Young's Double-Slit and Wheeler's Delayed-Choice Experiments: What's Really Happening at the Single-Quantum Level?

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A new 'wave-particle non-dualistic interpretation' at the single-quantum level, existing within the quantum formalism, is presented by showing the Schrödinger wave function as an '*instantaneous resonant spatial mode*' where a particle moves. For the first time, the position eigenstate of a particle is identified to be related to the absolute phase of the wave function in such a way that its position eigen values always lie on a classical trajectory, proving that the 'time parameter' is common to both classical and quantum mechanics. It's brought into light that the quantum formalism demands a different kind of boundary conditions to be imposed to the wave function unlike classical formalism and hence naturally yields the Born rule as a limiting case of the relative frequency of detection. This derivation of the Born rule automatically resolves the measurement problem. Also, these boundary conditions immediately expound Bohr's principle of complementarity at a single quantum level. Further, the non-duality naturally contains the required physical mechanism to elucidate why the Copenhagen interpretation is experimentally so successful. The single-quantum phenomenon is then used to unambiguously explain what's really going on in the Young double-slit experiment as anticipated by Feynmann and the same is again used to provide a causal explanation of Wheeler's delayed-choice experiment.

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I. INTRODUCTION

For nearly hundred years, there is no consensus about what kind of physical reality is being revealed by the quantum formalism irrespective of its ability to accurately predict the outcomes of innumerable experiments. It's an extremely successful theoretical description of Nature, especially in the atomic scale, where the classical mechanistic concepts seem to fail completely. Therefore, the exact interpretation of the quantum formalism is very important as it can naturally yield an intuitive visualization of the true picture of reality which will surely contribute to the deeper developments in the fundamental physics. Its one immediate application will be in quantum computers.

Consider Young's double-slit (YDS) experiment [1] with a single-quantum source (Fig. 1). Every quantum is fired at the YDS one-at-a-time. The time interval between any two consecutively fired quanta is chosen to be greater than the time of arrival of one quantum from the source to the screen. This choice guarantees that every quantum is independent of every other one and hence the behavior of an individual quantum becomes transparent. As a large number of quanta are being collected on the screen, an interference pattern, reminiscent of wave nature, gradually emerges out. If slit-1 (slit-2) is blocked, then a clump pattern corresponding to single-slit diffraction of slit-2 (slit-1), supposed to be of particle nature, occurs on the same screen. This implies that every individual quantum is aware of how many slits are opened.

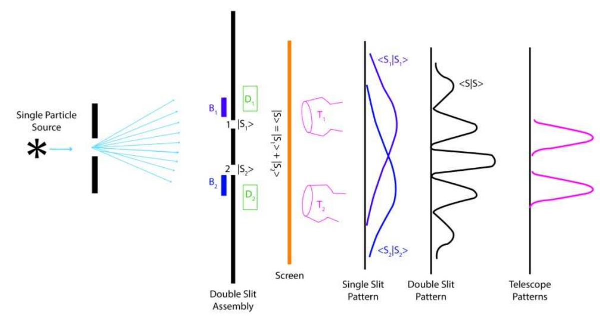

Figure 1. **Single-quantum Young's double-slit experiment**: A source shoots quanta, one at a time, towards a double-slit assembly. 1 and 2 represent two slits through which the state vectors $|S_1>$ and $|S_2>$ get excited and superposed as $|S>=|S_1>+|S_2>$. B_1 and B_2 are two blockers which can block either slit-1 or slit-2 at any time. D_1 and D_2 are two detectors useful to find out through which slit any quantum is passing towards the screen. Immediately behind the screen, a twin-telescopes, T_1 and T_2, is placed such that the quanta passing through slit-1 and slit-2 reach T_1 and T_2, respectively. After collecting a large number of quanta, the resulting distribution patterns at the screen and the telescopes were given at the right hand side. If both slits are opened, then the observed distribution is $<S|S>$. If slit-2 (slit-1) is blocked, then the distribution is $<S_1|S_1>$ ($<S_2|S_2>$).

The observed interference pattern suggests to infer that the quantum 'somehow' simultaneously passes through both the slits like a wave. However, this inference fails during the experimental observation, because, a quantum always appears going through either slit-1 or slit-2 like a particle but never simultaneously through both the

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slits like a wave. Also, two clump patterns appear on the screen instead of interference pattern as a confirmation of the observed particle behavior at the respective slits. This dual behavior of a quantum is summarized as the 'quantum enigma': "When quanta are watched, they appear as going through any one slit like particles; otherwise, they seem to go through both the slits like a wave". However, a particle is a localized entity present at some definite location whereas a wave, a delocalized one and both are incompatible with each other. Hence, a quantum is inferred to simultaneously posses those two mutually exclusive 'classical' behaviors in order to explain the quantum enigma and is known as the 'wave-particle duality' [2–5]. All material particles like photons, electrons, neutrons, atoms, molecules etc., are known to exhibit the duality [6–11].

But, "How does a wave, after passing through the YDS, collapse to a particle on the screen?" because, an individual quantum event is always observed as a well localized chunk. The mainstream Copenhagen interpretation [3–5] treats the wave function as unreal and as merely representing 'the probability of finding a particle at some location' [12]. However, such a treatment leads to a non-intuitive conclusion that a quantum is a *probability wave* until observed as a particle, clearly denying the pre-existing real world independent of observers. Therefore, the observation seems to play a very special role even though such a process is absent in the Hamiltonian describing a quantum. On the other hand, if the wave function is considered to be physically real, then it's a must to provide a definite mechanism for the collapse, which may demand the modification of the Schrödinger equation [13, 14], the very fundamental one for quantum mechanics. The collapse is avoided, for example, in the 'many-worlds' interpretation but at a cost of branching of the entire Universe into innumerable copies [15, 16]. There are various other interpretations of quantum formalism, like, Bohmian mechanics [17, 18], modal interpretation [19], relational interpretations [20], Consistent histories [21], transactional interpretation [22, 23], QBism [24] etc,. Although, each one of them is interesting by itself, none of them gives a derivation for the Born rule using the single-quantum events as it will be shown in the present article.

If the experimentally observed quantum phenomena can aptly be summarized by the Schrödinger equation, then a natural question to ask is, "What could the Nature be like such that the quantum formalism correctly predicts the experimental outcomes?" The answer is precisely the 'physical reality' of Schrödinger's wave function along with its relation to the observed particle. For the first time in Section-II, the mutually exclusive classical natures, wave and particle, are successfully united into a single entity which is named as wave-particle nonduality. It's an unique possible picture existing within the quantum formalism and is analogous to the situation of a moving test particle in curved space-time of the general relativity [25]. In the present article, only the time-independent non-relativistic quantum mechanics is considered, because, its interpretation naturally goes through time-dependent and relativistic cases. At this moment, it may be worth emphasizing that, the findings in the present article never go beyond the quantum formalism, but, only brings out the picture of reality hidden within the same.

All the quantum phenomena are actually found to take place in a complex vector space (CVS) rather than the usual three-dimensional Euclidean space (3DES), i.e., 3DES is insufficient to accommodate them; a classic example is the Stern-Gerlack experiment [2, 26]. Most crucially, the heart of quantum formalism is the canonical commutation relation, $[\hat{x}\ ,\ \hat{p}] = i\hbar$, which necessarily demands a CVS for its action; here, $\hat{x}$ is the position operator, $\hat{p}$ is the momentum operator, $i = \sqrt{-1}$ and $\hbar$ is the reduced Plank's constant. It captures the essence of de Broglie's hypothesis [27], "Every moving particle is associated with a wave nature of wavelength $\lambda = h/p$"; here, h is the Plank's constant and p is the momentum. Then, why the macroscopic objects, which are obviously composites of 'quantum entities', appear to live in 3DES? In Section-III, the actual space where Nature dwells is shown to be CVS but not 3DES. Also, quantum formalism answers how the CVS is 'effectively' perceived as 3DES. (The notion about the space around us as a CVS instead of the usual 3DES may sound weird. Let it be so! But, once this much of initial weirdness is accepted, then there will be no more weirdness in the quantum mechanics). In Section-IV, the absolute phase associated with the wave function is shown to be responsible for the experimental outcome of a definite eigenvalue of an observable.

It's a well-known fact that a number of physical parameters entering into quantum physics, like the frequency (ν) of a light source in the Plank-Einstein formula, $E = h\nu$, for the energy content of a photon [28, 29], the speed of light, etc., are all measured using the classical time. How the classical time is becoming suitable to measure the physical parameters needed to describe the quantum phenomena? As an answer, using the quantum formalism, the equality of classical and quantum mechanical times are explicitly shown for the first time in Section-V.

The Born rule, interpreting the square of the norm of a wave function as probability density for finding a particle, is experimentally successful quantum algorithm. Yet at the fundamental level, it's not really pleasing unless derived from the quantum formalism. In Section-VI, it's shown that a quantum state vector induces its own dual in a detector (or at a boundary) and interacts according to the inner-product. Collection of a large number of random inner-product events statistically yields the relative frequency of detection and hence, the Born rule. This statistical derivation shows that there is no direct and further irreducible equation for probability density like the Schrödinger equation for wave function, which implies the absence of probability for a single-quantum.

The same is explicitly proved using a sequential measurements [26] in Section-VII.

Prof. Feynmann famously said [2], "We choose to examine a phenomenon which is impossible, *absolutely* impossible, to explain in any classical way, and which has in it the heart of quantum mechanics. In reality, it contains the *only* mystery"; here, the 'phenomenon' stands for the wave-particle duality of a single quantum in the YDS experiment. Section-VIII contains an explanation of this mystery in a classical way.

According to Bohr's principle of complementarity [5, 30–32], depending on the experimental configuration, observation of wave nature excludes the simultaneous observation of particle nature and vice versa. For example, in the YDS experiment (Fig. 1), the presence of a screen or a twin telescopes corresponds to observe the classical wave or particle behavior, respectively. Alternatively, the same can also be viewed as a decision taken by the quantum, after 'somehow' sensing the configuration of the measuring device, to behave like a wave or a particle [4, 5]. This later view-point was examined in Wheeler's delayed-choice experiment (WDCE) [33]: the screen is quickly removed, exposing the twin telescopes, after a quantum has already passed the YDS. The expected interference pattern on the screen is lost and two clump patterns, one at each telescope, are formed. According to the duality, the quantum retroactively rearranges its past history of simultaneously passing through both the slits like a wave to that of passing through any one slit like a particle, resulting in the clump patterns. For the first time, an unambiguous causal explanation of WDCE is provided using the wave-particle non-duality in the Section-IX. Section-X contains the conclusions and the discussions.

II. SCHRÖDINGER'S WAVE FUNCTION AS AN INSTANTANEOUS RESONANT SPATIAL MODE

In this section, using the quantum formalism, a mathematical reasoning is developed to identify the physical nature of the Schrödinger wave function as an 'instantaneous resonant spatial mode' (IRSM). Consider the de Broglie case of a particle executing force-free motion in one-dimension (1D). Its classical Hamiltonian, H, is given by

$$H = \frac{p^2}{2m} = E, \tag{1}$$

where, p is the momentum, m is the mass and E is the total energy of the particle. The Hamiltonian equations of motion,

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \text{ and } \dot{p} = -\frac{\partial H}{\partial x} = 0, \tag{2}$$

yield the following solutions,

$$x(t) = \frac{p(0)}{m} t + x(0) \text{ and } p(t) = p(0), \tag{3}$$

where, $\dot{x}$ and $\dot{p}$ are the time (t) derivatives of x and p and $x(0)$ and $p(0)$ are constants of integration corresponding to the initial position and momentum at $t = 0$.

Replacing the classical variables, x and p, by operators, $\hat{x}$ and $\hat{p}$, respectively, Eq. (3) yields Heisenberg's equations of motion [26]:

$$\hat{x}(t) = \frac{t}{m}\hat{p}(0) + \hat{x}(0) \text{ and } \hat{p}(t) = \hat{p}(0). \tag{4}$$

Therefore, both the $x(0)$ and $p(0)$ must be treated as operators so that the validity of original position-momentum commutation relation remains unaffected,

$$\text{i.e.,} \quad [\hat{x}(t), \hat{p}(t)] = [\hat{x}(0), \hat{p}(0)] = i\hbar. \tag{5}$$

Hence, in the position representation, the Hamiltonian operator can be written in two equivalent ways as,

$$\text{either} \qquad \hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x(t)^2} = E, \tag{6}$$

$$\text{or} \qquad \hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x(0)^2} = E, \tag{7}$$

so that the free particle's eigenvalue equation is either

$$\hat{H}\psi(x(t)) = E\psi(x(t)) \text{ or } \hat{H}\psi(x(0)) = E\psi(x(0)). \tag{8}$$

Both $\psi(x(t))$ and $\psi(x(0))$ describe the same physical situation for the same energy eigenvalue E. Also, from Eq. (4), one has

$$[\hat{x}(0), \hat{x}(t)] = i\hbar\frac{t}{m} \; ; \; [\hat{p}(0), \hat{p}(t)] = 0, \tag{9}$$

where,

$$\begin{aligned} \hat{x}(0)|x(0)> &= x(0)|x(0)>, \\ \hat{x}(t)|x(t)> &= x(t)|x(t)> . \end{aligned} \tag{10}$$

The sets of position eigenvalues, $\{x(0)\}$ and $\{x(t)\}$, of $\hat{x}(0)$ and $\hat{x}(t)$ span the same 1D space. However, any given position eigenstate $|x(0)>$ is a linear superposition of all eigenstates of $\hat{x}(t)$ due to $[\hat{x}(0), \hat{x}(t)] \neq 0$, for any non-zero value of t and vice versa. In other words, the CVSs of $\hat{x}(0)$ and $\hat{x}(t)$ are twisted with respect to each other though they give rise to the respective sets of position eigenvalues spanning the same 1D Euclidean space.

The wave function, ψ, can be considered as a function on the set of all possible initial values, $\{x(0)\}$, or as a function on the set of all possible values at a later time, $\{x(t)\}$. But, for any dynamical process happening in any given space, the set of all possible initial values (or values at any other instant of time) is the space itself. Thus, one arrives at a conclusion that ψ is like a field on the 1D space. If the reference to the 1D space, i.e., the position representation, is ignored, then the state vector ($|\psi>$) corresponding to ψ ($=<x|\psi>$) itself must be viewed as a spatial mode in the CVS. Once a particle of momentum p appears (like the emission of an electron from a metal surface, appearance of a photon at a light source etc.), say at some position eigen value $x_p(0)$, then $\psi(x(0))$ appears instantaneously everywhere in the entire 1D space. At later time t, the position eigen value of the particle changes from $x_p(0)$ to $x_p(t)$ and the corresponding instantaneous eigen mode is $\psi(x(t))$. Therefore, ψ is indeed an IRSM and the Plank's constant, h, can be viewed as a kind of coupling parameter tying the IRSM and its particle. Here, the coupling means that the particle is actually free to move but always confined to its IRSM. This inseparable nature of IRSM and its resonant particle, which is like the eigenstate and its eigen value, is named as wave-particle non-duality.

III. 'EMPTY SPACE' - ACCORDING TO QUANTUM MECHANICS

Though the physical state of a quantum system is described by a complex vector in some 'abstract' Hilbert space, it's relevant to explain the experimentally observed data via the Born rule. The experimental setup seems to exist in the 3DES but still it can capture the information from the respective CVS. This clearly shows that there is a fundamental mismatch between the actual space where the quantum particles really live and our intuition about the particles as if present in 3DES.

In the Newtonian paradigm, a point is located in 3DES by specifying its coordinates by a rigid measuring rod and hence attaching the property of rigidity to the space itself. Such a rigid but empty space provides an absolutely passive and unchanging 'stage' for all the physical phenomena. But in Einstein's special theory of relativity, objects have an intimate connection with the space-time in such a way that their relative speeds never exceed the Cosmic speed limit in any inertial frame of reference. In general theory of relativity, space-time is directly related to the energy-momentum distribution and is dynamical. It bends, stretches, twists and even ripples and dictates the particle's motion to lie along a geodesic. It's very important to identify the actual space where a physical phenomenon is happening. Though the absolute space can be felt intuitively as nothingness, the important aspect to note is that its true nature is unavailable independent of the material phenomena happening in the same.

Quantum mechanics is reveling a profound and remarkable property of space (or space-time) itself which is quite different from that of general relativity. Keeping its formalism in mind, the following postulates are proposed.

Postulate-1: *'Empty space' where Nature dwells is an infinite dimensional complex vector space or equivalently the Hilbert space,* $\mathbf{H}$*, but not the 3DES.*

Postulate-2: *A precise set of elementary particles in* $\mathbf{H}$*, with well-defined properties and interactions among them, results in the macroscopic manifestation of matter with respect to which the eigenvalues of position operator 'effectively' form the 3DES.*

The position base representations of the tensor product of two or more vectors belonging to $\mathbf{H}$ are super-imposed on top of each other and can independently coexist in the same region of 3DES spanned by the eigenvalues of the position operator. Two or more classical waves, like ripples on the surface of water, never behave in this way.

Let '$\hat{\mathbf{r}}$' be the position operator, with eigenstates '$|\mathbf{r}>$' such that the set of all eigenvalues, $\{\mathbf{r}\}$, span the 3DES. When a free particle of definite momentum eigenvalue 'p' is created, then a resonant spatial mode '$<\mathbf{r}|\psi>$' instantaneously appears such that the particle's motion is completely confined to the same; here, $|\psi>\in\mathbf{H}$. Since $|\mathbf{r}>$ and $<\mathbf{r}|\psi>$ are in one-to-one correspondence, without loss of generality, the state vector itself can be called as an IRSM which has the representation:

$$|\psi>=\int d\mathbf{r}|\mathbf{r}><\mathbf{r}|\psi>. \tag{11}$$

By attaching a complex vector $|\mathbf{r}><\mathbf{r}|\psi>$ at every eigenvalue $\mathbf{r}$, the IRSM becomes intuitively visible. The particle itself will be present at some eigenvalue $\mathbf{r_p}$, carrying an eigen vector, $|\mathbf{r_p}><\mathbf{r_p}|\psi>$; here, the subscript $\mathbf{p}$ stands for particle. This particular non-dualistic classical picture is unavailable to the direct experience in every day world due to its complex-valuedness. This shows that the the quantum mechanical Hilbert space or the CVS is not some 'abstract' space but, the actual space where Nature dwells. The same picture also holds for non-free particles.

IV. PRINCIPLE OF MINIMUM PHASE TO REALIZE A DEFINITE EIGENSTATE

"*Questions about what decides whether the photon is to go through or not and how it changes its direction of polarization when it does go through can not be investigated by experiment and should be regarded as outside the domain of science*" - a profound statement by Prof. Dirac [34], which contains the key for the randomness and hence, the probability in quantum mechanics. The last part of the statement, " *...and should be regarded as outside the domain of science*", is resolved using the following example:

Consider a classical scenario of tossing a coin in 3DES. Using the Newtonian mechanics, it's possible, in principle, to predict exactly whether head or tail will occur on a horizontal flat ground. If there is an ignorance about some parameters involved in the dynamics of the coin, then probability can be invoked. Let $\hat{\mathbf{n}}$ be a normal vector to the head surface passing through the coin's center-of-mass and θ be an angle between $\hat{\mathbf{n}}$ and any parallel vector, $\hat{\mathbf{g}}$, to the gravitational force. Just before the landing of the coin, consider its position at a height $h\leq r$ above the ground surface; here, r is the radius of the coin. If

$-\pi/2 < \theta < \pi/2$, then head will be the out come. Otherwise, tail occurs for $\pi/2 < \theta < 3\pi/2$. Depending on the value of θ, the coin will jump into either head or tail state. Upon the outcome, $\hat{\mathbf{n}}$ will be pointing either parallel or anti-parallel to $\hat{\mathbf{g}}$ (notice that, if the space is a complex-Euclidean instead of 3DES, then θ corresponds to the phase angle between the respective complex vectors). But, this result will not hold in the limit $r \to 0$, because, the range of θ splitting into two halves is due to the non-zero value of r. Nevertheless, it can still be retained including in that limit by considering the coin in CVS instead of 3DES. If an element of CVS gets resolved into two components, then the coin has to enter into any one, because, it itself can't split. It naturally gets into energetically favored component making a minimum phase angle with $\hat{\mathbf{n}}$. If $\hat{\mathbf{n}}$ is replaced by electron's spin magnetic axis and $\hat{\mathbf{g}}$ by magnetic force direction, then the resulting situation is exactly identical to that of an electron in the Stern-Gerlac (SG) experiment [2, 26, 35]. Let's consider this case in detail:

Let $I_{\rm SG} = |\uparrow><\uparrow| + |\downarrow><\downarrow|$ be an unit operator in the CVS of the SG apparatus. The IRSM, say $|s>$, representing the spin state (spatial dependence is suppressed for simplicity) through which an electron is flying, encounters the SG apparatus:

$$|s>=|\uparrow><\uparrow|s>+|\downarrow><\downarrow|s> .$$

The electron's spin jumps into either $|\uparrow>$ or $|\downarrow>$ depending upon which complex number, either $<\uparrow|s>$ or $<\downarrow|s>$, has a minimum phase. Notice that, though the electron jumps into, say $|\uparrow>$, the empty orthogonal mode $|\downarrow>$ survives until the detection of electron. The absolute phase of $|s>$ determines the eigenstate where the electron will be detected.

If the IRSM, say $|\psi>$, encounters a CVS of an operator having continuous orthogonal eigenstates, then the particle, without any jump, naturally enters into an eigenstate having the same phase as that of $|\psi>$. As an example, consider the Eq. (11): a particle will be present in a state $|\mathbf{r_p}><\mathbf{r_p}|\psi>$ whose phase is exactly the same as that of $|\psi>$.

V. EQUALITY OF CLASSICAL AND QUANTUM MECHANICAL TIMES

In the Section-II, it's shown that a particle moves in its IRSM but nothing was said about its motion along some trajectory, if exists. In order to uncover the particle's motion, the propagators [26] are derived in a new way using the Heisenberg equations of motion, because, (i) the IRSM is a solution of the time-independent Schrödinger wave equation and (ii) the time-dependent Schrödinger wave equation is not explicitly considered in the present article. Application of the principle of minimum phase to the propagator results in the classical path of least action on which the position eigen values of particle-state lie. From this, the equality of classical and quantum mechanical times arises as shown below:

Substitution from Eq. (4) into the second part of Eq. (10) results,

$$(\hat{x}(0)+\frac{t}{m}\hat{p}(0))|x(t)>=x(t)|x(t)>, \qquad (12)$$

which can be expressed as a first order partial differential equation,

$$\left(-i\hbar\frac{t}{m}\frac{\partial}{\partial x(0)}+x(0)-x(t)\right)<x(0)|x(t)>=0, \quad (13)$$

by using an unit operator, $\int dx(0)|x(0)><x(0)|$, in the position basis at time $t=0$, having a solution,

$$<x(0)|x(t)>=e^{\left\{-\frac{im}{2\hbar t}[x^2(0)-2x(0)x(t)+\alpha]\right\}}, \qquad (14)$$

where, $-\frac{im}{2\hbar t}\alpha$ is a constant of integration. Similarly, making use of the identity operator, $\int dx(t)|x(t)><x(t)|$, at time t in the first part of Eq. (10) results in the equation,

$$\left(i\hbar\frac{t}{m}\frac{\partial}{\partial x(t)}+x(t)-x(0)\right)<x(t)|x(0)>=0, \qquad (15)$$

having a solution,

$$<x(t)|x(0)>=e^{\left\{\frac{im}{2\hbar t}[x^2(t)-2x(0)x(t)+\beta]\right\}}, \qquad (16)$$

where, $\frac{im}{2\hbar t}\beta$ is another constant of integration. Using $<x(t)|x(0)>=<x(0)|x(t)>^\star$, a property of complex numbers, in Eqs. (14) and (16) yields,

$$x^2(t)-2x(0)x(t)+\beta=x^2(0)-2x(0)x(t)+\alpha^\star, \quad (17)$$

whose solution is,

$$\beta=\sigma+x^2(0)\ ;\alpha^\star=\sigma+x^2(t), \qquad (18)$$

where, $\star$ stands for complex conjugation and σ is a constant. Hence, Eq. (16) can be rewritten as,

$$<x(t)|x(0)>=e^{\left\{\sigma'+\frac{im}{2\hbar t}(x(t)-x(0))^2\right\}}, \qquad (19)$$

with $\sigma'=\frac{im}{2\hbar t}\sigma$. From the requirement,

$$\lim_{t\to 0}<x(t)|x(0)>=\delta(x(t)-x(0)), \qquad (20)$$

an inference $e^{\sigma'}=\sqrt{\frac{m}{2\pi i\hbar t}}$ can be made, but it works only for free particle. The following is a general procedure:

Considering the unit operators in the position basis at time t and at $t=0$ as,

$$\begin{aligned}
I &= \int dx(t)|x(t)><x(t)| \\
&= \iiint \{dx'(0)dx''(0)dx(t)|x'(0)><x'(0)|x(t)> \\
&\quad \times <x(t)|x''(0)><x''(0)|\} \\
&= \iiint \{dx'(0)dx''(0)dx(t)|x'(0)> \\
&\quad \times F(x(t),x'(0),x''(0))<x''(0)|\}, \qquad (21)
\end{aligned}$$

where,

$$F \equiv e^{\left\{\sigma'+\sigma'^{\star}+\frac{im}{\hbar t}\left[x'(0)-x''(0)\right]x(t)+\frac{im}{2\hbar t}\left(x'(0)-x''(0)\right)^2\right\}},$$

such that,

$$\int dx(t)F(x(t),x'(0),x''(0)) = e^{2\sigma'_R}\frac{2\pi\hbar t}{m}\delta(x'(0)-x''(0)),$$

yielding, $e^{\sigma'_R} = \sqrt{\frac{m}{2\pi\hbar t}}$; here, $\sigma'_R = (\sigma'+\sigma'^{\star})/2 = \mathrm{Re}\{\sigma'\}$ is the real part of σ'. Now, Eq. (19) becomes,

$$< x(t)|x(0) >= \sqrt{\frac{m}{2\pi\hbar t}}e^{i\left\{\sigma'_I+\frac{m}{2\hbar t}(x(t)-x(0))^2\right\}}, \qquad (22)$$

where, $\sigma'_I = (\sigma' - \sigma'^{\star})/(2i) = \mathrm{Im}\{\sigma'\}$ is the imaginary part of σ', which can be evaluated from the requirement given in Eq. (20):

$$\begin{aligned}\delta(x(t)-x(0)) &= \lim_{t\to 0} < x(t)|x(0) > \\ &= \lim_{t\to 0} e^{i\sigma'_I}\sqrt{\frac{m}{2\pi\hbar t}}e^{\left\{\frac{im}{2\hbar t}(x(t)-x(0))^2\right\}} \\ &= e^{i\sigma'_I}i^{\frac{1}{2}}\delta(x(t)-x(0)), \qquad (23)\end{aligned}$$

implying $e^{i\sigma'_I}i^{\frac{1}{2}} = 1$. Hence,

$$< x(t)|x(0) >= \sqrt{\frac{m}{2\pi i\hbar t}}e^{\left\{\frac{im}{2\hbar t}(x(t)-x(0))^2\right\}}. \qquad (24)$$

If the time parameter varies from t_1 to t_2 instead of 0 to t, then the above equation becomes,

$$\begin{aligned}< x(t_2)|x(t_1) >=&\sqrt{\frac{m}{2\pi i\hbar(t_2-t_1)}} \\ &\times e^{\left\{\frac{im}{2\hbar(t_2-t_1)}(x(t_2)-x(t_1))^2\right\}}. \qquad (25)\end{aligned}$$

Similar analysis can be carried out for a simple harmonic oscillator:

$$\begin{aligned}< x(t_2)|x(t_1) >=& \sqrt{\frac{m}{2\pi i\hbar \sin(t_2-t_1)}} \\ &\times e^{\frac{im\omega}{2\hbar\sin(\omega(t_2-t_1))}G(x(t_2),x(t_1);t_2,t_1)}, \qquad (26)\end{aligned}$$

where,

$$G \equiv \left(x^2(t_2)+x^2(t_1)\right)\cos(\omega(t_2-t_1)) - 2x(t_2)x(t_1).$$

If $t_2 - t_1 = \Delta t \to 0$, then the Eq. (25) or (26) can be written as

$$\begin{aligned}&\lim_{t_2\to t_1} < x(t_2)|x(t_1) > \\ &= \sqrt{\frac{m}{2\pi i\hbar\Delta t}}\exp\left\{\frac{im\Delta t}{2\hbar}\left(\frac{x(t_2)-x(t_1)}{\Delta t}\right)^2\right. \\ &\quad\left.-\frac{i\Delta t}{2\hbar}\left[V(x(t_2))+V(x(t_1))\right]\right\} \\ &= \sqrt{\frac{m}{2\pi i\hbar\Delta t}}\exp\left\{\frac{i}{\hbar}\int_{t_1}^{t_2}dtL(\dot{x}(t),x(t))\right\}. \qquad (27)\end{aligned}$$

Now, consider the energy eigenstate,

$$\begin{aligned}|\psi >=& \int dx(0)|x(0) >< x(0)|\psi > \\ =& \iint dx(0)dx(t)|x(0) >< x(0)|x(t) >< x(t)\psi > (28)\end{aligned}$$

The particle will be present at some particular eigenstate $|x_p(0 >$ whose phase $\mathrm{ph}\{< x_p(0)|\psi >\}$ at time $t = 0$ is same as $\mathrm{ph}\{|\psi >\}$. This criterion yields the following relation from Eq. (28):

$$\begin{aligned}&\mathrm{ph}\{|\psi >\} = \mathrm{ph}\{< x_p(0)|\psi >\} \\ &= \mathrm{ph}\{< x_p(0)|x_p(t) >< x_p(t)|\psi >\} \\ &= \mathrm{ph}\{< x_p(0)|x_p(t) >\} + \mathrm{ph}\{< x_p(t)|\psi >\}, \quad (29)\end{aligned}$$

where, $\mathrm{ph}\{< x_p(t)|\psi >\}$ is the phase of the particle state at t. These phases at $t = 0$ and t will be different,

$$\mathrm{ph}\{< x_p(0)|\psi >\} \neq \mathrm{ph}\{< x_p(t)|\psi >\}, \qquad (30)$$

but, any infinitesimal variation of phase at $t = 0$ results in the corresponding variation of phase at t,

$$\delta\{\mathrm{ph} < x_p(0)|\psi >\} = \delta\{\mathrm{ph} < x_p(t)|\psi >\}. \qquad (31)$$

Applying the above condition to Eq. (29) results,

$$\delta\{\mathrm{ph} < x_p(0)|x_p(t) >\} = 0, \qquad (32)$$

which, in turn, implies

$$\delta\{\mathrm{ph} < x_p(t_1)|x_p(t_2) >\} = 0. \qquad (33)$$

Application of Eq. (33) to Eq. (27) yields the classical least action principle,

$$\delta\int_{t_1}^{t_2}dtL(\dot{x}_p(t),x_p(t)) = 0, \qquad (34)$$

which explicitly shows that the position eigenvalue of a particle state always, as a function of time, lie on a classical path. In fact, this result is independent of whether the physical system is microscopic or macroscopic and proves that the time parameter is common to both the quantum and the classical mechanics. If a particle can be inferred to be at location 'A' and later at 'B', then A and B are connected by the particle-path which obeys the principle of least action. Note that, in YDS experiment, no particle will be found in the regions of dark fringes because the IRSM vanishes there, which in turn implies that no classical paths, formed by the position eigen values of the particle states, are available from any slit to any dark fringe. Even though the result in Eq. (34) is proved here for free particle and harmonic oscillator, its universal validity can be verified by noting the additive property of phase in Eq. (29) and time-interval independence of Eq. (33). One immediate example of this result is the traces of particle trajectories photographed in the detectors like Wilson's chamber.

VI. INNER-PRODUCT INTERACTION AND THE DERIVATION OF BORN'S RULE

It's well-known that the square of amplitude of a classical wave is proportional to its intensity. But, such an intensity can't be claimed for an IRSM, though it obeys the Schrödinger wave equation, because, the non-dualistic picture of a particle flying in its own IRSM is not analogous to any classical wave.

Wherever the particle will be ending up, there a dual mode gets induced which interacts with the IRSM according to the inner-product. This interaction can be found within the quantum formalism. Let an IRSM, $|\psi>$, encounters a detector screen and gets scattered into some other state, say $|\psi'>$. This process can be described by associating an operator, $\hat{O} = |\psi'><\psi|$:

$$\hat{O}|\psi>=<\psi|\psi>|\psi'> . \tag{35}$$

If the scattered state is discarded or it's a null-state, then the particle must have interacted or got absorbed, respectively, at some location in the region of $<\psi|\psi>$.

Precisely due to the inner-product interaction and the equality of classical and quantum mechanical times, the CVS 'effectively' appears to be 3DES.

If the detector states do not span $|\psi>$ completely and is associated with a projection operator $\hat{P}$, then the IRSM seen by the detector is $|\psi_D>=\hat{P}|\psi>$ and the excited dual vector is $<\psi_D|$. Therefore, the interaction region for the particle detection is $<\psi_D|\psi_D>$.

Consider the boundary conditions to an IRSM of a free particle confined in a 1D box of length 'L' placed along the X-axis, i.e., $0 \leq x \leq L$. In the spirit of Eq. (35), the dual modes excited at the boundaries interact with the IRSM, $|\psi>$:

$$<\psi|\psi>|_{x=0}=<\psi|\psi>|_{x=L}=0. \tag{36}$$

Note that the above type of boundary conditions don't exist in the classical mechanics, though they eventually give rise to the classical boundary conditions i.e., $<x|\psi>|_{x=0}=<x|\psi>|_{x=L}=0$, which in turn imply,

$$\begin{aligned} <\psi|\psi> &= \int_0^L dx <\psi|x><x|\psi> \\ &= \int_0^L dx \quad |<x|\psi>|^2 \neq \infty. \end{aligned} \tag{37}$$

The above integral must converge in order to have any physical interpretation and is the well-known Born's rule [2–4, 12]. Here, the aim is just to show that it naturally emerges from the non-duality.

For a free particle, the initial boundary condition is a point in the CVS where the momentum originated and remains unaltered as long as the particle sustains with the same momentum. This condition can be justified based on our common sense experience by considering a distant star located at some millions of light years away. While looking, it's not actually appearing as it is right now but how it was some millions of years ago. Right now, anything might have happened like it might have exploded, swallowed by a black-hole, etc. All these details don't appear except the star before millions of years. This simply implies that the origin of a photon remains unaffected. The final boundary condition depends on where the particle will end up and need not be a fixed boundary condition. It can be changed randomly before the arrival of the particle.

If the particle's momentum undergoes a sudden change, then the corresponding IRSM completely disappears and a new IRSM corresponding to new momentum instantly appears. The origin of new IRSM lies at the spatial point where the particle gained the new momentum. *This is simply a reflection of the nature of eigenvalue equation along with the boundary conditions.*

If the IRSM, $|\psi>$, encounters a CVS of a detector A spanned by orthogonal eigenstates, $|a_i>; i=1,2,3,\cdots$, of an observable, $\hat{A}$, then the particle enters into one of the eigenstate, say $|a_p>$, which makes a minimum phase with $|\psi>$. Other eigenstates remain empty but present ontologically. During the detection, the particle will be naturally found in $|a_p>$ with an eigenvalue a_p, because, all other orthogonal empty states contribute zero outcome. The IRSM in A's CVS,

$$|\psi>=\sum_i |a_i><a_i|\psi>, \tag{38}$$

interacts with its excited dual-mode, $<\psi|$ as,

$$<\psi|\psi>=\sum_i <\psi|a_i><a_i|\psi> \xrightarrow[\text{at A}]{\text{Detection}} |<a_p|\psi>|^2,$$

yielding the eigenvalue a_p. The particle itself contributes a point to the function $|<a_p|\psi>|^2$. Note that, this physical mechanism is indeed in one-to-one correspondence with the 'wave function collapse' advocated in the Copenhagen interpretation [3–5]. Further, repeating the detection procedure on several identical particle states having different initial phases and normalizing the outcomes by the total number of particles yield a relative frequency of detection (RFD) for various eigenvalues, a_i. In the limit of infinite number of particles, the RFD coincides with $|<a_i|\psi>|^2$. In other words,

$$<\psi|\psi>=\sum_i |<a_i|\psi>|^2=1, \tag{39}$$

which is the Born rule. Therefore,

$$\text{RFD} \xrightarrow[\text{detection events}]{\text{Infinite number of}} \text{The Born rule}$$

and hence, the Copenhagen interpretation is completely contained within the present non-dualistic interpretation of quantum mechanics. Though, a single quantum phenomenon can be deterministically described, the unavailability of the information about the absolute phase of the IRSM due to the inner-product interaction forces experiments to observe only the RFD. Here, it's worth recollecting Born's Probabilistic Interpretation [3]:

"*The wave function determines only the probability that a particle - which brings with itself energy and momentum - takes a path; but no energy and no momentum pertains to the wave*". Notice that, except for the notion of probability, this statement is in exact agreement with the spirit of wave-particle non-duality which recognizes the Schrödinger wave function as an IRSM.

Suppose that, instead of $\hat{A}$, the same IRSM, $|\psi>$, encounters a different observable, $\hat{B}$, whose CVS is spanned by the eigenstates, say $|b_i>$:

$$|\psi>=\sum_i |b_i><b_i|\psi>, \tag{40}$$

and the particle will be present in some eigenstate, $|b_p>$, which makes a minimum phase with $|\psi>$. The inner-product interaction at the detector is,

$$<\psi|\psi>=\sum_i <\psi|b_i><b_i|\psi> \xrightarrow[\text{at B}]{\text{Detection}} |<b_p|\psi>|^2,$$

yielding the eigenvalue b_p and the particle itself contributes a point to $|<b_p|\psi>|^2$. Therefore, it's the measuring device, either A or B where the inner-product interaction happens, decides which property, either a_p or b_p, of the quantum to be observed. This is actually Bohr's principle of complementarity [30–32], but, at a single-quantum level. However, notice that, *the non-dualistic picture of a particle flying in its own IRSM is further irreducible and is independent of any measuring device.* When $\hat{A}$ and $\hat{B}$ are not commuting, the entire CVS of A will have a non-zero twist with respect to the entire CVS of B in such a way that any eigenstate from the former will have unavoidable non-zero projections along more than one eigenstate of the later and vice versa. Therefore, *through the experiment, only the preexisting properties of the quantum can be observed. However, the values of the observed properties may get altered due to the act of observation.*

Consider the IRSM given in Eq. (11): the particle will be present in a state, $|\mathbf{r_p}><\mathbf{r_p}|\psi>$, whose phase is exactly same as $|\psi>$. Therefore, its interaction with the excited dual, $<\psi|$, in a detector is,

$$<\psi|\psi>=\int d\mathbf{r}<\psi|\mathbf{r}><\mathbf{r}|\psi> \xrightarrow{\text{Detector}} |<\mathbf{r_p}|\psi>|^2,$$

because, except for the particle state $|\mathbf{r_p}><\mathbf{r_p}|\psi>$, the remaining orthogonal ones, $|\mathbf{r}><\mathbf{r}|\psi>$, are empty.

Finally, notice that the *measurement problem* doesn't exist in the quantum formalism due to the inner product interaction and the principle of minimum phase. There is no distinction between microscopic and macroscopic physical systems, because, non-duality recognizes all of them as represented by suitable CVSs.

VII. SEQUENTIAL SELECTIVE MEASUREMENTS

Consider three sequential detectors A, B and C representing the observables $\hat{A}$, $\hat{B}$ and $\hat{C}$, whose eigenstates and eigenvalues are $|a_i>$, $|b_j>$and $|c_k>$ and a_i, b_j and c_k, respectively [26]; here, $i, j, k = 1, 2, 3, \cdots$. Let A, B and C select some particular states $|a'_i>$, $b'_j>$ and $|c'_j>$ and reject the rest. Let $|\psi>$ be an IRSM in which a quantum is flying. The IRSM is resolved in A's CVS as,

$$|\psi>=\sum_i <a_i|\psi>|a_i>, \tag{41}$$

and only $|a'_i>$ component comes out. If the initial phase of $|\psi>$ makes a minimum phase with $|a'_i>$, then the particle passes on to B. Hence, the mode $|\tilde{a}_i>$ ($\equiv <a'_i|\psi>$ $|a'_i>$) encounters B's CVS and gets resolved as,

$$|\tilde{a}_i>=\sum_j <b_j|\tilde{a}_i>|b_j>. \tag{42}$$

B allows only $<b'_j|\tilde{a}_i>|b'_j>$, which encounters C's CVS:

$$<b'_j|\tilde{a}_i>|b'_j>=<b'_j|\tilde{a}_i>\sum_k <c_k|b'_j>|c_k>. \tag{43}$$

Now, C projects out only $<b'_j|\tilde{a}_i><c'_k|b'_j>|c'_k>$ which interacts with its excited dual $<\tilde{a}_i|b'_j><b'_j|c'_k><c'_k|$ as $|<b'_j|\tilde{a}_i>|^2|<c'_k|b'_j>|^2$. Out of a large number of identical particle-states with different initial phases, the RFD at C is,

$$<\tilde{a}_i|\tilde{a}_i> \xrightarrow[\text{at C}]{\text{RFD}} |<b'_j|\tilde{a}_i>|^2|<c'_k|b'_j>|^2. \tag{44}$$

Suppose, B allows all $|b_j>$ states to pass through. Then C will encounter a superposition:

$$\sum_j <b_j|\tilde{a}_i>|b_j>=|\tilde{a}_i>. \tag{45}$$

In B's CVS, though only $|b'_j>$ contains the particle, all other empty modes do exist ontologically and if unblocked, then they contribute at C:

$$\begin{aligned}|\tilde{a}_i> &= \sum_k \left(\sum_j <b_j|\tilde{a}_i><c_k|b_j>\right)|c_k> \\ &= \sum_k <c_k|\tilde{a}_i>|c_k>. \end{aligned} \tag{46}$$

Now, as usual, C projects out only $<c'_k|\tilde{a}_i>|c'_k>$, yielding a RFD,

$$<\tilde{a}_i|\tilde{a}_i> \xrightarrow[\text{at C}]{\text{RFD}} |<\tilde{a}_i|c'_k>|^2, \tag{47}$$

which is entirely different from Eq. (44). Therefore, the ontological presence of an empty mode can have a physically observable effect.

In Eq. (44), if $|<b'_j|\tilde{a}_i>|^2$ is regarded as a probability for the particle to go through the $|b'_j>$ route in B and $|<c'_k|b'_j>|^2$ as a probability of finding the same at C, then they obey the usual rule of probability multiplication. If the probability is really in play here, then its total, say

$P(c'_k)$, for the particle to arrive at C through all possible routes in B,

$$P(c'_k) = \sum_j | < b'_j|\tilde{a}_i > |^2 | < c'_k|b'_j > |^2, \qquad (48)$$

must be the same as without B. But, in the absence of B, Eq. (47) gives the total probability of finding the particle at C, which is entirely different from Eq. (48). This is a clear proof for the absence of probability in quantum mechanics. Only the RFD arises at a detector when repeated measurements are made on identical states (modulo overall absolute phase). If the existence of a particle is inferred by probability, that too, in the absence of observation, then it will not yield the correct picture of a single-quantum.

As a corollary, consider the case of commuting $\hat{A}$ and $\hat{B}$. Any given eigenstate of $\hat{A}$ can have a projection only along any one of the eigenstates of $\hat{B}$ and the remaining projections along all others become zero, i.e., $< a'_i|b'_j >= \delta_{ij}$; here, δ_{ij} is the Kronecker delta. Therefore, it doesn't matter whether B is present or absent. Only in this case, Eq. (48) becomes exactly the same as Eq. (47):

$$\sum_j | < b'_j|\tilde{a}_i > |^2 | < c'_k|b'_j > |^2 = | < \tilde{a}_i|c'_k > |^2. \qquad (49)$$

VIII. YOUNG'S DOUBLE-SLIT EXPERIMENT: WHAT'S REALLY HAPPENING?

In this section, the YDS experiment (Fig. 1) performed with a single-particle source is considered in order to elucidate the actual behavior of an individual quantum.

Each particle is shot onto the screen through the YDS, only after the registration of the previous one. Classically, the particles were expected to leave a pattern of two strips on the screen, as some of them pass through slit-1 and the others through slit-2, because, they were thought to be moving in the 3DES. But, according to non-duality, the particles actually move in their own IRSMs obeying Schrödinger's equation and hence an interference pattern occurs.

Let $|S_0>$ be the IRSM of a particle created at the source. The projector, $\hat{P}_{\rm yds}$, associated with the YDS assembly is,

$$\hat{P}_{\rm yds} = \sum_{i=1}^{2} \int d\mathbf{r}_i^{(i)} |\mathbf{r}_i^{(i)} >< \mathbf{r}_i^{(i)}|, \qquad (50)$$

where, $\{|\mathbf{r}_1^{(1)} >\}$ and $\{|\mathbf{r}_2^{(2)} >\}$ are position bases for slit-1 and slit-2, respectively. The IRSM, $|S>$, excited towards the screen from the YDS is given by the projection:

$$\begin{aligned} |S> &= \hat{P}_{\rm ds}|S_0> \\ &= \sum_{i=1}^{2} \int d\mathbf{r}_i^{(i)} |\mathbf{r}_i^{(i)} >< \mathbf{r}_i^{(i)}|S_0> \\ &\equiv |S_1> + |S_2> . \end{aligned} \qquad (51)$$

As explained in the Section-VI, the induced dual-mode, $< S|$, in the screen interacts with the IRSM as,

$$< S|S >= \sum_{i,j=1}^{2} < S_i|S_j > . \qquad (52)$$

Note that, the above inner-product interaction happens instantaneously the moment a particle is created, but its effect remains unfelt until the hit of the particle on the screen. As elucidated in the Section-IV, depending on the initial phase of the IRSM, the particle passes either through slit-1 or slit-2. If particle's momentum changes either due absorption or scattering at the screen, then the entire IRSM disappears such that the particle contributes a point to $< S|S >$.

The next particle appears at the source along with its IRSM whose absolute phase will be different from the previous one. However, its interaction region, $< S|S >$, is the same as all previous ones. The hits of particles on the screen occur randomly at different locations due to different initial phases. The randomness in the phase is due to its dependence on the nature of source. After a large collection of particles, an interference pattern emerges out which is nothing but the construction of the function $| < \mathbf{r}|S > |^2$ with individual points; here, $|\mathbf{r}>$ is the eigenstate of position operator, $\hat{\mathbf{r}}$, with eigenvalues, $\mathbf{r}$, which span the detector screen. At this moment, it's worth recollecting a philosophical saying, "*It is necessary for the very existence of science that the same conditions always produce the same result*" - this statement is in perfect agreement even in the quantum domain, because, the occurrence of many identical initial states with the same absolute phase is impossible.

A remark follows: after collecting a large number of particles, the resulting interference intensity is indeed equivalent to the interference intensity produced by a 'macroscopic matter-wave' (MaMW). Therefore, each and every individual particle may be inferred as an averaged MaMW i.e., as a 'microscopic matter wave' (MiMW). If inferred this way, then MiMW has to collapse to a particle at some region of its own interference pattern. This leads to wave-particle duality which demands the instant collapse of entire wave function to a point. Inherent randomness and probability become unavoidable while describing the collapse though they arise due to the nature of particle source. This can be juxtaposed with a situation of black body radiation. While calculating the Plank radiation formula [28], the statistically averaged energy of a photon is found to be $\bar{\epsilon} = h\nu/[\exp(h\nu/KT) - 1]$; here, h is Plank's constant, ν is the frequency associated with the photon, K is the Boltzmann constant and T is the absolute temperature. However, this never implies the existence of any photon with energy $\bar{\epsilon}$. But, the usage of $\bar{\epsilon}$ yields correct results provided the experiment is done with a large number of photons, because, $\bar{\epsilon}$ is reflecting the statistical nature of observation. Therefore, an individual event need not carry the statistical average which, of course, yields a right value if used along with a huge data. In other words, a particle itself never behaves like a wave though it is associated with the de Broglie wave nature. Otherwise, the YDS interference pattern obtained with macroscopic molecules of definite internal structure [8, 9, 11] can't be explained unambiguously.

If slit-1 (slit-2) is blocked, then a clump pattern corresponding to a single slit diffraction of slit-2 (slit-1), $< S_2|S_2 >$ ($< S_1|S_1 >$), is produced on the screen. According to wave-particle duality, a single slit diffraction is attributed to the particle nature whereas, the double-slit interference to the wave nature. But, in non-duality, the particle always moves in its IRSM whether through a single slit or a double- slit. Therefore, if a detector observes through which slit a particle is going, then it will always appear through either slit-1 or slit-2. As mentioned in the Section-VI, any momentum changing interaction of the particle with the detector's probe will result in the disappearance of $|S >$, which had two origins, one at each slit. A new IRSM, either $|S'_1 >$ or $|S'_2 >$, of new momentum appears with a single origin where the interaction took place in the vicinity of the respective slit. Its interaction with the detector screen is either $< S'_1|S'_1 >$ or $< S'_2|S'_2 >$. Therefore, in the presence of detectors, clump patters occur and in their absence, the interference comes back. This particular property viz, '*the disappearance of interference pattern and the appearance of clump patterns whenever the quanta are watched through which slit they are going*', is an ultimate proof for the underlying particle nature. Otherwise, the disappearance of interference pattern is impossible. Therefore, the quantum enigma is an inference drawn based on the wave-particle duality by visualizing the quantum phenomenon in 3DES.

Here, it's worth mentioning that a given particle simultaneously carries both position and momentum eigenvalues. Consider the YDS experiment as an example: a particle will be found at some location of the detector screen due to the momentum it carries. However, the position and momentum state vectors of the particle live in their own respective CVSs which have a non-zero twist with respect to each other due to the commutation relation, $[\hat{x}, \hat{p}] = i\hbar$. Therefore, according to Bohr's principle of complementarity at a single-quantum level as given in Section-VI, the position measurement excludes the momentum measurement and vice versa. This situation is totally different from the classical scenario where the vector spaces of both the position and momentum are untwisted due to $[\hat{x}, \hat{p}] = 0$ and hence their superimposition on top of each other can be regarded as a single space.

Nevertheless, a free quantum can not be detected inside a width, Δx, less than the de Brogle wavelength which is characteristic of its momentum. Any such detection will change the IRSM itself. The minimum width Δx_{min} inside which a particle can be found, without changing its momentum eigenvalue, is $\Delta x_{min} = \lambda$ which yields $p\ \Delta x_{min} = h$ (using de Broglie's relation). If the IRSM encounters a vertical slit of width Δy, then from Bragg's law, for first order diffraction, $\Delta y\ \sin\theta = \lambda$ implying that $\Delta y\ p\ \geq h$. So, there exists a maximum momentum, $P_{max} = h/\Delta y$, a particle can have such that at least a first order diffraction can be observed. Any momentum greater than P_{max} will not exhibit any diffraction except the zeroth order. These are equivalents of the Heisenberg uncertainty relations [2–4] in the case of quantum phenomena at the single-quantum level.

Also, it's easy to check that the macroscopic objects naturally yield clump patterns matching our day-to-day worldly experience because, their de Broglie wave length is extremely small when compared to the size of the object, the dimensions of slits and their separation.

IX. CAUSALITY IN WHEELER'S DELAYED-CHOICE EXPERIMENT

As already mentioned in Section-I, if the screen in YDS experiment (Fig. 1) is quickly replaced while a quantum is in mid-flight after passing the double-slit, then the state encountered jointly by the twin telescopes, T_1 and T_2, is

$$\begin{aligned}(\hat{T}_1 + \hat{T}_2)|S > &= (\hat{T}_1 + \hat{T}_2)(|S_1 > + |S_2 >) \\ &= \hat{T}_1|S_1 > + \hat{T}_2|S_2 > \\ &= |\tilde{S}_1 > + |\tilde{S}_2 > \equiv |\tilde{S} >, \end{aligned} \tag{53}$$

where, $\hat{T}_1$ and $\hat{T}_2$ are operators associated with the telescopes and $\hat{T}_1|S_2 >= \hat{T}_2|S_1 >= 0$, because, T_1 and T_2 are tightly focused on slit-1 and slit-2, respectively. Let's take the quantum to be a photon. The replacement of screen by telescopes can be viewed as a change of either final boundary conditions of IRSM or CVS where $|S >$ is represented. Now, the old IRSM, $|S >$, changes to a new one, $|\tilde{S} >$, but their initial boundary conditions (origins) remain the same. Wherever be the position of photon during the replacement, it continues to fly from there through $|\tilde{S} >$. *The photon's motion is always continues even though the boundary condition or the representation of its IRSM changes suddenly.* The continuity in photon's motion is governed by conserved quantities. Since, the CVSs of telescopes are orthogonal to each other, the observed RFD is,

$$< \tilde{S}|\tilde{S} >=< \tilde{S}_1|\tilde{S}_1 > + < \tilde{S}_2|\tilde{S}_2 >, \tag{54}$$

which corresponds to clump patterns. Note that, *the non-duality preserves causality.*

Consider again the same YDS experiment with polarization filters P_1 and P_2 instead of the blockers B_1 and B_2, respectively. In this case, the IRSM is,

$$|S >>= |S_1 > |P_1 > + |S_2 > |P_2 >, \tag{55}$$

where, $|P_1 >$ and $|P_2 >$ correspond to photon's polarization states. The interaction of IRSM with its excited dual at the screen is,

$$<< S|S >>= \sum_{i,j=1}^{2} < S_i|S_j >< P_i|P_j > . \tag{56}$$

If $|P_1>$ and $|P_2>$ are orthogonal, $<P_1|P_2>=0$, then the interference vanishes even though $<S_1|S_2>\neq 0$.

If $|P_1>=|H>$ and $|P_2>=|V>$, then Eq. (55) becomes,

$$|S>>=|S_1>|H>+|S_2>|V>, \tag{57}$$

where, $|H>$ and $|V>$ are horizontal and vertical polarization states, respectively. Insertion of a 45^o polarization rotator (PR), with an unit operator $I_{\rm pr}=|\bar{H}><\bar{H}|+|\bar{V}><\bar{V}|$, just before the screen, changes the representation of IRSM:

$$\begin{aligned}|S>> &= <\bar{H}|S>>|\bar{H}>+<\bar{V}|S>>|\bar{V}> \\ &\equiv |\bar{S}_1>|\bar{H}>+|\bar{S}_2>|\bar{V}>\end{aligned} \tag{58}$$

where,

$$|\bar{H}>=(|H>+|V>)/\sqrt{2}; |\bar{V}>=(-|H>+|V>)/\sqrt{2}$$

$$|\bar{S}_1>=<\bar{H}|S>>=(|S_1>+|S_2>)/\sqrt{2} \tag{59}$$

and

$$|\bar{S}_2>=<\bar{V}|S>>=-(|S_1>-|S_2>)/\sqrt{2}. \tag{60}$$

It's clear from Eq. (59) and Eq. (60) that the photon passing through the slit-1 or slit-2 will be present in either $|\bar{S}_1>$ or $|\bar{S}_2>$, respectively.

Now, inserting a Wollaston prism (WP), with unit operator $I_{\rm wp}=|H><H|+|V><V|$, between PR and the screen, changes again the representation of the photon state:

$$\begin{aligned}|S>> &= (<H|\bar{H}>|\bar{S}_1>+<H|\bar{V}>|\bar{S}_2>)|H> \\ &\quad +(<V|\bar{H}>|\bar{S}_1>+<V|\bar{V}>|\bar{S}_2>)|V> \\ &= \frac{1}{\sqrt{2}}[(|\bar{S}_1>-|\bar{S}_2>)|H>+(|\bar{S}_1>+|\bar{S}_2>)|V>] \\ &= |S_1>|H>+|S_2>|V>,\end{aligned} \tag{61}$$

whose interaction with its dual at the screen is,

$$\begin{aligned}<<S|S>> &= \frac{1}{2}\sum_{i,j=1}^{2}(-1)^{(i+j)}<\bar{S}_i|\bar{S}_j><H|H> \\ &\quad +\frac{1}{2}\sum_{i,j=1}^{2}<\bar{S}_i|\bar{S}_j><V|V> \\ &= <\bar{S}_1|\bar{S}_1>+<\bar{S}_2|\bar{S}_2> \\ &= <S_1|S_1>+<S_2|S_2>,\end{aligned} \tag{62}$$

showing that the clump patterns are intact even in the presence of both PR and WP. Note that, the above equation never implies $<S_i|S_i>=<\bar{S}_i|\bar{S}_i>$.

The RFDs of two orthogonal components, $|H>$ and $|V>$, from the WP can be detected by two independent detectors, say H and V:

$$RFD_{\rm H}=\frac{1}{2}\sum_{i,j=1}^{2}(-1)^{(i+j)}<\bar{S}_i|\bar{S}_j>=<S_1|S_1> \tag{63}$$

and

$$RFD_{\rm V}=\frac{1}{2}\sum_{i,j=1}^{2}<\bar{S}_i|\bar{S}_j>=<S_2|S_2>. \tag{64}$$

Therefore, a photon present in the $|H>$ component of the WP will contribute a point to the anti-interference pattern given in Eq. (63) and the one in $|V>$ to interference pattern in Eq. (64). Also, these equations correctly predict that the photon initially entered through slit-1 or slit-2 of YDS will be detected by H or V, respectively, due to the law of conservation of momentum. In the absence of PR, the usual clump patterns corresponding to YDS will be formed at H and V. The role of PR is simply to replace the clump patterns by the respective anti-interference and interference structures. If the screen is used for detection instead of H and V, then these structures disappear into each other yielding the clump patterns as shown in Eq. (62).

Note that, the PR can be randomly introduced or removed before a photon passes through the same. This technique was used in an experiment by Jacques et al. [36, 37], where the Mach-Zehnder interferometer is used instead of YDS assembly. Similar experiment using single atoms [38] and its quantum versions [39, 40] where the detector itself is described by a quantum state [41] were also done. As it was already explained, even during the random changes of representations or boundary conditions for the IRSM, the photon flies continuously.

X. CONCLUSIONS AND DISCUSSIONS

The physical reality of the wave function remained as a mystery since its discovery by Prof. E. Schrödinger and the same is brought to light for the first time by the present non-dualistic interpretation of quantum mechanics. It's shown to be an Instantaneous Resonant Spatial Mode (IRSM) where a quantum particle flies due to the constants of motion. The IRSM and its particle always coexist together as a single entity, which is named as wave-particle non-duality. As demanded by the quantum formalism, if the underlying space of the Nature is recognized as a complex vector space rather than the usual Euclidean, then the quantum phenomena at the single-quantum level become causal and deterministic exactly like the classical phenomena. In a nutshell, the Universe is fundamentally quantum mechanical even though it 'effectively' appears to be classical at a large scale. The unavoidable absolute phase of the wave function is shown to be responsible for the outcome of a definite eigenvalue of an observable in a given experiment. Moreover, the position eigenstate, containing a quantum, is shown to be related to the same absolute phase in such a way that its eigenvalue, which varies with time, always lies on a classical path of least action, proving the commonality of classical and quantum mechanical times. Unknown to the classical formalism, the quantum formalism demands the consideration of inner-product of the state vector with its dual while imposing the boundary conditions to the wave function. This naturally yields the relative frequency of detection and hence, the Born rule, showing that the statistical nature of doing experiment is the actual reason behind the success of the probabilistic interpretation. Moreover, the Copenhagen interpretation is completely contained within the non-duality. Also, a causal explanation of Wheeler's delayed-choice experiment is provided for the first time. In the Young's double-slit experiment, if the screen is suddenly replaced by a twin-telescopes while the particle has already crossed the double-slit, then the IRSM or the representation of particle's state vector changes accordingly but, the particle itself flies continuously. The same logic is applicable, for example, in the case of hydrogen atom. When it absorbs or emits a photon, the electron motion will remain continuous though the energy eigenstate changes. In the case of quantum tunneling, the particle itself simply moves in its IRSM, though it appears to an observer as if tunneling through the potential barrier. With respect to non-duality, the measurement problem doesn't exist.

The work reported here is a confirmation of what Einstein said, "God does not play dice". The standard quantum mechanics is universally valid and the classical mechanics emerges out only 'effectively', but not as a limiting case of Plank's constant becoming zero. Since, the non-duality is a visualization of the nature of reality reflected within the quantum formalism, it will go through both time-dependent and relativistic quantum mechanics. In the relativistic case, the IRSM is such that, apart from obeying the usual quantum mechanical commutation relations, it takes care of the cosmic speed limit of its resonant particle, though it itself can change instantaneously.

Another mystery of the quantum world, untouched in the present article, is the entanglement of two or more particles [42, 43]. It's worth mentioning that the non-duality is capable of providing the physical mechanism for Einstein's spooky action-at-a-distance [42] by making use of the nature of IRSM and will be reported elsewhere. The wave-particle non-duality will surely bring consensus about the kind of physical reality being reflected by the quantum formalism. Undoubtedly, it will further enhance the deeper understanding of Nature's working at the most fundamental level, particularly in the direction of quantum gravity.

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