Division by Zero Calculus in Equations and Inequalities

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Abstract: In this paper, we will examine the division by zero calculus from the viewpoints of equations and inequalities as a starting new idea.

Key Words: Zero, division by zero, division by zero calculus, $0/0 = 1/0 = z/0 = \tan(\pi/2) = \log 0 = 0$, complex analysis, Laurent expansion, equation, inequality.

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1 Division by zero and division by zero calculus

For the long history of division by zero, see [2, 22]. The division by zero with mysterious and long history was indeed trivial and clear as in the followings.

By the concept of the Moore-Penrose generalized solution of the fundamental equation $ax = b$, the division by zero was trivial and clear as $b/0 = 0$ in the **generalized fraction** that is defined by the generalized solution of the equation $ax = b$. Here, the generalized solution is always uniquely determined and the theory is very classical. See [6] for example.

Division by zero is trivial and clear also from the concept of repeated subtraction - H. Michiwaki.

Recall the uniqueness theorem by S. Takahasi on the division by zero. See [6, 32].

The simple field structure containing division by zero was established by M. Yamada ([9]). For a simple introduction, see H. Okumura [19].

Many applications of the division by zero to Wasan geometry were given by H. Okumura. See [13, 14, 15, 16, 17, 18] for example.

As the number system containing the division by zero, the Yamada field structure is perfect. However, for applications of the division by zero to **functions**, we need the concept of the division by zero calculus for the sake of uniquely determinations of the results and for other reasons.

For example, for the typical linear mapping

$$W = \frac{z-i}{z+i}, \tag{1.1}$$

it gives a conformal mapping on $\{\mathbf{C} \setminus \{-i\}\}$ onto $\{\mathbf{C} \setminus \{1\}\}$ in one to one and from

$$W = 1 + \frac{-2i}{z-(-i)}, \tag{1.2}$$

we see that $-i$ corresponds to 1 and so the function maps the whole $\{\mathbf{C}\}$ onto $\{\mathbf{C}\}$ in one to one.

Meanwhile, note that for

$$W = (z-i)\cdot\frac{1}{z+i}, \tag{1.3}$$

we should not enter $z = -i$ in the way

$$[(z-i)]_{z=-i}\cdot\left[\frac{1}{z+i}\right]_{z=-i} = (-2i)\cdot 0 = 0. \tag{1.4}$$

However, in many cases, the above two results will have practical meanings and so, we will need to consider many ways for the application of the division by zero and we will need to check the results obtained, in some practical viewpoints. We referred to this delicate problem with many examples in the references.

Therefore, we will introduce the division by zero calculus. For any Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{-1} C_n(z-a)^n + C_0 + \sum_{n=1}^{\infty} C_n(z-a)^n, \tag{1.5}$$

we **define** the identity, (by the division by zero)

$$f(a) = C_0. \tag{1.6}$$

(Note that here, there is no problem on any convergence of the expansion (1.5) at the point $z = a$, because all the terms $(z-a)^n$ are zero at $z = a$ for $n \neq 0$.)

Apart from the motivation, we define the division by zero calculus by (1.6). With this assumption, we can obtain many new results and new ideas. However, for this assumption we have to check the results obtained whether they are reasonable or not. By this idea, we can avoid any logical problems. – In this point, the division by zero calculus may be considered as a fundamental assumption like an axiom.

The division by zero calculus opens a new world since Aristotele-Euclid. See, in particular, [3] and also the references for recent related results.

On February 16, 2019 Professor H. Okumura introduced the surprising news in Research Gate:

José Manuel Rodríguez Caballero
Added an answer
In the proof assistant Isabelle/HOL we have $x/0 = 0$ for each number x. This is advantageous in order to simplify the proofs. You can download this proof assistant here: **https://isabelle.in.tum.de/**.

J.M.R. Caballero kindly showed surprisingly several examples by the system that

$$\tan\frac{\pi}{2} = 0,$$

$$\log 0 = 0,$$

$$\exp\frac{1}{x}(x = 0) = 1,$$

and others. Furthermore, for the presentation at the annual meeting of the Japanese Mathematical Society at the Tokyo Institute of Technology:

March 17, 2019 : 9: 45-10: 00 in Complex Analysis Session, *Horn torus models for the Riemann sphere from the viewpoint of division by zero* with [3],

he kindly sent the kind message:

It is nice to know that you will present your result at the Tokyo Institute of Technology. Please remember to mention Isabelle/HOL, which is a software in which x/0 = 0. This software is the result of many years of research and a millions of dollars were invested in it. If x/0 = 0 was false, all these money was for nothing. Right now, there is a team of mathematicians formalizing all the mathematics in Isabelle/HOL, where x/0 = 0 for all x, so this mathematical relation is the future of mathematics. https://www.cl.cam.ac.uk/ lp15/Grants/Alexandria/

Meanwhile, on ZERO, the authors S. K. Sen and R. P. Agarwal [29] published its long history and many important properties of zero. See also R. Kaplan [5] and E. Sondheimer and A. Rogerson [31] on the very interesting books on zero and infinity. In particular, for the fundamental relation of zero and infinity, we stated the simple and fundamental relation in [27] that

The point at infinity is represented by zero; and zero is the definite complex number and the point at infinity is considered by the limiting idea as an ideal point of one point compactification
and that is represented geometrically with the horn torus model [3].

S. K. Sen and R. P. Agarwal [29] referred to the paper [6] in connection with division by zero, however, their understandings on the paper seem to be not suitable (not right) and their ideas on the division by zero seem to be traditional, indeed, they stated as the conclusion of the introduction of the book that:

"Thou shalt not divide by zero" remains valid eternally.

However, in [26] we stated simply based on the division by zero calculus that

We Can Divide the Numbers and Analytic Functions by Zero with a Natural Sense.

They stated in the book many meanings of zero over mathematics, deeply.

By the new idea of the division by zero calculus, we can consider the values of analytic functions at isolated singular points. This fact will give great impacts in equations and inequalities. In this paper we will state the fact with simple examples as the initial starting point.

2 Equations

For the equation

$$\frac{x-4y+2z}{x} = \frac{2x+7y-4z}{y} = \frac{4x+10y-6z}{z} = k,$$

from $k = 1$, we have the solution with parameter λ

$$x = y = \lambda, \quad z = 2\lambda.$$

We obtain also the natural solution

$$x = y = z = 0.$$

However, then $k = 0$.

For the equation

$$x - x = \frac{x}{x} \tag{2.1}$$

(Nathaniel Andika: 2019.6.22.05:36 in Quora), we have the solution $x = 0$.

On the history of mathematics, we have the nature that in order to solve equations, we extended the number system; for example, in order to solve the equation $x^2 = -1$, we introduced the complex numbers by introducing i. On this history, we can consider that in order to solve the fundamental equation (2.1) we introduced the division by zero $0/0 = 0$ by giving the meaning of $\frac{x}{x}$ at the point $x = 0$.

In particular, note that the division by zero calculus is not almighty. The notation

$$\Delta(x) = \frac{x}{x} = \begin{cases} 0 & \text{for } x = 0 \\ 1 & \text{for } x \neq 0 \end{cases} \tag{2.2}$$

will be convenient in connection with the Dirac delta function $\delta(x)$.

For the formula

$$\frac{d^n}{dz^n}\log z = (-1)^{n-1}(n-1)!z^{-n}$$

([1], page 69, 4.1.47), we have, for $n = 1$, of course

$$\frac{d}{dz}\log z = \frac{1}{z}.$$

For $n = 0$, by the division by zero calculus, we have, noting $z! = \Gamma(z+1)$

$$\frac{d^0}{dz^0}\log z = \log z$$

$$= (-1)^{-1}\left(\frac{1}{n} + \cdots\right)\left(1 - n\log z + \frac{1}{2!}(-n\log z)^2 + \cdots\right)$$

$$= \log z.$$

From the identity

$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n=-\infty}^{+\infty}\frac{1}{(z-n)^2},$$

by using the expansion

$$\pi^2\left(\frac{1}{\pi z} + \frac{\pi z}{6} + \cdots\right)^2$$

$$= \pi^2\left(\frac{1}{\pi^2 z^2} + \frac{1}{3} + \cdots\right),$$

we have the identity, from the division by zero calculus, immediately, for $z = 0$

$$\zeta(2) = \frac{\pi^2}{6}.$$

As a typical example in A. Kaneko ([4], page 11) in the theory of hyperfunction theory we see that for non-integers λ, we have

$$x_+^\lambda = \left[\frac{-(-z)^\lambda}{2i\sin\pi\lambda}\right] = \frac{1}{2i\sin\pi\lambda}\{(-x+i0)^\lambda - (-x-i0)^\lambda\}$$

where the left hand side is a Sato hyperfunction and the middle term is the representative analytic function whose meaning is given by the last term. For an integer n, Kaneko derived that

$$x_+^n = \left[-\frac{z^n}{2\pi i} \log(-z) \right],$$

where log is a principal value on $\{-\pi < \arg z < +\pi\}$. Kaneko stated there that by taking a finite part of the Laurent expansion, the formula is derived. Indeed, we have the expansion, around an integer n,

$$\frac{-(-z)^\lambda}{2i \sin \pi\lambda}$$

$$= \frac{-z^n}{2\pi i} \frac{1}{\lambda - n} - \frac{z^n}{2\pi i} \log(-z)$$

$$- \left(\frac{\log^2(-z) z^n}{2\pi i \cdot 2!} + \frac{\pi z^n}{2i \cdot 3!} \right) (\lambda - n) + ...$$

([4], page 220).

However, we can derive the result from the division by zero calculus, immediately.

Meanwhile, M. Morimoto derived this result by using the Gamma function with the elementary means in [11], pages 60-62.

For example, we have the identity

$$\frac{1}{(x-a)(x-b)(x-c)} = \frac{1}{(c-b)(a-c)(x-a)}$$

$$+ \frac{1}{(b-c)(b-a)(x-b)} + \frac{1}{(c-a)(c-b)(x-c)}.$$

By the division by zero calculus, the first term in the right hand side is zero for $x = a$, and

$$\frac{1}{(b-c)(b-a)(a-b)} + \frac{1}{(c-a)(c-b)(a-c)}.$$

This result is the same as

$$\frac{1}{(x-a)(x-b)(x-c)} (a),$$

by the divison by zero calculus.

For the identity

$$\frac{1}{x(a+x)^2} = \frac{1}{a^2 x} - \frac{1}{a(a+x)^2} - \frac{1}{a^2(a+x)},$$

we have the identity as $\frac{1}{x^3}$ for $a = 0$.

For the identity

$$f(z) = \Pi_{j=1}^{n}(z - z_j),$$

we have the identity

$$\left[\frac{f'(z)}{f(z)}\right]_{z=z_1} = \frac{1}{z_1 - z_2} + ... + \frac{1}{z_1 - z_n}.$$

For the identity

$$\frac{mx+n}{ax^2+2bx+c}$$

$$= \frac{m}{2a}\frac{2ax+2b}{ax^2+2bx+c} + \frac{an-bm}{a} - \frac{1}{ax^2+2bx+c},$$

for $a = 0$, we have

$$\frac{mx+n}{2bx+c} = \frac{x(bx+c)}{(2bx+c)^2} + \frac{2bnx+nc+bmx^2}{(2bx+c)^2}.$$

For the identity

$$I_n = (-1)^n n! \frac{1}{(a^2+x^2)^{(n+1)/2}} \sin(n+1)\theta$$

$$= \frac{(-1)^n n!}{2i}\left[\frac{1}{(x-ai)^{n+1}} - \frac{1}{(x+ai)^{n+1}}\right], z = x + iy = e^{i\theta},$$

we have, for $x = ai$

$$[I_n]_{x=ai} = \frac{(-1)^n n!}{2^{n+2} i^n}.$$

In the identity, for $-\pi \le x \le \pi$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos nx}{n^2-a^2} = \frac{\pi \cos ax}{2a \sin a\pi} - \frac{1}{2a^2},$$

for $a=0$, we have

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cos nx}{n^2}=\frac{1}{12}(\pi^2-3x^2).$$

For the identities, for $0\le x\le 2\pi$

$$\sum_{n=1}^{\infty}\frac{1}{n^2+a^2}\cos(nx)=\frac{\pi}{2a\sinh(a\pi)}\cosh[a(\pi-x]]-\frac{1}{2a^2},$$

and

$$\sum_{n=1}^{\infty}\frac{1}{n^2-a^2}\cos(nx)=\frac{\pi}{2a\sin(a\pi)}\cos[a(\pi-x]]+\frac{1}{2a^2},$$

for $a=0$, we have

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\cos(nx)=\frac{1}{12}(3x^2-6\pi x+2\pi^2).$$

In the identity

$$\frac{1}{x}-\frac{{}_nC_1}{x+1}+\frac{{}_nC_2}{x+2}+\cdots+(-1)^n\frac{{}_nC_n}{x+n}$$
$$=\frac{n!}{x(x+1)(x+2)\cdots(x+n)},$$

from the singular points, we obtain many identities, for example, from $x=0$, we obtain the identity

$$-{}_nC_1+\frac{{}_nC_2}{2}+\cdots+(-1)^n\frac{{}_nC_n}{n}$$
$$=-\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right).$$

We can derive many identities in this way.

For the simple Clairaut differential equation

$$y=px+\frac{1}{p},\quad p=\frac{dy}{dx},$$

we have the general solution

$$y = Cx + \frac{1}{C}, \tag{2.3}$$

with a general constant C and the singular solution

$$y^2 = 4x.$$

Note that we have also the solution $y = 0$ from the general solution, by the division by zero $1/0 = 0$ from $C = 0$ in (2.3).

In general, for the applications of the division by zero calculus to differential equations, see [20].

3 Inequalities

For the problem

$$f(x) = \frac{1}{(x-1)(x-2)} < 0,$$

we have the solution

$$1 < x < 2$$

in the usual sense. However, note that by the division by zero calculus

$$f(1) = -1$$

and

$$f(2) = -1.$$

Therefore, we have the solution

$$1 \leq x \leq 2.$$

Meanwhile, we know

Growth Lemma ([21], 267 page) *For the polynomial*

$$P(z) = a_0 + a_1 z + ... + a_n z^n (a_0, a_n \neq 0, n > 1)$$

we have the inequality with a sufficient r*, for* $|z| \geq r$

$$\frac{|a_n|}{2}|z|^n \leq |P(z)| \leq \frac{3|a_n|}{2}|z|^n.$$

At the point at infinity, since $P(z)$ takes the value a_0, the inequality is not valid more.

In the inequality

$$\pi < \frac{\sin \pi x}{x(1-x)} \leq 4 \quad (0 < x < 1)$$

([1], page 75, 4.3.82), the function takes π at $x = 0, 1$ and so we have the inequality

$$\pi \leq \frac{\sin \pi x}{x(1-x)} \leq 4 \quad (0 \leq x \leq 1).$$

Therefore, for inequalities, for the values of singular points by means of the division by zero calculus, we have to check the values, case by case.

4 Conclusion

The division by zero calculus requests the essential arrangements for equations and inequalities in analytic functions.

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