial \xi}[\mathcal{V}(x,\xi)]\right) = 0 \quad \text{(A3-1)}.$$

Since, by definition, the quaternion constant,

$$0 + Q^*0 = 0 + 0 = 0,$$

it immediately follows from Eq. (A3-1) that

$$\frac{\partial}{\partial x}[\mathcal{U}(x,\xi)] - \frac{\partial}{\partial \xi}[\mathcal{V}(x,\xi)] = 0 \;\&\; \frac{\partial}{\partial x}[\mathcal{V}(x,\xi)] + \frac{\partial}{\partial \xi}[\mathcal{U}(x,\xi)] = 0$$

$$\therefore \frac{\partial}{\partial x}[\mathcal{U}(x,\xi)] = \frac{\partial}{\partial \xi}[\mathcal{V}(x,\xi)] \;\&\; \frac{\partial}{\partial x}[\mathcal{V}(x,\xi)] = -\frac{\partial}{\partial \xi}[\mathcal{U}(x,\xi)],$$

as required. Q.E.D. [*]

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[*] N.B. This particular result is logically consistent with the contents of Theorem TII-12 having been enunciated on pages 186-190 of Pearson [5].

A4. Reduction of the Series Expansions of Quaternion Hypercomplex Functions, restricted to Smooth Arcs embedded in q-Space, about a Non-singular Point to Their Real Valued Analogues.

Theorem T(A4)-1.

Let there exist a quaternion hypercomplex function,

$$f(q) = f(x + iy + j\hat{x} + k\hat{y})$$

$$= u_1(x,y,\hat{x},\hat{y}) + iv_1(x,y,\hat{x},\hat{y}) + ju_2(x,y,\hat{x},\hat{y}) + kv_2(x,y,\hat{x},\hat{y}),$$

which is differentiable in x, y, $\hat{x}$ & $\hat{y}$ respectively such that its concomitant series expansion, restricted to a smooth arc, $\mathcal{C}$, embedded in q-space about a non-singular point, t_0, namely –

$$f(q(t)) = \sum_{n=0}^{\infty} \left[\frac{d^n}{dt^n}[f(q(t))]\right]_{t=t_0} (t-t_0)^n/n!,$$

may be evaluated for each parametric variable, $t \in (a,b) \subseteq \mathbb{R}$.

Hence, it may be proven that this particular series expansion reduces to

$$f(x(t)) = \sum_{n=0}^{\infty} \left[\frac{d^n}{dt^n}[f(x(t))]\right]_{t=t_0} (t-t_0)^n/n! \in \mathbb{R}, \ \forall t \in (a,b) \subseteq \mathbb{R},$$

provided that the real valued function,

$$v_1(x,0,0,0) = u_2(x,0,0,0) = v_2(x,0,0,0) = 0. \ [\underline{*}]$$

* * * * *

PROOF:–

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Since the independent variable, $q = x + iy + j\hat{x} + k\hat{y}$, is defined on a smooth arc, C, we may initially write

$$q = q(t);\ x = x(t);\ y = y(t);\ \hat{x} = \hat{x}(t)\ \&\ \hat{y} = \hat{y}(t),\ \forall t \in (a, b) \subseteq \mathbb{R},$$

thereby implying that

$$q = x(t) + iy(t) + j\hat{x}(t) + k\hat{y}(t)\ \&$$

$$f(q(t)) = u_1(x(t), y(t), \hat{x}(t), \hat{y}(t)) + iv_1(x(t), y(t), \hat{x}(t), \hat{y}(t)) + ju_2(x(t), y(t), \hat{x}(t), \hat{y}(t)) + kv_2(x(t), y(t), \hat{x}(t), \hat{y}(t)).$$

Now, let the point,

$$(y, \hat{x}, \hat{y}) = (y(t), \hat{x}(t), \hat{y}(t)) = (0,0,0),$$

whereupon we immediately deduce that

$$q(t) = x(t) + i.0 + j.0 + k.0 = x(t)\ \&$$

$$f(q(t)) = f(x(t)) = u_1(x(t), 0,0,0) + iv_1(x(t), 0,0,0) + ju_2(x(t), 0,0,0) + kv_2(x(t), 0,0,0)$$
$$= \sum_{n=0}^{\infty} \left[\tfrac{d^n}{dt^n}[f(x(t))]\right]_{t=t_0} (t - t_0)^n / n!.$$

By stipulating that the real valued function,

$$v_1(x,0,0,0) = u_2(x,0,0,0) = v_2(x,0,0,0) = 0,$$

then, in view of the preceding statements, it likewise follows that the real valued parametric function,

..../21

$v_1(x(t),0,0,0) = u_2(x(t),0,0,0) = v_2(x(t),0,0,0) = 0,$

and hence the function,

$$f(x(t)) = u_1(x(t),0,0,0) = \sum_{n=0}^{\infty}\left[\frac{d^n}{dt^n}[f(x(t))]\right]_{t=t_0}(t-t_0)^n/n! \in \mathbb{R} \ [\underline{*}],$$

$\forall t \in (a,b) \subseteq \mathbb{R}$, since the real valued function, $u_1(x,0,0,0)$, automatically implies that the concomitant function, $u_1(x(t),0,0,0) \in \mathbb{R}$, bearing in mind that the 'nth' derivative,

$$\begin{aligned}\frac{d^n}{dt^n}[f(x(t))] &= \frac{d^{n-1}}{dt^{n-1}}\left(\frac{d}{dt}[f(x(t))]\right)\\ &= \left[\frac{d^{n-1}}{dt^{n-1}}\left(\frac{d}{dt}[f(q(t))]\right)\right]_{(y(t),\hat{x}(t),\hat{y}(t))=(0,0,0)},\end{aligned}$$

is defined on the interval, $(a,b) \subseteq \mathbb{R}$, $\forall n \in \{0,1,2,\ldots\ldots,\infty\}$, since the parametric first derivative,

$$\frac{d}{dt}[f(q(t))] = \frac{\partial}{\partial x}(f(q))\frac{dx}{dt} + \frac{\partial}{\partial y}(f(q))\frac{dy}{dt} + \frac{\partial}{\partial \hat{x}}(f(q))\frac{d\hat{x}}{dt} + \frac{\partial}{\partial \hat{y}}(f(q))\frac{d\hat{y}}{dt},$$

is similarly defined on the same interval via Pearson [5], as required. Q.E.D.

[$\underline{*}$] N.B. This particular statement is valid provided that the series,

$$\sum_{n=0}^{\infty}\left[\frac{d^n}{dt^n}[f(q(t))]\right]_{t=t_0}(t-t_0)^n/n!,$$

likewise converges whenever $|t-t_0| < r_0 \Longrightarrow t \in (t_0-r_0, t_0+r_0) \subseteq (a,b) \subseteq \mathbb{R}$.

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IV. BIBLIOGRAPHY.

[1] Quaternion – Wikipedia [https://en.wikipedia.org/wiki/Quaternion].

[2] O. Dzagnidze; On the Differentiability of Quaternion Functions [arXiv:1203.5619v1 [math.cv] 26 Mar 2012].

[3] A. Sudbery; Quaternionic Analysis [Math. Proc. Camb. Phil. Soc. (1979), 85, 199–225].

[4] S. C. Pearson; An Introduction to Functions of a Quaternion Hypercomplex Variable [31st March 1984; 161 handwritten foolscap pages]. [*]

[5] S. C. Pearson; A Supplementary Discourse on the Classification and Calculus of Quaternion Hypercomplex Functions [5th March 2001; 316 handwritten foolscap pages]. [*]

Underscore

[*] Web site address:– http://vixra.org/author/stephen_c_pearson.

[*] This particular paper will be published via the aforesaid web site address in due course.

*   *   *   *   *

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III. COPY OF AUTHOR'S ORIGINAL MONOGRAPH No.2.

(1) Title of Monograph:-

"Supplementary Notes in Relation to the Proofs of Two Theorems pertaining to the Integration of Quaternion Hypercomplex Functions."

Re:- A Specific Quaternion Analogue of the Cauchy-Goursat Theorem.

(2) Date of Completion:- 21st May 2019.

(3) Total number of A4 pages to follow = 14.

(4) Referee's Assessment:-

For further details, the reader should accordingly refer to the contents of **Appendix A2**.

* * * * * * * *

Supplementary Notes in Relation to the Proofs of Two Theorems pertaining to the Integration of Quaternion Hypercomplex Functions.

Re:- A Specific Quaternion Analogue of the Cauchy-Goursat Theorem.

I. Enunciation of Theorem TII-25 in the Author's Paper [P1].

With reference to the proof of the above mentioned theorem, the author has re-examined a couple of statements, which in his opinion are not sufficiently rigorous and have subsequently been rectified, as is evident from the contents of Subsection 1B.

1A. Copy of Preamble plus Initial Portion of Proof.

Theorem TII-25.

Let there exist a simple closed contour, $C = \bigcup_{n=1}^{N} K_{n}$, such that each of its component smooth arcs, K_n, is accordingly represented by the equation,

$$q_n(t) = x_n(t) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\xi_n(t) \;,\; \forall t \in [a_n, b_n] \;\&\; \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R},$$

such that the corresponding endpoints,

(i) $q_{m+1}(a_{m+1}) = q_m(b_m)$, $\forall m \in \{1, 2, 3, \ldots\ldots\ldots, N-1\}$,

(N is some finite positive integer)

AND

(ii) $q_1(a_1) = q_N(b_N)$.

Hence, it may be shown that, if the quasi-complex function,

..../2

$$f(q) = f(x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\xi) = \mathcal{U}(x,\xi) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\mathcal{V}(x,\xi),$$

is analytic at every point interior to and on the simple closed contour, C, then the definite integral,

$$\int_C f(q)\,dq = 0,$$

likewise exists, if and only if the contour, C, is sufficiently small so as to be entirely contained within the δ-neighbourhood of any fixed point, $q(t_0)$, located on the contour. This result shall otherwise be referred to as the

quaternion analogue of the Cauchy-Goursat Theorem.

* * *

PROOF:-

From the preceding Theorem TII-13 and Definition DII-10, we recall that a quasi-complex function,

$$f(q) = f(x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\xi) = \mathcal{U}(x,\xi) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\mathcal{V}(x,\xi),$$

is analytic at each point, $q = q_0 = q(t_0)$, on a simple closed contour, $C = \bigcup_{n=1}^{N} K_n$, if and only if the first derivative, $\frac{d}{dq}(f(q))$, exists not only at $q(t_0)$ but also at every point within some δ-neighbourhood of $q(t_0)$. From Definition DII-9, we also obtain

..../3

N.B. For reasons of brevity, the ensuing statements from pages 236 – 239 of paper [P1] have been omitted.

-240-

Bearing in mind the provisions of Theorem TII-18, we similarly perceive that the modulus,

$$\left|\int_C f(q)\,dq\right| = \left|\sum_{n=1}^{N}\int_{a_n}^{b_n}\phi(q_n(t))(q_n(t)-q(t_0))\tfrac{d}{dt}[q_n(t)]\,dt\right|$$

$$\leq \sum_{n=1}^{N}\left|\int_{a_n}^{b_n}\phi(q_n(t))(q_n(t)-q(t_0))\tfrac{d}{dt}[q_n(t)]\,dt\right|$$

$$\leq \sum_{n=1}^{N}\int_{a_n}^{b_n}\left|\phi(q_n(t))(q_n(t)-q(t_0))\tfrac{d}{dt}[q_n(t)]\right|dt.$$

But since, as previously stated, the moduli,

$$|\phi(q_n(t))| = \left|\frac{f(q_n(t))-f(q(t_0))}{q_n(t)-q(t_0)} - \tfrac{d}{dq}(f(q))\bigg|_{q=q(t_0)}\right| < \epsilon,$$

$$|q_n(t)-q(t_0)| < \delta,$$

1B. Revised Final Portion of Proof.

[Continuation of the last statement in Subsection 1A.]

and also by definition the moduli,

$$|\phi(q_n(t))| < |\phi(q_n(t))| + 1;$$

$$|\tfrac{d}{dt}[q_n(t)]| < |\tfrac{d}{dt}[q_n(t)]| + 1,$$

such that the modulus product,

$$|\phi(q_n(t))(q_n(t) - q(t_0))\tfrac{d}{dt}[q_n(t)]| = |\phi(q_n(t))||q_n(t) - q(t_0)||\tfrac{d}{dt}[q_n(t)]|$$

$$< (|\phi(q_n(t))| + 1)|q_n(t) - q(t_0)|(|\tfrac{d}{dt}[q_n(t)]| + 1)$$

$$< (\epsilon + 1)\delta(|\tfrac{d}{dt}[q_n(t)]| + 1) \qquad (a_n \leq t \leq b_n),$$

it therefore follows from real variable analysis that the definite integral,

$$\int_{a_n}^{b_n} |\phi(q_n(t))(q_n(t) - q(t_0))\tfrac{d}{dt}[q_n(t)]|\,dt < \int_{a_n}^{b_n} (\epsilon + 1)\delta(|\tfrac{d}{dt}[q_n(t)]| + 1)\,dt$$

$$= (\epsilon + 1)\delta \int_{a_n}^{b_n} (|\tfrac{d}{dt}[q_n(t)]| + 1)\,dt = (\epsilon + 1)\delta\Big(\int_{a_n}^{b_n} |\tfrac{d}{dt}[q_n(t)]|\,dt + \int_{a_n}^{b_n} 1\,dt\Big)$$

$$= (\epsilon + 1)\delta\Big(\int_{a_n}^{b_n} |\tfrac{d}{dt}[q_n(t)]|\,dt + b_n - a_n\Big).$$

Subsequently, the modulus,

$$\Big|\int_C f(q)\,dq\Big| \leq \sum_{n=1}^{N} \int_{a_n}^{b_n} |\phi(q_n(t))(q_n(t) - q(t_0))\tfrac{d}{dt}[q_n(t)]|\,dt$$

…/5

$$< \sum_{n=1}^{N} (\epsilon+1)\delta\left(\int_{a_n}^{b_n} \left|\frac{d}{dt}[q_n(t)]\right| dt + b_n - a_n\right)$$

$$= (\epsilon+1)\delta \sum_{n=1}^{N} \left(\int_{a_n}^{b_n} \left|\frac{d}{dt}[q_n(t)]\right| dt + b_n - a_n\right)$$

$$= \delta(\epsilon+1)\left[\sum_{n=1}^{N} \int_{a_n}^{b_n} \left|\frac{d}{dt}[q_n(t)]\right| dt + \sum_{n=1}^{N}(b_n - a_n)\right]$$

$$\therefore \left|\int_C f(q)\,dq\right| < \delta(\epsilon+1)\left[\sum_{n=1}^{N} \int_{a_n}^{b_n} \left|\frac{d}{dt}[q_n(t)]\right| dt + \sum_{n=1}^{N}(b_n - a_n)\right] \qquad (1).$$

Since the absolute value,

$$\left|\frac{d}{dt}[q_n(t)]\right| \geq 0,$$

we recall from the established definitions and theorems thus pertaining to the integration of real valued functions that the definite integral,

$$\int_{a_n}^{b_n} \left|\frac{d}{dt}[q_n(t)]\right| dt \geq 0 \Longrightarrow \sum_{n=1}^{N} \int_{a_n}^{b_n} \left|\frac{d}{dt}[q_n(t)]\right| dt \geq 0$$

$$\Longrightarrow \sum_{n=1}^{N} \int_{a_n}^{b_n} \left|\frac{d}{dt}[q_n(t)]\right| dt = \left|\sum_{n=1}^{N} \int_{a_n}^{b_n} \left|\frac{d}{dt}[q_n(t)]\right| dt\right|.$$

Similarly, since each of the intervals, $[a_n, b_n]$, implies that $b_n \geq a_n$, we likewise deduce that

$$b_n \geq a_n \Longrightarrow b_n - a_n \geq 0 \Longrightarrow \sum_{n=1}^{N}(b_n - a_n) \geq 0$$

$$\Longrightarrow \sum_{n=1}^{N}(b_n - a_n) = \left|\sum_{n=1}^{N}(b_n - a_n)\right|.$$

..../6

Subsequently, inequality (1) can be written as

$$\left|\int_C f(q)\,dq\right| < \delta(\epsilon+1)\left[\left|\sum_{n=1}^{N}\int_{a_n}^{b_n}\left|\frac{d}{dt}[q_n(t)]\right|dt\right| + \left|\sum_{n=1}^{N}(b_n-a_n)\right|\right] \quad (2).$$

From the established definitions and theorems thus pertaining to the limits of real valued functions, we perceive that by setting the variable, $X = b_n$, the one sided limits,

$$\left\{\begin{array}{l}\lim\limits_{X\downarrow a_n}[X] = a_n; \\ \\ \lim\limits_{X\downarrow a_n}\left[\int_{a_n}^{X}\left|\frac{d}{dt}[q_n(t)]\right|dt\right] = 0\end{array}\right\} \Longrightarrow \left\{\begin{array}{l}\lim\limits_{b_n\downarrow a_n}[b_n] = a_n; \\ \\ \lim\limits_{b_n\downarrow a_n}\left[\int_{a_n}^{b_n}\left|\frac{d}{dt}[q_n(t)]\right|dt\right] = 0\end{array}\right\},$$

whereupon there exist real numbers, $\lambda_{1n} > 0$; $\lambda_{2n} > 0$; $\mu_1 > 0$ & $\mu_2 > 0$, such that we obtain the following pairs of inequalities, namely –

$$|b_n - a_n| < \frac{\mu_1}{N} \quad (3a),$$

whenever $a_n \leq b_n < a_n + \lambda_{1n}$;

$$\left|\int_{a_n}^{b_n}\left|\frac{d}{dt}[q_n(t)]\right|dt\right| < \frac{\mu_2}{N} \quad (3b),$$

whenever $a_n \leq b_n < a_n + \lambda_{2n}$. [*]

With regard to inequalities (3a) & (3b), it immediately follows that the absolute value,

$$\left|\sum_{n=1}^{N}\int_{a_n}^{b_n}\left|\frac{d}{dt}[q_n(t)]\right|dt\right| \leq \sum_{n=1}^{N}\left|\int_{a_n}^{b_n}\left|\frac{d}{dt}[q_n(t)]\right|dt\right| < \sum_{n=1}^{N}\frac{\mu_2}{N} = \frac{1}{N}\sum_{n=1}^{N}\mu_2$$

$$= \frac{1}{N}(\mu_2 N) = \mu_2$$

..../7

$$\therefore \left|\sum_{n=1}^{N}\int_{a_n}^{b_n}\left|\tfrac{d}{dt}[q_n(t)]\right|dt\right| < \mu_2 \quad (4a),$$

and similarly the absolute value,

$$\left|\sum_{n=1}^{N}(b_n - a_n)\right| \leq \sum_{n=1}^{N}|b_n - a_n| < \sum_{n=1}^{N}\frac{\mu_1}{N} = \frac{1}{N}\sum_{n=1}^{N}\mu_1 = \frac{1}{N}(\mu_1 N) = \mu_1$$

$$\therefore \left|\sum_{n=1}^{N}(b_n - a_n)\right| < \mu_1 \quad (4b).$$

Substitution of inequalities (4a) & (4b) into inequality (2) thus yields

$$\left|\int_C f(q)\,dq\right| < \delta(\epsilon+1)\left[\left|\sum_{n=1}^{N}\int_{a_n}^{b_n}\left|\tfrac{d}{dt}[q_n(t)]\right|dt\right| + \left|\sum^{N}(b_n - a_n)\right|\right]$$

$$< \delta(\epsilon+1)(\mu_2+\mu_1) = \delta(\epsilon+1)(\mu_1+\mu_2)$$

$$\therefore \left|\int_C f(q)\,dq\right| < \delta(\epsilon+1)(\mu_1+\mu_2) \quad (5).$$

Finally, since the simultaneous existence of the limits,

$$\lim_{b_n \downarrow a_n}[b_n] = a_n\,;\ \lim_{b_n \downarrow a_n}\left[\int_{a_n}^{b_n}\left|\tfrac{d}{dt}[q_n(t)]\right|dt\right] = 0,\ \forall n \in \{1,2,\ldots\ldots\ldots,N\},$$

and the first derivative,

$\frac{d}{dq}(f(q)) = \lim_{\delta q \to 0}\left[\frac{f(q+\delta q) - f(q)}{\delta q}\right], \forall q \in R(t_0)$, the region bounded by the simple closed contour, C,

implies that the real numbers, $\delta > 0$; $\epsilon > 0$; $\lambda_{1n} > 0$; $\lambda_{2n} > 0$; $\mu_1 > 0$ & $\mu_2 > 0$, can be made arbitrarily small, we therefore conclude that the term on the right side of inequality (5) can likewise be made arbitrarily small and hence the definite integral,

$\int_C f(q)\,dq = 0$, as required. Q.E.D.

[*] N.B.

Since the inequality,

$$0 < |q_n(t) - q(t_0)| < \delta, \ \forall t \in [a_n, b_n] \ \& \ n \in \{1, 2, \ldots\ldots\ldots, N\},$$

also implies, after making the appropriate algebraic substitutions, that the endpoints, $q_n(a_n)$ & $q_n(b_n)$, of each smooth arc, K_n, are characterised by the inequalities,

$$0 < |q_n(a_n) - q(t_0)| < \delta \ \& \ 0 < |q_n(b_n) - q(t_0)| < \delta,$$

we accordingly observe that their simultaneous existence in conjunction with the inequalities,

$$a_n \leq b_n < a_n + \lambda_{1n}; \ a_n \leq b_n < a_n + \lambda_{2n}; \ |b_n - a_n| < \frac{\mu_1}{N} \ \&$$

$$\left| \int_{a_n}^{b_n} \left| \frac{d}{dt}[q_n(t)] \right| dt \right| < \frac{\mu_2}{N},$$

likewise implies that both the length of the simple closed contour, C, namely -

$$\mathcal{L}(C) = \sum_{n=1}^{N} \int_{a_n}^{b_n} \left| \frac{d}{dt}[q_n(t)] \right| dt \longrightarrow 0,$$

and also the length of each interval, $[a_n, b_n]$, namely -

$$\mathcal{L}([a_n, b_n]) = b_n - a_n \longrightarrow 0,$$

whenever the endpoints, $q_n(a_n)$ & $q_n(b_n) \longrightarrow q(t_0)$.

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II. Enunciation of Theorem TI-3 in the Author's Paper [P2].

With reference to the proof of the above mentioned theorem, the author has provided an additional discourse, as is evident from the contents of Subsection 2B.

2A. Copy of Preamble plus Proof.

Theorem TI-3.

Let there exist a simple closed contour, $C = \bigcup_{n=1}^{N} K_n$, such that each of its constituent smooth arcs, K_n, is accordingly represented by the equation,

$$q_n(t) = x_n(t) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\xi_n(t),\ \forall t \in [a_n, b_n]\ \&\ \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R},$$

insofar as the corresponding endpoints,

(i) $q_{m+1}(a_{m+1}) = q_m(b_m),\ \forall m \in \{1, 2, 3, \ldots\ldots\ldots, N-1\}$,

(N is some finite positive integer)

AND

(ii) $q_1(a_1) = q_N(b_N)$.

Hence, it may be shown that, if the quasi-complex function,

-4-

$$f(q) = f\left(x + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\xi\right) = \mathcal{U}(x,\xi) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right]\mathcal{V}(x,\xi),$$

is analytic at every point interior to and on the simple closed contour, C, then the definite integral,

$$\int_C f(q)\,dq = 0,$$

likewise exists. This result shall accordingly be referred to as the 'generalised' quaternion analogue of the Cauchy-Goursat Theorem.

* * * * *

PROOF:-

In order to facilitate the proof of this particular theorem, we will invoke the principle of mathematical induction.

N.B. For reasons of brevity, the ensuing statements from pages 4-10 of paper [P2] have been omitted.

Finally, by increasing the magnitude of the positive integers, N; M(1);........; M(N), it therefore follows from Fig. 5 that the respective sizes of the contours, $\partial R[1,1]$; ....; $\partial R[N, M(N)]$, will inevitably be reduced and hence, in accordance with Theorem TI-1, if each of these contours is sufficiently small so as to be entirely contained within the δ-neighbourhood of any fixed point located on the contour, then every concomitant definite integral,

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$$\int_{\partial R[n,m]} f(q)\,dq = 0 \quad [\underline{*}],$$

-11-

thereby implying, after making the appropriate algebraic substitutions, that the definite integral,

$$\int_C f(q)\,dq = \sum_{n=1}^{N}\sum_{m=1}^{M(n)} 0 = \sum_{n=1}^{N} 0 = 0, \text{ as required. } \underline{\text{Q.E.D}}.$$

[*] N.B.

Since the function, $f(q)$, is analytic at every point interior to and on the simple closed contour, $C = \partial R$, as specified in the preamble to this proof, it must therefore be analytic at every point interior to and on each simple closed contour, $\partial R[n,m]$, as is evident from Fig. 5.

2B. Additional Discourse.

With reference to the last statement recorded in Subsection 2A, namely –

"Finally, by increasing the magnitude of the positive integers, ..., then every concomitant definite integral,

$$\int_{\partial R[n,m]} f(q)\,dq = 0 \quad [\underline{*}],$$

thereby implying, ..., as required. Q.E.D.", the author suggests that the process of partitioning the region, R, into its constituent subregions, $R[n,m]$, can be further clarified by means of the following discourse:-

* * * * *

Let each subregion,

$R[n,m] \subset \{s \,|\, |s - q_{(n,m)}(t_0)| < \delta_{(n,m)}(N_0)\}$

$\subset \prod = \langle 1, (i\lambda_1 + j\lambda_2 + k\lambda_3)/\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2} \rangle,$

where $q_{(n,m)}(t_0)$ denotes a fixed point on the simple closed contour, $\partial R[n,m]$, and the real number, $\delta_{(n,m)}(N_0)$, is characterised by the inequality,

$0 < \delta_{(n,m)}(N_0 + 1) < \delta_{(n,m)}(N_0), \ \forall N_0 \in \mathbb{N}$, the set of positive integers (1).

From <u>Fig. 5</u> we perceive that the <u>total number of subregions, $R[n,m]$</u>,

$T_N = M(1) + \ldots\ldots\ldots + M(N) = \sum_{n=1}^{N} M(n) \geq \sum_{n=1}^{N} 1 = N,$

bearing in mind that, by definition, every positive integer, $M(n) \geq 1$, whereupon it immediately follows that

$T_N = \sum_{n=1}^{N} M(n) \geq N.$

Moreover, the existence of the limit,

$\lim_{N \to \infty} (N) = \infty,$

implies that there exists a positive integer, N_0, and a real number, $W > 0$, such that

..../13

$N > W$ & $N > N_0$,

and hence we may write

$$T_N = \sum_{n=1}^{N} M(n) \geq N > W \;\&\; N > N_0 \Longrightarrow \lim_{N \to \infty}(T_N) = \lim_{N \to \infty}\left[\sum_{n=1}^{N} M(n)\right] = \infty.$$

From the preceding statements, it is apparent that, if the total number of subregions, $R[n,m]$, namely $-$ T_N, is made arbitrarily large by simultaneously increasing the magnitude of N_0 and W, then the real number, $\delta_{(n,m)}(N_0)$, will likewise become arbitrarily small by virtue of inequality (1), which also implies the existence of the limit,

$$\lim_{N_0 \to \infty}\left[\delta_{(n,m)}(N_0)\right] = 0,$$

since the sequence, $\delta_{(n,m)}(N_0)$, is <u>monotonically approaching its greatest lower bound, 0</u>. Subsequently, we conclude that the size of each simple closed contour, $\partial R[n,m] \subset R[n,m]$, will inevitably be reduced thereby satisfying the criteria specified in <u>Theorem TI-1</u>, as previously indicated.

III. BIBLIOGRAPHY.

[P1] S. C. Pearson; A Supplementary Discourse on the Classification and Calculus of Quaternion Hypercomplex Functions [5th March 2001; 316 handwritten foolscap pages]. [*]

[P2] S. C. Pearson; Supplementary Notes pertaining to a Specific Quaternion Analogue of the Cauchy-Goursat Theorem [6th March 2019; 16 handwritten A4 pages]. [*]

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* * * * *

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IV. COPY OF AUTHOR'S ORIGINAL MONOGRAPH No.3.

(1) Title of Monograph:-

"Supplementary Notes in Relation to the Proof of One Theorem pertaining to the Integration of Quaternion Hypercomplex Functions."

Re:- A Specific Quaternion Analogue of the Cauchy-Goursat Theorem.

(2) Date of Completion:- 29th May 2019.

(3) Total number of A4 pages to follow = 6.

(4) Referee's Assessment:-

For further details, the reader should accordingly refer to the contents of **Appendix A2**.

* * * * * * * *

Supplementary Notes in Relation to the Proof of One Theorem pertaining to the Integration of Quaternion Hypercomplex Functions.

Re:– A Specific Quaternion Analogue of the Cauchy–Goursat Theorem.

I. Enunciation of Theorem as a Supplement to the Preceding Monograph No. 2.

With reference to those statements made on page 8 of the above mentioned monograph, we will accordingly enunciate the following additional theorem:–

Theorem TI–1.

Let there exist a simple closed contour, $C = \bigcup_{n=1}^{N} K_n$, such that each of its component smooth arcs, K_n, is accordingly represented by the equation,

$$q_n(t) = x_n(t) + \left[\frac{i\lambda_1 + j\lambda_2 + k\lambda_3}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}\right] \xi_n(t), \ \forall t \in [a_n, b_n] \ \& \ \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R},$$

insofar as the corresponding endpoints,

(i) $q_{m+1}(a_m + 1) = q_m(b_m)$, $\forall m \in \{1, 2, \ldots\ldots, N-1\}$,

(N is some finite positive integer)

AND

(ii) $q_1(a_1) = q_N(b_N)$.

Hence, it may be shown that if $q(t_0)$ is an arbitrary fixed point located on the simple closed contour, C, then the difference between the endpoints, $q_n(a_n)$ & $q_n(b_n)$, namely –

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$$\Delta_{end}(K_n) = q_n(b_n) - q_n(a_n) \longrightarrow 0,$$

whenever $q_n(a_n)$ & $q_n(b_n) \longrightarrow q(t_0)$.

* * * * *

PROOF:-

From the author's paper [1] we perceive that both the differentiability and continuity of the quaternion parametric function, $q_n(t)$, throughout the interval, $[a_n, b_n]$, will subsequently guarantee the existence of the one sided limit,

$$\lim_{t \downarrow a_n}[q_n(t)] = q_n(a_n) \Longrightarrow \lim_{b_n \downarrow a_n}[q_n(b_n)] = q_n(a_n),$$

after setting the variable, $t = b_n$, insofar as there exist real numbers, $\xi_n > 0$ & $\eta_n > 0$, such that we obtain the pair of inequalities,

$$|q_n(b_n) - q_n(a_n)| < \eta_n \quad (1);$$

$$a_n \leq b_n < a_n + \xi_n. \quad [\underline{*}]$$

Since the aforesaid inequality from page 8 of Monograph No. 2, namely -

$$0 < |q_n(t_0) - q(t_0)| < \delta, \ \forall t \in [a_n, b_n] \ \& \ n \in \{1, 2, \ldots\ldots, N\},$$

also implies, after making the appropriate algebraic substitutions, that the endpoints, $q_n(a_n)$ & $q_n(b_n)$, of each smooth arc, K_n, are characterised by the inequalities,

$$0 < |q_n(a_n) - q(t_0)| < \delta \ \& \ 0 < |q_n(b_n) - q(t_0)| < \delta,$$

we deduce that squaring each term therein respectively yields

..../3

$$0 < |q_n(a_n) - q(t_0)|^2 < \delta^2 \ \& \ 0 < |q_n(b_n) - q(t_0)|^2 < \delta^2$$

and hence their resultant addition gives

$$0 < |q_n(a_n) - q(t_0)|^2 + |q_n(b_n) - q(t_0)|^2 < 2\delta^2 \Longrightarrow$$

$$0 < \sqrt{|q_n(a_n) - q(t_0)|^2 + |q_n(b_n) - q(t_0)|^2} < \sqrt{2}\,\delta \qquad (2).$$

Now, in accordance with the contents of Appendix A1, we observe that the quaternion norm,

$$\|(q_n(a_n), q_n(b_n)) - (q(t_0), q(t_0))\| = \|(q_n(a_n) - q(t_0), q_n(b_n) - q(t_0))\|$$

$$= \sqrt{|q_n(a_n) - q(t_0)|^2 + |q_n(b_n) - q(t_0)|^2},$$

whereupon inequality (2) can be written as

$$0 < \|(q_n(a_n), q_n(b_n)) - (q(t_0), q(t_0))\| < \sqrt{2}\,\delta \qquad (3).$$

Subsequently, the simultaneous occurrence of inequalities (1) & (3) guarantees the existence of the limit,

$$\lim_{(q_n(a_n), q_n(b_n)) \to (q(t_0), q(t_0))} \left[q_n(b_n) - q_n(a_n) \right] = 0,$$

and hence we conclude that the difference between the endpoints, $q_n(a_n)$ & $q_n(b_n)$, namely –

$$\Delta_{end}(K_n) = q_n(b_n) - q_n(a_n) \longrightarrow 0,$$

whenever $q_n(a_n)$ & $q_n(b_n) \longrightarrow q(t_0)$, as required. Q.E.D.

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[*] N.B.

Let there exist a quaternion parametric function, $f(q(t))$, which is continuous throughout the interval, $[a, b]$, such that for any point, $t_0 \in (a, b)$, the limit,

$$\lim_{t \to t_0} [f(q(t))] = f(q(t_0)),$$

exists, whereupon there also exist real numbers, $\delta^* > 0$ & $\epsilon^* > 0$, which generate the following pair of inequalities, namely –

$$|f(q(t)) - f(q(t_0))| < \epsilon^* \quad (1*);$$

$$|t - t_0| < \delta^*.$$

Furthermore, since the inequality,

$$|t - t_0| < \delta^* \Longrightarrow \begin{cases} t_0 - \delta^* < t \leq t_0 & (2a*); \\ t_0 \leq t < t_0 + \delta^* & (2b*), \end{cases}$$

we subsequently deduce that the simultaneous occurrence of inequalities (1*) & (2a*) implies the existence of the one sided limit,

$$\lim_{t \uparrow t_0} [f(q(t))] = f(q(t_0)),$$

and similarly the simultaneous occurrence of inequalities (1*) & (2b*) likewise implies the existence of the one sided limit,

$$\lim_{t \downarrow t_0} [f(q(t))] = f(q(t_0)).$$

Clearly, in view of the preceding statements, we have now validated the assertions made in the first paragraph of this proof.

II. APPENDICES.

A1. Basic Properties of Quaternion Norms.

In order to define the basic properties of quaternion norms, the author has subsequently referred to their more familiar analogues having been enunciated in various articles and papers pertaining to functions of several complex variables, as is evident from the contents of Appendix A2.

A2. Copy of Extract from Bremermann [2].

1. THE SPACE OF n-TUPLES OF COMPLEX NUMBERS C^n

From the familiar complex numbers we may form n-tuples. The collection of all n-tuples $z = (z_1, \cdots, z_n)$ of complex numbers $z_1, \cdots, z_n$ is denoted by C^n. We make it a linear vector space by introducing addition

$$z^{(1)} + z^{(2)} = (z_1^{(1)}, \cdots, z_n^{(1)}) + (z_1^{(2)}, \cdots, z_n^{(2)}) = (z_1^{(1)} + z_1^{(2)}, \cdots, z_n^{(1)} + z_n^{(2)});$$

and multiplication with a complex scalar λ

$$\lambda z = \lambda(z_1, \cdots, z_n) = (\lambda z_1, \cdots, \lambda z_n).$$

The addition is associative and commutative because it is defined as addition of the components, which are complex numbers. Analogously the multiplication by a scalar is distributive.

We leave it to the reader to verify that all the axioms of a linear vector space are satisfied.

1.1 The C^n becomes a Banach space by introducing a norm $\|\ \|$ satisfying: (1) $\|z\| > 0$ if $z \neq 0$; (2) $\|z^{(1)} + z^{(2)}\| \leq \|z^{(1)}\| + \|z^{(2)}\|$; (3) $\|\lambda z\| = (|\lambda|\,\|z\|)$, where λ is a complex number; (4) the C^n is complete with respect to the norm; that is, if for a sequence $\{z^{(j)}\}$, $z^{(j)} \in C^n$ we have $\|z^{(j)} - z^{(k)}\|$ tending to zero as j and k tend to infinity, then there exists an element $z^{(0)} \in C^n$ such that

$$\lim_{j\to\infty} \|z^{(j)} - z^{(0)}\| = 0.$$

Examples of Norms. The euclidean norm: $||z||_e^2 = |z_1|^2 + \cdots + |z_n|^2$. The maximum norm: $||z||_m = \max\{|z_1|, \cdots, |z_n|\}$. Every norm induces a topology if one defines as neighborhoods of a point $z^{(0)}$ the point sets

$$\{z \mid ||z - z^{(0)}|| < \epsilon; \epsilon > 0\}.$$

It is easy to show (the reader may carry out the proof) that: For any norm $|| \quad ||$ there exist two numbers $\rho > 0$ and $\sigma > 0$ such that for any $z \in C^n$ we have

$$\rho||z||_m \leq ||z|| \leq \sigma||z||_m,$$

where $|| \quad ||_m$ is the maximum norm.

III. BIBLIOGRAPHY.

[1] S. C. Pearson; An Introduction to Functions of a Quaternion Hypercomplex Variable [31st March 1984; 161 handwritten foolscap pages]. [*]

Underscore

[*] Web site address:- http://vixra.org/author/stephen_c_pearson.

[2] H. J. Bremermann; Several Complex Variables [viz. Article No. 1 in Hirschman [3]].

[3] I. I. Hirschman, Jr. (Editor); Studies in Real and Complex Analysis (Studies in Mathematics - Volume 3); The Mathematical Association of America Inc. [1965].

* * * * *

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V. APPENDICES.

A1. Referee’s Assessment of Monograph No.1.

This monograph had been previously refereed on 4th July 2018, as is evident from the contents of the author’s antecedent submission, namely -

“A Supplementary Discourse on the Classification and Calculus of Quaternion Hypercomplex Functions - ADDITIONAL REMARKS pertaining to PART 1/10.”,

having been published under the **‘VIXRA’ Mathematics subheading:-** ***‘Functions and Analysis’***.

A2. Referee’s Assessment of Monograph Nos.2 & 3.

Both of these monographs were refereed on 1st July 2019, as is evident from the contents of the next page.

*   *   *   *   *   *   *   *

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1st July 2019.

TO WHOM IT MAY CONCERN.

In June of this year, Mr. Stephen C. Pearson had very kindly provided me with copies of two mathematical papers, thus entitled -

***"Supplementary Notes in Relation to the Proofs of Two Theorems pertaining to the Integration of Quaternion Hypercomplex Functions."* [*];**

***"Supplementary Notes in Relation to the Proof of One Theorem pertaining to the Integration of Quaternion Hypercomplex Functions."* [*]**

[*] *Re:- A Specific Quaternion Analogue of the Cauchy-Goursat Theorem.* ,

which he had completed on 21st & 29th May 2019 respectively.

Both papers are being jointly presented as an addendum to Mr. Pearson's antecedent submissions, which I had previously refereed, as is evident from the contents of his 'VIXRA' author web page.

Having examined these particular Supplementary Notes, I commend Mr. Pearson for his diligence in further clarifying the proofs of the aforesaid theorems, bearing in mind their significance with regard to the analytic properties of quaternion hypercomplex functions.

Stephen J. Crothers

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