Integrals – Identities - Pi

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Abstract. We give some integrals for Pi.

Notation: $\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 3.14159265358979323846264338327950288419716 9...$

1. Integrals

$$\pi = \int_0^{\infty} \left(\frac{\pi}{4} - \tan^{-1}\left(\frac{x^2}{8+\sqrt{64+x^4}} \right) \right) dx = \int_0^{\infty} \tan^{-1}\left(\frac{8}{x^2+\sqrt{64+x^4}} \right) dx \quad (1)$$

$$\pi = \int_0^{\infty} \sin^{-1}\left(\sqrt{\frac{32}{64+x^4+x^2\sqrt{64+x^4}}} \right) dx \quad (2)$$

$$\pi = \int_0^{\infty} \cos^{-1}\left(\frac{x^2+\sqrt{64+x^4}}{\sqrt{128+2x^4+2x^2\sqrt{64+x^4}}} \right) dx \quad (3)$$

$$\pi = 2\int_0^{\pi/4} \sqrt{\cot x - \tan x}\, dx \quad (4)$$

$$\pi = 2\int_0^1 \frac{\sqrt{1-x^2}}{\sqrt{x}\left(1+x^2\right)} dx \quad (5)$$

$$\pi = 4\int_0^1 \frac{\sqrt{1-x^4}}{1+x^4} dx \quad (6)$$

$$\pi = 4\int_0^1 \sqrt[4]{\frac{2\left(1-x^2\right)}{1+2x^2+\sqrt{1+8x^2}}}\, dx \quad (7)$$

$$\pi = \int_0^{\sqrt{2}} \left(\sqrt[4]{\frac{8 - x^2 + 4\sqrt{4 - 2x^2}}{x^2}} - \sqrt[4]{\frac{8 - x^2 - 4\sqrt{4 - 2x^2}}{x^2}} \right) dx \tag{8}$$

2. Integrals - Identities

If $u \geq 0, v = \tan^{-1}\left(\frac{8}{u^2 + \sqrt{64 + u^4}} \right)$, *then*

$$\pi = u \tan^{-1}\left(\frac{8}{u^2 + \sqrt{64 + u^4}} \right) + 2\int_v^{\pi/4} \sqrt{\cot x - \tan x}\, dx + \int_u^{\infty} \tan^{-1}\left(\frac{8}{x^2 + \sqrt{64 + x^4}} \right) dx \tag{9}$$

$$\pi + u \tan^{-1}\left(\frac{8}{u^2 + \sqrt{64 + u^4}} \right) = 2\int_0^{v} \sqrt{\cot x - \tan x}\, dx + \int_0^{u} \tan^{-1}\left(\frac{8}{x^2 + \sqrt{64 + x^4}} \right) dx \tag{10}$$

If $0 \leq u \leq 1, v = \frac{\sqrt{1 - u^4}}{1 + u^4}$, *then*

$$\pi = 4u \frac{\sqrt{1 - u^4}}{1 + u^4} + 4\int_u^1 \frac{\sqrt{1 - x^4}}{1 + x^4} dx + 4\int_v^1 \sqrt[4]{\frac{2\left(1 - x^2\right)}{1 + 2x^2 + \sqrt{1 + 8x^2}}}\, dx \tag{11}$$

$$\pi + 4u \frac{\sqrt{1 - u^4}}{1 + u^4} = 4\int_0^u \frac{\sqrt{1 - x^4}}{1 + x^4} dx + 4\int_0^v \sqrt[4]{\frac{2\left(1 - x^2\right)}{1 + 2x^2 + \sqrt{1 + 8x^2}}}\, dx \tag{12}$$

If $f(n,k) = \int_0^1 \left(\frac{2(1 - x)^4 - 1}{3} \right)^{n-k} \left(\frac{6x - 4x^2 + x^3}{4} \right)^k \sqrt{x}\, dx$, *then*

$$\pi = \frac{16}{3} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{2k}{k} \frac{(-1)^{n-k} 2^{-2k}}{1 - 2k} f(n,k) \tag{13}$$

References

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