The number Pi and the integral: $\int (\tanh x)^a \frac{dx}{x}$

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abstract
This note presents some formulas for Pi .

1. Introduction

The number π is defined by :

$$\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 3.141592... \tag{1}$$

This note presents some series for π .

2. The arctangent function

If $0 \le x < 1$, then

$$\tan^{-1} x = \sum_{n=0}^{\infty} A_n B_n(x) \tag{2}$$

where

$$A_{n+1} = -A_n - \frac{2}{(2n+1)(2n+3)} \quad , A_0 = 1 \tag{3}$$

$$B_n(x) = \int_0^{\tanh^{-1} x} \frac{(\tanh u)^{2n+1}}{u} du \tag{4}$$

Remarks :

$$\tanh u = \frac{e^u - e^{-u}}{e^u + e^{-u}} \quad , e = \sum_{n=0}^{\infty} \frac{1}{n!} = 2.71828... \tag{5}$$

$$A_n = \left\{1, -\frac{5}{3}, \frac{23}{15}, -\frac{167}{105}, \frac{491}{315}, -\frac{5471}{3465}, \ldots\right\} \quad (6)$$

$$A_n = (-1)^n \left(1 - 2\sum_{k=1}^{n} \frac{(-1)^k}{4k^2 - 1}\right) \quad (7)$$

$$|A_n| \to \frac{\pi}{2} \;, n \to \infty \quad (8)$$

3. Formulas

$$\pi = 6\sum_{n=0}^{\infty} A_n \, B_n\left(1/\sqrt{3}\right) \quad (9)$$

$$B_n\left(1/\sqrt{3}\right) = \int_0^{\frac{1}{2}\ln\left(2+\sqrt{3}\right)} \frac{(\tanh u)^{2n+1}}{u} du \quad (10)$$

$$\pi = 8\sum_{n=0}^{\infty} A_n \, B_n\left(\sqrt{2}-1\right) \quad (11)$$

$$B_n\left(\sqrt{2}-1\right) = \int_0^{\frac{1}{2}\ln\left(\sqrt{2}+1\right)} \frac{(\tanh u)^{2n+1}}{u} du \quad (12)$$

$$\pi = 12\sum_{n=0}^{\infty} A_n \, B_n\left(2-\sqrt{3}\right) \quad (13)$$

$$B_n\left(2-\sqrt{3}\right) = \int_0^{\frac{1}{4}\ln 3} \frac{(\tanh u)^{2n+1}}{u} du \quad (14)$$

References

1. Arndt, J., and Haenel, C.: π unleashed. Springer-Verlag, 2001.
2. Bailey, D. H., Borwein, J. M., Borwein, P. B., and Plouffe, S.: The quest for Pi. Math. Intelligencer, 19, 1997 , 50-57.