Compton-Wavelength Minimum Position Uncertainty Due to Diversion of Momentum Uncertainty to Increased Particle Number

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Abstract

In single-particle quantum mechanics there is no impediment to a particle's wave function having a significant amplitude for arbitrarily short wavelengths, i.e., for arbitrarily large momenta, so the iconic single-particle uncertainty relation permits a particle's position to be arbitrarily accurately ascertained. Once a single-particle theory is second quantized, however, the physics of imparting a particle's wave function with ever larger momenta eventually encounters stiff competition from the formation of multiparticle states wherein none of the individual particles is characterized by high momentum. In fact, the single-particle uncertainty principle itself is modified by the presence of the expectation value of the particle number operator on its right side. Since the threshold for the high-momentum-diverting formation of additional particles is set by the particle's rest mass, it stands to reason that particles have an irreducible position uncertainty of the order of their Compton wavelength, for which we develop a specific model.

Introduction

The uncertainty relation of *single-particle* quantum mechanics permits the particle's *position* to be *arbitrarily accurately ascertained* by imparting a significant amplitude *for sufficiently large momenta* to the particle's wave function, i.e., by imparting a significant amplitude *for sufficiently short wavelengths* to that wave function. The *multiparticle second quantization of relativistic free single-particle quantum mechanics*, however, *alerts* us to the physical fact that *attempting* to impart a significant amplitude *for particle momenta appreciably in excess of the particle's rest mass m times c* to a particle's wave function *imparts a significant amplitude for the existence of multiparticle states whose individual-particle wave functions are characteristized by momenta much smaller than mc*. This *diversion to increased particle number* of wavelengths appreciably shorter than $\hbar/(mc)$ in individual-particle wave functions of course *results in a minimum position uncertainty of individual particles of the order of $\hbar/(mc)$*, which is their Compton wavelength.

The multiparticle second quantization of single-particle quantum mechanics formally signals an *extension* of the underlying single-particle theory's iconic uncertainty relation because the result of commuting matching components of the second-quantized particle position and momentum operators comes out to be *$i\hbar$ times the particle number operator $\widehat{N}$ instead of $i\hbar$ alone*. The particle number expectation value $\langle\widehat{N}\rangle$ *consequently occurs on the right side of the multiparticle uncertainty relation*, which is,

$$\big\langle|\Delta\widehat{\mathbf{r}}|^2\big\rangle^{\frac{1}{2}}\big\langle|\Delta\widehat{\mathbf{p}}|^2\big\rangle^{\frac{1}{2}} \geq (\sqrt{3}/2)\hbar\langle\widehat{N}\rangle. \tag{1a}$$

To estimate $\langle\widehat{N}\rangle$ for a relativistic free particle of rest mass m and initial momentum uncertainty $|\Delta\mathbf{p}|_{\text{initial}}$, we note that its initial energy could produce states of up to $\mathrm{N}_{|\Delta\mathbf{p}|_{\text{initial}}}$ particles of rest mass m, where,

$$\mathrm{N}_{|\Delta\mathbf{p}|_{\text{initial}}} = \big(E_{|\Delta\mathbf{p}|_{\text{initial}}}\big)/\big(mc^2\big) = \big(m^2c^4 + (c|\Delta\mathbf{p}|_{\text{initial}})^2\big)^{\frac{1}{2}}/\big(mc^2\big) = \big(1 + (|\Delta\mathbf{p}|_{\text{initial}}/(mc))^2\big)^{\frac{1}{2}}. \tag{1b}$$

So if we set the $\big\langle|\Delta\widehat{\mathbf{p}}|^2\big\rangle^{\frac{1}{2}}$ of Eq. (1a) to $|\Delta\mathbf{p}|_{\text{initial}}$, we would suppose that that should be *accompanied* by setting the $\langle\widehat{N}\rangle$ of Eq. (1a) to $\mathrm{N}_{|\Delta\mathbf{p}|_{\text{initial}}}$. Carrying this out produces,

$$\big\langle|\Delta\widehat{\mathbf{r}}|^2\big\rangle^{\frac{1}{2}} \geq (\sqrt{3}/2)\hbar\big(\mathrm{N}_{|\Delta\mathbf{p}|_{\text{initial}}}\big)/\big(|\Delta\mathbf{p}|_{\text{initial}}\big) = (\sqrt{3}/2)\hbar\big((|\Delta\mathbf{p}|_{\text{initial}})^{-2} + (mc)^{-2}\big)^{\frac{1}{2}}. \tag{1c}$$

The Eq. (1c) result *reduces to the familiar single-particle uncertainty relation if the particle's initial momentum uncertainty $|\Delta\mathbf{p}|_{\text{initial}}$ was much smaller than its rest mass m times c*. But no matter what the particle's initial momentum uncertainty $|\Delta\mathbf{p}|_{\text{initial}}$ was, the particle's Eq. (1c) position uncertainty $\big\langle|\Delta\widehat{\mathbf{r}}|^2\big\rangle^{\frac{1}{2}}$ *won't be less than* $(\sqrt{3}/2)(\hbar/(mc))$; this is the particle's *minimum position uncertainty*.

We now review the second quantization of single-particle quantum mechanics, with emphasis on those aspects which enter into the derivation of the Eq. (1a) multiparticle uncertainty relation.

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Single-particle second quantization and the multiparticle uncertainty relation

It is technically tenable to regard the single-particle complex-valued linear Schrödinger equation, namely,

$$i\hbar\, d\psi/dt = H(\widehat{\mathbf{r}},\, \widehat{\mathbf{p}})\psi, \tag{2a}$$

in either configuration representation, i.e.,

$$\begin{gathered} i\hbar\, d\psi(\mathbf{r})/dt = \textstyle\int \langle \mathbf{r}|H(\widehat{\mathbf{r}}, \widehat{\mathbf{p}})|\mathbf{r}'\rangle \psi(\mathbf{r}')\, d^3\mathbf{r}', \\ \text{where } \langle \mathbf{r}|\widehat{\mathbf{r}}|\mathbf{r}'\rangle = \mathbf{r}\, \delta^{(3)}(\mathbf{r}-\mathbf{r}') \ \text{ and } \ \langle \mathbf{r}|\widehat{\mathbf{p}}|\mathbf{r}'\rangle = -i\hbar\nabla_{\mathbf{r}}\delta^{(3)}(\mathbf{r}-\mathbf{r}'), \end{gathered} \tag{2b}$$

or momentum representation, i.e.,

$$\begin{gathered} i\hbar\, d\psi(\mathbf{p})/dt = \textstyle\int \langle \mathbf{p}|H(\widehat{\mathbf{r}}, \widehat{\mathbf{p}})|\mathbf{p}'\rangle \psi(\mathbf{p}')\, d^3\mathbf{p}', \\ \text{where } \langle \mathbf{p}|\widehat{\mathbf{r}}|\mathbf{p}'\rangle = i\hbar\nabla_{\mathbf{p}}\delta^{(3)}(\mathbf{p}-\mathbf{p}') \ \text{ and } \ \langle \mathbf{p}|\widehat{\mathbf{p}}|\mathbf{p}'\rangle = \mathbf{p}\, \delta^{(3)}(\mathbf{p}-\mathbf{p}'), \end{gathered} \tag{2c}$$

as *two classical Hamiltonian field equations* for, respectively, either the real-valued canonical pair of fields,

$$\chi(\mathbf{r}) = k\,\psi(\mathbf{r}) + k^*\psi^*(\mathbf{r}), \quad \phi(\mathbf{r}) = -i(k\,\psi(\mathbf{r}) - k^*\psi^*(\mathbf{r})), \tag{2d}$$

where k is an unspecified nonzero complex-valued constant, or the real-valued canonical pair of fields,

$$\chi(\mathbf{p}) = k\,\psi(\mathbf{p}) + k^*\psi^*(\mathbf{p}), \quad \phi(\mathbf{p}) = -i(k\,\psi(\mathbf{p}) - k^*\psi^*(\mathbf{p})). \tag{2e}$$

The Eq. (2e) momentum-representation case implies, via its Eq. (2c) Schrödinger equation, that,

$$\begin{gathered} d\chi(\mathbf{p})/dt = k\, d\psi(\mathbf{p})/dt + k^* d\psi^*(\mathbf{p})/dt = \\ (-i/\hbar)\big[k \textstyle\int \langle \mathbf{p}|H(\widehat{\mathbf{r}}, \widehat{\mathbf{p}})|\mathbf{p}'\rangle \psi(\mathbf{p}')\, d^3\mathbf{p}' - k^* \int \psi^*(\mathbf{p}')\langle \mathbf{p}'|H(\widehat{\mathbf{r}}, \widehat{\mathbf{p}})|\mathbf{p}\rangle\, d^3\mathbf{p}'\big], \\ d\phi(\mathbf{p})/dt = -i(k\, d\psi(\mathbf{p})/dt - k^* d\psi^*(\mathbf{p})/dt) = \\ -(1/\hbar)\big[k \textstyle\int \langle \mathbf{p}|H(\widehat{\mathbf{r}}, \widehat{\mathbf{p}})|\mathbf{p}'\rangle \psi(\mathbf{p}')\, d^3\mathbf{p}' + k^* \int \psi^*(\mathbf{p}')\langle \mathbf{p}'|H(\widehat{\mathbf{r}}, \widehat{\mathbf{p}})|\mathbf{p}\rangle\, d^3\mathbf{p}'\big]. \end{gathered} \tag{2f}$$

where in the second term on the right side of both of the Eq. (2f) equations we inserted the fact that,

$$(\langle \mathbf{p}|H(\widehat{\mathbf{r}}, \widehat{\mathbf{p}})|\mathbf{p}'\rangle)^* = \langle \mathbf{p}'|H(\widehat{\mathbf{r}}, \widehat{\mathbf{p}})|\mathbf{p}\rangle \tag{2g}$$

which follows from the Hermitian nature of the single-particle Hamiltonian operator $H(\widehat{\mathbf{r}}, \widehat{\mathbf{p}})$. To make Eq. (2f) *into two linear equations for* $\chi(\mathbf{p})$ *and* $\phi(\mathbf{p})$, we *invert* Eq. (2e) to obtain,

$$\psi(\mathbf{p}) = (\chi(\mathbf{p}) + i\phi(\mathbf{p}))/(2k) \qquad \psi^*(\mathbf{p}) = (\chi(\mathbf{p}) - i\phi(\mathbf{p}))/(2k^*), \tag{2h}$$

which we substitute into Eq. (2f). The result *is independent of* k, and can be written,

$$\begin{gathered} d\chi(\mathbf{p})/dt = \big[\textstyle\int \big(-i/\sqrt{2\hbar}\big)\langle \mathbf{p}|H(\widehat{\mathbf{r}}, \widehat{\mathbf{p}})|\mathbf{p}'\rangle \big((\chi(\mathbf{p}') + i\phi(\mathbf{p}'))/\sqrt{2\hbar}\big)\, d^3\mathbf{p}' + \\ \textstyle\int \big((\chi(\mathbf{p}') - i\phi(\mathbf{p}'))/\sqrt{2\hbar}\big)\langle \mathbf{p}'|H(\widehat{\mathbf{r}}, \widehat{\mathbf{p}})|\mathbf{p}\rangle \big(i/\sqrt{2\hbar}\big)\, d^3\mathbf{p}'\big], \\ d\phi(\mathbf{p})/dt = -\big[\textstyle\int \big(1/\sqrt{2\hbar}\big)\langle \mathbf{p}|H(\widehat{\mathbf{r}}, \widehat{\mathbf{p}})|\mathbf{p}'\rangle \big((\chi(\mathbf{p}') + i\phi(\mathbf{p}'))/\sqrt{2\hbar}\big)\, d^3\mathbf{p}' + \\ \textstyle\int \big((\chi(\mathbf{p}') - i\phi(\mathbf{p}'))/\sqrt{2\hbar}\big)\langle \mathbf{p}'|H(\widehat{\mathbf{r}}, \widehat{\mathbf{p}})|\mathbf{p}\rangle \big(1/\sqrt{2\hbar}\big)\, d^3\mathbf{p}'\big]. \end{gathered} \tag{2i}$$

With the square roots in Eq. (2i) as written, it can be recognized *that functional derivatives with respect to* $\phi(\mathbf{p})$ *and* $\chi(\mathbf{p})$ *of a single bilinear functional occur on the right sides of the two equations in* Eq. (2i), i.e.,

$$\begin{gathered} d\chi(\mathbf{p})/dt = \delta\Big(\textstyle\int \big((\chi(\mathbf{p}) - i\phi(\mathbf{p}))/\sqrt{2\hbar}\big)\langle \mathbf{p}|H(\widehat{\mathbf{r}}, \widehat{\mathbf{p}})|\mathbf{p}'\rangle \big((\chi(\mathbf{p}') + i\phi(\mathbf{p}'))/\sqrt{2\hbar}\big)\, d^3\mathbf{p}' d^3\mathbf{p}\Big)/\delta\phi(\mathbf{p}), \\ d\phi(\mathbf{p})/dt = -\delta\Big(\textstyle\int \big((\chi(\mathbf{p}) - i\phi(\mathbf{p}))/\sqrt{2\hbar}\big)\langle \mathbf{p}|H(\widehat{\mathbf{r}}, \widehat{\mathbf{p}})|\mathbf{p}'\rangle \big((\chi(\mathbf{p}') + i\phi(\mathbf{p}'))/\sqrt{2\hbar}\big)\, d^3\mathbf{p}' d^3\mathbf{p}\Big)/\delta\chi(\mathbf{p}). \end{gathered} \tag{2j}$$

Eq. (2j) presents the Eq. (2c) momentum-representation Schrödinger equation for the complex-valued field $\psi(\mathbf{p})$ *as two classical Hamiltonian field equations for the real-valued canonical pair of fields* $[\chi(\mathbf{p}), \phi(\mathbf{p})]$, where that field pair is given in terms of $\psi(\mathbf{p})$ and k by Eq. (2e), *and the classical Hamiltonian functional of that canonical field pair* $[\chi(\mathbf{p}), \phi(\mathbf{p})]$ *is the real-valued bilinear k-independent functional*,

$$H_{\rm cl}[\chi(\mathbf{p}), \phi(\mathbf{p}); \langle\mathbf{p}|H(\widehat{\mathbf{r}}, \widehat{\mathbf{p}})|\mathbf{p}'\rangle] \stackrel{\rm def}{=} \int\big((\chi(\mathbf{p}) - i\phi(\mathbf{p}))/\sqrt{2\hbar}\big)\langle\mathbf{p}|H(\widehat{\mathbf{r}}, \widehat{\mathbf{p}})|\mathbf{p}'\rangle\big((\chi(\mathbf{p}') + i\phi(\mathbf{p}'))/\sqrt{2\hbar}\big)\, d^3\mathbf{p}' d^3\mathbf{p}. \tag{2k}$$

Since from Eq. (2h), $\big((\chi(\mathbf{p}) + i\phi(\mathbf{p}))/\sqrt{2\hbar}\big) = k\sqrt{2/\hbar}\,\psi(\mathbf{p})$, the presentation of the Eq. (2k) classical Hamiltonian field functional *as a functional of* $\psi(\mathbf{p})$ is,

$$H_{\rm cl}[\psi(\mathbf{p}); \langle\mathbf{p}|H(\widehat{\mathbf{r}}, \widehat{\mathbf{p}})|\mathbf{p}'\rangle] = |k|^2(2/\hbar)\int \psi^*(\mathbf{p})\langle\mathbf{p}|H(\widehat{\mathbf{r}}, \widehat{\mathbf{p}})|\mathbf{p}'\rangle\psi(\mathbf{p})\, d^3\mathbf{p}' d^3\mathbf{p} = |k|^2(2/\hbar)\big(\psi, H(\widehat{\mathbf{r}}, \widehat{\mathbf{p}})\psi\big).$$

Thus $H_{\rm cl}[\psi(\mathbf{p}); \langle\mathbf{p}|H(\widehat{\mathbf{r}}, \widehat{\mathbf{p}})|\mathbf{p}'\rangle]$ equals $|k|^2(2/\hbar)$ *times the expectation value of the single-particle Hamiltonian operator* $H(\widehat{\mathbf{r}}, \widehat{\mathbf{p}})$. Since *an expectation value isn't tied to any particular single-particle representation, such as the single-particle momentum representation we've been using, the* Eq. (2j) *classical Hamiltonian field theory is fully equivalent to the underlying single-particle quantum mechanics*, so *its quantization* is of considerable interest. Quantization of a classical Hamiltonian theory *proceeds from the Poisson-bracket relations of its canonical variables*, so we need *the Poisson-bracket relations of the canonical field pair* $[\chi(\mathbf{p}), \phi(\mathbf{p})]$. First, however, we note from the above result for $H_{\rm cl}[\psi(\mathbf{p}); \langle\mathbf{p}|H(\widehat{\mathbf{r}}, \widehat{\mathbf{p}})|\mathbf{p}'\rangle]$ *that the most convenient choice for k is* $\sqrt{\hbar/2}$. This choice for k turns Eqs. (2h) and (2e) respectively into,

$$\psi(\mathbf{p}) = \big((\chi(\mathbf{p}) + i\phi(\mathbf{p}))/\sqrt{2\hbar}\big), \quad \psi^*(\mathbf{p}) = \big((\chi(\mathbf{p}) - i\phi(\mathbf{p}))/\sqrt{2\hbar}\big),$$
$$\chi(\mathbf{p}) = \sqrt{\hbar/2}(\psi(\mathbf{p}) + \psi^*(\mathbf{p})), \quad \phi(\mathbf{p}) = -i\sqrt{\hbar/2}(\psi(\mathbf{p}) - \psi^*(\mathbf{p})), \tag{2l}$$

while it of course turns the above classical Hamiltonian field functional $H_{\rm cl}[\psi(\mathbf{p}); \langle\mathbf{p}|H(\widehat{\mathbf{r}}, \widehat{\mathbf{p}})|\mathbf{p}'\rangle]$ into simply,

$$H_{\rm cl}[\psi(\mathbf{p}); \langle\mathbf{p}|H(\widehat{\mathbf{r}}, \widehat{\mathbf{p}})|\mathbf{p}'\rangle] = \int \psi^*(\mathbf{p})\langle\mathbf{p}|H(\widehat{\mathbf{r}}, \widehat{\mathbf{p}})|\mathbf{p}'\rangle\psi(\mathbf{p}')\, d^3\mathbf{p}' d^3\mathbf{p} = \big(\psi, H(\widehat{\mathbf{r}}, \widehat{\mathbf{p}})\psi\big) = \big\langle H(\widehat{\mathbf{r}}, \widehat{\mathbf{p}})\big\rangle_\psi. \tag{2m}$$

In classical Hamiltonian field theory, *the antisymmetric Poisson bracket of two functionals* $\mathcal{F}_1[\chi(\mathbf{p}), \phi(\mathbf{p})]$ *and* $\mathcal{F}_2[\chi(\mathbf{p}), \phi(\mathbf{p})]$ *of the canonical field pair* $[\chi(\mathbf{p}), \phi(\mathbf{p})]$ *is*,

$$\{\mathcal{F}_1, \mathcal{F}_2\} = \int\big[\big(\delta\mathcal{F}_1/\delta\chi(\mathbf{p}'')\big)\big(\delta\mathcal{F}_2/\delta\phi(\mathbf{p}'')\big) - \big(\delta\mathcal{F}_2/\delta\chi(\mathbf{p}'')\big)\big(\delta\mathcal{F}_1/\delta\phi(\mathbf{p}'')\big)\big]d^3\mathbf{p}''. \tag{3a}$$

Each *member* of the canonical field pair $[\chi(\mathbf{p}), \phi(\mathbf{p})]$ *is itself such a functional because*,

$$\chi(\mathbf{p}) = \int \delta^{(3)}(\mathbf{p} - \mathbf{p}'')\chi(\mathbf{p}'')\, d^3\mathbf{p}'', \quad \phi(\mathbf{p}) = \int \delta^{(3)}(\mathbf{p} - \mathbf{p}'')\phi(\mathbf{p}'')\, d^3\mathbf{p}'', \tag{3b}$$

which implies that,

$$\delta\chi(\mathbf{p})/\delta\chi(\mathbf{p}'') = \delta^{(3)}(\mathbf{p} - \mathbf{p}''), \quad \delta\phi(\mathbf{p})/\delta\phi(\mathbf{p}'') = \delta^{(3)}(\mathbf{p} - \mathbf{p}''), \quad \delta\chi(\mathbf{p})/\delta\phi(\mathbf{p}'') = 0, \quad \delta\phi(\mathbf{p})/\delta\chi(\mathbf{p}'') = 0. \tag{3c}$$

From Eqs. (3a) and (3c), the Poisson-bracket relations of the canonical field pair $[\chi(\mathbf{p}), \phi(\mathbf{p})]$ are,

$$\{\chi(\mathbf{p}), \chi(\mathbf{p}')\} = \{\phi(\mathbf{p}), \phi(\mathbf{p}')\} = 0, \quad \{\chi(\mathbf{p}), \phi(\mathbf{p}')\} = \delta^{(3)}(\mathbf{p} - \mathbf{p}'), \quad \{\phi(\mathbf{p}), \chi(\mathbf{p}')\} = -\delta^{(3)}(\mathbf{p} - \mathbf{p}'). \tag{3d}$$

Eqs. (3d) and (2l) then yield the Poisson-bracket relations of $\psi(\mathbf{p})$ and $\psi^*(\mathbf{p})$ as follows,

$$\{\psi(\mathbf{p}), \psi(\mathbf{p}')\} = \big\{\big((\chi(\mathbf{p}) + i\phi(\mathbf{p}))/\sqrt{2\hbar}\big), \big((\chi(\mathbf{p}') + i\phi(\mathbf{p}'))/\sqrt{2\hbar}\big)\big\} = (1/(2\hbar))\big[\{\chi(\mathbf{p}), \chi(\mathbf{p}')\} - \{\phi(\mathbf{p}), \phi(\mathbf{p}')\} + i\big(\{\chi(\mathbf{p}), \phi(\mathbf{p}')\} + \{\phi(\mathbf{p}), \chi(\mathbf{p}')\}\big)\big] = 0, \tag{3e}$$

and,

$$\{\psi^*(\mathbf{p}), \psi^*(\mathbf{p}')\} = \big\{\big((\chi(\mathbf{p}) - i\phi(\mathbf{p}))/\sqrt{2\hbar}\big), \big((\chi(\mathbf{p}') - i\phi(\mathbf{p}'))/\sqrt{2\hbar}\big)\big\} = (1/(2\hbar))\big[\{\chi(\mathbf{p}), \chi(\mathbf{p}')\} - \{\phi(\mathbf{p}), \phi(\mathbf{p}')\} - i\big(\{\chi(\mathbf{p}), \phi(\mathbf{p}')\} + \{\phi(\mathbf{p}), \chi(\mathbf{p}')\}\big)\big] = 0. \tag{3f}$$

However,

$$\{\psi(\mathbf{p}), \psi^*(\mathbf{p}')\} = \big\{\big((\chi(\mathbf{p}) + i\phi(\mathbf{p}))/\sqrt{2\hbar}\big), \big((\chi(\mathbf{p}') - i\phi(\mathbf{p}'))/\sqrt{2\hbar}\big)\big\} = (1/(2\hbar))\big[\{\chi(\mathbf{p}), \chi(\mathbf{p}')\} + \{\phi(\mathbf{p}), \phi(\mathbf{p}')\} - i\big(\{\chi(\mathbf{p}), \phi(\mathbf{p}')\} - \{\phi(\mathbf{p}), \chi(\mathbf{p}')\}\big)\big] = (-i/\hbar)\,\delta^{(3)}(\mathbf{p} - \mathbf{p}'). \tag{3g}$$

Quantization of $[\chi(\mathbf{p}),\phi(\mathbf{p})]$ *converts these real-valued classical fields into* $\mathbf{p}$*-parameterized Hermitian operators* $(\widehat{\chi}(\mathbf{p}),\widehat{\phi}(\mathbf{p}))$ *whose commutator values equal* $(i\hbar)$ *times their corresponding classical Poisson-bracket values*. So from Eq. (3d) we obtain the following *quantized-field canonical commutation relations*,

$$\big[\widehat{\chi}(\mathbf{p}),\widehat{\chi}(\mathbf{p}')\big]=\big[\widehat{\phi}(\mathbf{p}),\widehat{\phi}(\mathbf{p}')\big]=0,\quad \big[\widehat{\chi}(\mathbf{p}),\widehat{\phi}(\mathbf{p}')\big]=i\hbar\delta^{(3)}(\mathbf{p}-\mathbf{p}'),\quad \big[\widehat{\phi}(\mathbf{p}),\widehat{\chi}(\mathbf{p}')\big]=-i\hbar\delta^{(3)}(\mathbf{p}-\mathbf{p}'). \tag{4a}$$

From Eqs. (4a) and (2l), *or directly from* Eqs. (3e)–(3g), we obtain *the equivalent* $\widehat{\psi}(\mathbf{p})$ *commutation relations*,

$$\big[\widehat{\psi}(\mathbf{p}),\widehat{\psi}(\mathbf{p}')\big]=\big[\widehat{\psi}^{\,\dagger}(\mathbf{p}),\widehat{\psi}^{\,\dagger}(\mathbf{p}')\big]=0,\quad \big[\widehat{\psi}(\mathbf{p}),\widehat{\psi}^{\,\dagger}(\mathbf{p}')\big]=\delta^{(3)}(\mathbf{p}-\mathbf{p}'). \tag{4b}$$

The Eq. (4b) *commutation relations* impose *Bose-Einstein statistics* on the second quantization of single-particle quantum mechanics. To *instead* impose *Fermi-Dirac statistics* on that second quantization, Eq. (4b) *is replaced* by the following *anticommutation relations*,

$$\widehat{\psi}(\mathbf{p})\widehat{\psi}(\mathbf{p}')+\widehat{\psi}(\mathbf{p}')\widehat{\psi}(\mathbf{p})=\widehat{\psi}^{\,\dagger}(\mathbf{p})\widehat{\psi}^{\,\dagger}(\mathbf{p}')+\widehat{\psi}^{\,\dagger}(\mathbf{p}')\widehat{\psi}^{\,\dagger}(\mathbf{p})=0,\ \ \widehat{\psi}(\mathbf{p})\widehat{\psi}^{\,\dagger}(\mathbf{p}')+\widehat{\psi}^{\,\dagger}(\mathbf{p}')\widehat{\psi}(\mathbf{p})=\delta^{(3)}(\mathbf{p}-\mathbf{p}'). \tag{4c}$$

There is *a key commutation lemma* which is *independent* of whether Eq. (4b) commutation-quantization or Eq. (4c) anticommutation-quantization is selected. *Assuming* Eq. (4b), we have that,

$$\begin{gathered}\big[\widehat{\psi}^{\,\dagger}(\mathbf{p})\widehat{\psi}(\mathbf{p}'),\widehat{\psi}^{\,\dagger}(\mathbf{p}'')\big]=\widehat{\psi}^{\,\dagger}(\mathbf{p})\widehat{\psi}(\mathbf{p}')\widehat{\psi}^{\,\dagger}(\mathbf{p}'')-\widehat{\psi}^{\,\dagger}(\mathbf{p}'')\widehat{\psi}^{\,\dagger}(\mathbf{p})\widehat{\psi}(\mathbf{p}')=\\ \widehat{\psi}^{\,\dagger}(\mathbf{p})\big(\widehat{\psi}(\mathbf{p}')\widehat{\psi}^{\,\dagger}(\mathbf{p}'')-\widehat{\psi}^{\,\dagger}(\mathbf{p}'')\widehat{\psi}(\mathbf{p}')\big)+\big(\widehat{\psi}^{\,\dagger}(\mathbf{p})\widehat{\psi}^{\,\dagger}(\mathbf{p}'')-\widehat{\psi}^{\,\dagger}(\mathbf{p}'')\widehat{\psi}^{\,\dagger}(\mathbf{p})\big)\widehat{\psi}(\mathbf{p}')=\\ \widehat{\psi}^{\,\dagger}(\mathbf{p})\big[\widehat{\psi}(\mathbf{p}'),\widehat{\psi}^{\,\dagger}(\mathbf{p}'')\big]+\big[\widehat{\psi}^{\,\dagger}(\mathbf{p}),\widehat{\psi}^{\,\dagger}(\mathbf{p}'')\big]\widehat{\psi}(\mathbf{p}')=\widehat{\psi}^{\,\dagger}(\mathbf{p})\delta^{(3)}(\mathbf{p}'-\mathbf{p}'').\end{gathered} \tag{4d}$$

Now *assuming* Eq. (4c) *instead of* Eq. (4b), we have that,

$$\begin{gathered}\big[\widehat{\psi}^{\,\dagger}(\mathbf{p})\widehat{\psi}(\mathbf{p}'),\widehat{\psi}^{\,\dagger}(\mathbf{p}'')\big]=\widehat{\psi}^{\,\dagger}(\mathbf{p})\widehat{\psi}(\mathbf{p}')\widehat{\psi}^{\,\dagger}(\mathbf{p}'')-\widehat{\psi}^{\,\dagger}(\mathbf{p}'')\widehat{\psi}^{\,\dagger}(vp)\widehat{\psi}(\mathbf{p}')=\\ \widehat{\psi}^{\,\dagger}(\mathbf{p})\big(\widehat{\psi}(\mathbf{p}')\widehat{\psi}^{\,\dagger}(\mathbf{p}'')+\widehat{\psi}^{\,\dagger}(\mathbf{p}'')\widehat{\psi}(\mathbf{p}')\big)-\big(\widehat{\psi}^{\,\dagger}(\mathbf{p})\widehat{\psi}^{\,\dagger}(\mathbf{p}'')+\widehat{\psi}^{\,\dagger}(\mathbf{p}'')\widehat{\psi}^{\,\dagger}(\mathbf{p})\big)\widehat{\psi}(\mathbf{p}')=\widehat{\psi}^{\,\dagger}(\mathbf{p})\delta^{(3)}(\mathbf{p}'-\mathbf{p}''),\end{gathered} \tag{4e}$$

which is exactly the same result as Eq. (4d). Taking the Hermitian conjugate of this lemma yields,

$$\big[\widehat{\psi}^{\,\dagger}(\mathbf{p})\widehat{\psi}(\mathbf{p}'),\widehat{\psi}^{\,\dagger}(\mathbf{p}'')\big]^{\dagger}=\big[\widehat{\psi}(\mathbf{p}''),\widehat{\psi}^{\,\dagger}(\mathbf{p}')\widehat{\psi}(\mathbf{p})\big]=\widehat{\psi}(\mathbf{p})\delta^{(3)}(\mathbf{p}'-\mathbf{p}''),$$

which by reversing the order of the commutator implies,

$$\big[\widehat{\psi}^{\,\dagger}(\mathbf{p}')\widehat{\psi}(\mathbf{p}),\widehat{\psi}(\mathbf{p}'')\big]=-\widehat{\psi}(\mathbf{p})\delta^{(3)}(\mathbf{p}'-\mathbf{p}''),$$

which upon interchanging the $\mathbf{p}'$ and $\mathbf{p}$ labels becomes,

$$\big[\widehat{\psi}^{\,\dagger}(\mathbf{p})\widehat{\psi}(\mathbf{p}'),\widehat{\psi}(\mathbf{p}'')\big]=-\widehat{\psi}(\mathbf{p}')\delta^{(3)}(\mathbf{p}-\mathbf{p}''). \tag{4f}$$

Eqs. (4d)–(4f) *are key commutation lemmas that are independent of the selection of* Eq. (4b) *commutation-quantization or* Eq. (4c) *anticommutation-quantization*. Since Eqs. (4d)–(4f) *underpin all of the calculations which we shall henceforth undertake here, we needn't here be concerned with whether* Eq. (4b) *commutation-quantization or* Eq. (4c) *anticommutation-quantization is selected*. Eqs. (4d)–(4f) imply *a further very important commutation lemma*. We apply *the commutator identity*,

$$[A,bc]=Abc-bcA=(Ab-bA)c+b(Ac-cA)=[A,b]c+b[A,c], \tag{4g}$$

to the case that,

$$A=\widehat{\psi}^{\,\dagger}(\mathbf{p})\widehat{\psi}(\mathbf{p}'),\ \ b=\widehat{\psi}^{\,\dagger}(\mathbf{p}'')\ \text{ and }\ c=\widehat{\psi}(\mathbf{p}'''),$$

to obtain,

$$\begin{gathered}\big[\widehat{\psi}^{\,\dagger}(\mathbf{p})\widehat{\psi}(\mathbf{p}'),\widehat{\psi}^{\,\dagger}(\mathbf{p}'')\widehat{\psi}(\mathbf{p}''')\big]=\big[\widehat{\psi}^{\,\dagger}(\mathbf{p})\widehat{\psi}(\mathbf{p}'),\widehat{\psi}^{\,\dagger}(\mathbf{p}'')\big]\widehat{\psi}(\mathbf{p}''')+\widehat{\psi}^{\,\dagger}(\mathbf{p}'')\big[\widehat{\psi}^{\,\dagger}(\mathbf{p})\widehat{\psi}(\mathbf{p}'),\widehat{\psi}(\mathbf{p}''')\big]=\\ \widehat{\psi}^{\,\dagger}(\mathbf{p})\widehat{\psi}(\mathbf{p}''')\delta^{(3)}(\mathbf{p}'-\mathbf{p}'')-\widehat{\psi}^{\,\dagger}(\mathbf{p}'')\widehat{\psi}(\mathbf{p}')\delta^{(3)}(\mathbf{p}-\mathbf{p}'''),\end{gathered} \tag{4h}$$

where the last step follows from the commutation lemmas of Eqs. (4d)–(4f).

Conventional quantization of the Eq. (2m) classical Hamiltonian field functional $H_{\mathrm{cl}}[\psi(\mathbf{p}); \langle\mathbf{p}|H(\widehat{\mathbf{r}},\widehat{\mathbf{p}})|\mathbf{p}'\rangle]$ replaces the commuting $\psi^*(\mathbf{p})$ and $\psi(\mathbf{p}')$ in Eq. (2m) *by quantized* $\widehat{\psi}^{\,\dagger}(\mathbf{p})$ *placed left of* $\widehat{\psi}(\mathbf{p}')$. Consequently, *conventional second quantization* $\widehat{H}(\widehat{\mathbf{r}},\widehat{\mathbf{p}})$ *of the single-particle Hamiltonian operator* $H(\widehat{\mathbf{r}},\widehat{\mathbf{p}})$ *is explicitly*,

$$\widehat{H}(\widehat{\mathbf{r}},\widehat{\mathbf{p}}) = \int \widehat{\psi}^{\,\dagger}(\mathbf{p})\langle\mathbf{p}|H(\widehat{\mathbf{r}},\widehat{\mathbf{p}})|\mathbf{p}'\rangle\widehat{\psi}(\mathbf{p}')\, d^3\mathbf{p}' d^3\mathbf{p}. \tag{4i}$$

Eq. (4i) is the prototype *for conventional second quantization* $\widehat{F}(\widehat{\mathbf{r}},\widehat{\mathbf{p}})$ *of any function* $F(\widehat{\mathbf{r}},\widehat{\mathbf{p}})$ *of the single-particle quantized vector canonical pair* $(\widehat{\mathbf{r}},\widehat{\mathbf{p}})$*; that conventional second quantization of* $F(\widehat{\mathbf{r}},\widehat{\mathbf{p}})$ *is explicitly*,

$$\widehat{F}(\widehat{\mathbf{r}},\widehat{\mathbf{p}}) = \int \widehat{\psi}^{\,\dagger}(\mathbf{p})\langle\mathbf{p}|F(\widehat{\mathbf{r}},\widehat{\mathbf{p}})|\mathbf{p}'\rangle\widehat{\psi}(\mathbf{p}')\, d^3\mathbf{p}' d^3\mathbf{p}. \tag{4j}$$

Given the Eq. (4i) second-quantized Hamiltonian $\widehat{H}(\widehat{\mathbf{r}},\widehat{\mathbf{p}})$, the quantized field $\widehat{\psi}(\mathbf{p})$ *still ought to satisfy an equation of motion which reflects the single-particle Schrödinger equation that pertains to* $H(\widehat{\mathbf{r}},\widehat{\mathbf{p}})$ *and* $\psi(\mathbf{p})$. Of course the single-particle Schrödinger equation that pertains to $H(\widehat{\mathbf{r}},\widehat{\mathbf{p}})$ and $\psi(\mathbf{p})$ is Eq. (2c), i.e.,

$$i\hbar\, d\psi(\mathbf{p})/dt = \int \langle\mathbf{p}|H(\widehat{\mathbf{r}},\widehat{\mathbf{p}})|\mathbf{p}'\rangle\psi(\mathbf{p}')\, d^3\mathbf{p}'. \tag{4k}$$

The *time derivative of the quantized field* $\widehat{\psi}(\mathbf{p})$, however, *is given by the Heisenberg equation of motion which pertains to the second-quantized Hamiltonian* $\widehat{H}(\widehat{\mathbf{r}},\widehat{\mathbf{p}})$ *of* Eq. (4i) *and* $\widehat{\psi}(\mathbf{p})$, namely by,

$$d\widehat{\psi}(\mathbf{p})/dt = (-i/\hbar)[\widehat{\psi}(\mathbf{p}), \widehat{H}(\widehat{\mathbf{r}},\widehat{\mathbf{p}})] = (+i/\hbar)[\widehat{H}(\widehat{\mathbf{r}},\widehat{\mathbf{p}}), \widehat{\psi}(\mathbf{p})]. \tag{4l}$$

From Eq. (4l) and the Eq. (4i) *integral form of* $\widehat{H}(\widehat{\mathbf{r}},\widehat{\mathbf{p}})$, together with *the* Eq. (4f) *lemma*, we obtain,

$$\begin{gathered} d\widehat{\psi}(\mathbf{p})/dt = (+i/\hbar) \int [\widehat{\psi}^{\,\dagger}(\mathbf{p}'')\widehat{\psi}(\mathbf{p}'), \widehat{\psi}(\mathbf{p})]\langle\mathbf{p}''|H(\widehat{\mathbf{r}},\widehat{\mathbf{p}})|\mathbf{p}'\rangle\, d^3\mathbf{p}' d^3\mathbf{p}'' = \\ (-i/\hbar)\int \widehat{\psi}(\mathbf{p}')\delta^{(3)}(\mathbf{p}''-\mathbf{p})\langle\mathbf{p}''|H(\widehat{\mathbf{r}},\widehat{\mathbf{p}})|\mathbf{p}'\rangle\, d^3\mathbf{p}' d^3\mathbf{p}'' = (-i/\hbar)\int\langle\mathbf{p}|H(\widehat{\mathbf{r}},\widehat{\mathbf{p}})|\mathbf{p}'\rangle\widehat{\psi}(\mathbf{p}')\, d^3\mathbf{p}', \end{gathered} \tag{4m}$$

which implies,

$$i\hbar\, d\widehat{\psi}(\mathbf{p})/dt = \int\langle\mathbf{p}|H(\widehat{\mathbf{r}},\widehat{\mathbf{p}})|\mathbf{p}'\rangle\widehat{\psi}(\mathbf{p}')\, d^3\mathbf{p}'. \tag{4n}$$

Eq. (4n) *indeed reflects the single-particle Schrödinger equation given by* Eq. (4k) (and also by Eq. (2c)).

Regarding the Eq. (4j) second quantization of functions $F(\widehat{\mathbf{r}},\widehat{\mathbf{p}})$ of the single-particle quantized vector canonical pair $(\widehat{\mathbf{r}},\widehat{\mathbf{p}})$, *the most basic such functions are nonzero constants*. We obviously *don't need to separately consider cases* where the nonzero constant *isn't unity*, and since $\langle\mathbf{p}|1|\mathbf{p}'\rangle = \delta^{(3)}(\mathbf{p}-\mathbf{p}')$,

$$\widehat{1} = \int \widehat{\psi}^{\,\dagger}(\mathbf{p})\widehat{\psi}(\mathbf{p})\, d^3\mathbf{p}. \tag{5a}$$

The commutators of the operator $\widehat{1}$ *with products of* n *quantized* $\widehat{\psi}^{\,\dagger}(\mathbf{p}_k)$ *fields,* $k = 1, 2, \ldots,$ *are of special interest*. When $n = 1$, the commutator is obtained from the lemma of Eqs. (4d) and (4e),

$$\big[\,\widehat{1},\, \widehat{\psi}^{\,\dagger}(\mathbf{p}_k)\big] = \int [\widehat{\psi}^{\,\dagger}(\mathbf{p})\widehat{\psi}(\mathbf{p}), \widehat{\psi}^{\,\dagger}(\mathbf{p}_k)]\, d^3\mathbf{p} = \int \widehat{\psi}^{\,\dagger}(\mathbf{p})\delta^{(3)}(\mathbf{p}-\mathbf{p}_k)\, d^3\mathbf{p} = \widehat{\psi}^{\,\dagger}(\mathbf{p}_k). \tag{5b}$$

An "n implies $(n+1)$" step is set up by applying the Eq. (4g) commutator identity and Eq. (5b), i.e.,

$$\begin{gathered} \big[\,\widehat{1},\, \big(\widehat{\psi}^{\,\dagger}(\mathbf{p}_1)\cdots\widehat{\psi}^{\,\dagger}(\mathbf{p}_n)\widehat{\psi}^{\,\dagger}(\mathbf{p}_{n+1})\big)\big] = \big[\,\widehat{1},\, \big(\widehat{\psi}^{\,\dagger}(\mathbf{p}_1)\cdots\widehat{\psi}^{\,\dagger}(\mathbf{p}_n)\big)\widehat{\psi}^{\,\dagger}(\mathbf{p}_{n+1})\big] = \\ \big[\,\widehat{1},\, \big(\widehat{\psi}^{\,\dagger}(\mathbf{p}_1)\cdots\widehat{\psi}^{\,\dagger}(\mathbf{p}_n)\big)\big]\widehat{\psi}^{\,\dagger}(\mathbf{p}_{n+1}) + \big(\widehat{\psi}^{\,\dagger}(\mathbf{p}_1)\cdots\widehat{\psi}^{\,\dagger}(\mathbf{p}_n)\big)\big[\,\widehat{1},\, \widehat{\psi}^{\,\dagger}(\mathbf{p}_{n+1})\big] = \\ \big[\,\widehat{1},\, \big(\widehat{\psi}^{\,\dagger}(\mathbf{p}_1)\cdots\widehat{\psi}^{\,\dagger}(\mathbf{p}_n)\big)\big]\widehat{\psi}^{\,\dagger}(\mathbf{p}_{n+1}) + \big(\widehat{\psi}^{\,\dagger}(\mathbf{p}_1)\cdots\widehat{\psi}^{\,\dagger}(\mathbf{p}_n)\big)\widehat{\psi}^{\,\dagger}(\mathbf{p}_{n+1}), \end{gathered} \tag{5c}$$

where Eq. (5b) was applied to obtain the last equality. We *now assume the commutator result*,

$$\big[\,\widehat{1}, \big(\widehat{\psi}^{\,\dagger}(\mathbf{p}_1)\cdots\widehat{\psi}^{\,\dagger}(\mathbf{p}_n)\big)\big] = n\big(\widehat{\psi}^{\,\dagger}(\mathbf{p}_1)\cdots\widehat{\psi}^{\,\dagger}(\mathbf{p}_n)\big), \tag{5d}$$

which Eq. (5b) *shows is true for* $n = 1$. Substituting Eq. (5d) into the last line of Eq. (5c) yields,

$$\begin{gathered} \big[\,\widehat{1},\, \big(\widehat{\psi}^{\,\dagger}(\mathbf{p}_1)\cdots\widehat{\psi}^{\,\dagger}(\mathbf{p}_n)\widehat{\psi}^{\,\dagger}(\mathbf{p}_{n+1})\big)\big] = \\ n\big(\widehat{\psi}^{\,\dagger}(\mathbf{p}_1)\cdots\widehat{\psi}^{\,\dagger}(\mathbf{p}_n)\big)\widehat{\psi}^{\,\dagger}(\mathbf{p}_{n+1}) + \big(\widehat{\psi}^{\,\dagger}(\mathbf{p}_1)\cdots\widehat{\psi}^{\,\dagger}(\mathbf{p}_n)\big)\widehat{\psi}^{\,\dagger}(\mathbf{p}_{n+1}) = \\ (n+1)\big(\widehat{\psi}^{\,\dagger}(\mathbf{p}_1)\cdots\widehat{\psi}^{\,\dagger}(\mathbf{p}_n)\big)\widehat{\psi}^{\,\dagger}(\mathbf{p}_{n+1}) = (n+1)\big(\widehat{\psi}^{\,\dagger}(\mathbf{p}_1)\cdots\widehat{\psi}^{\,\dagger}(\mathbf{p}_n)\widehat{\psi}^{\,\dagger}(\mathbf{p}_{n+1})\big), \end{gathered} \tag{5e}$$

which shows that if Eq. (5d) is true for n, then it is true for $(n+1)$, completing its proof by induction.

The *zero-particle state* $|0\rangle$ is such that $\widehat{\psi}(\mathbf{p})|0\rangle = 0$ for every $\mathbf{p}$, so from the Eq. (5a) rendition of $\widehat{1}$, $\widehat{1}\,|0\rangle = 0$ *as well*. That property of $\widehat{1}$, combined with its induction-proved Eq. (5d) commutator result, yields,

$$\widehat{1}\big(\widehat{\psi}^{\,\dagger}(\mathbf{p}_1)\cdots\widehat{\psi}^{\,\dagger}(\mathbf{p}_n)\big)|0\rangle = n\big(\widehat{\psi}^{\,\dagger}(\mathbf{p}_1)\cdots\widehat{\psi}^{\,\dagger}(\mathbf{p}_n)\big)|0\rangle. \tag{5f}$$

Since a $\widehat{\psi}^{\,\dagger}(\mathbf{p}_k)$ field *creates a momentum-*$\mathbf{p}_k$ *particle*, the state $\big(\widehat{\psi}^{\,\dagger}(\mathbf{p}_1)\cdots\widehat{\psi}^{\,\dagger}(\mathbf{p}_n)\big)|0\rangle$ *has* n *particles*, and Eq. (5f) shows that $\widehat{1}$ *yields that particle number*. Therefore $\widehat{1}$ is referred to as *the particle number operator*, and *it is customarily denoted as* $\widehat{N}$.

We next apply the Eq. (4h) commutation lemma to show that *given two functions* $F_1(\widehat{\mathbf{r}},\widehat{\mathbf{p}})$ *and* $F_2(\widehat{\mathbf{r}},\widehat{\mathbf{p}})$ *of the single-particle quantized vector canonical pair* $(\widehat{\mathbf{r}},\widehat{\mathbf{p}})$, the commutator of their second quantizations, i.e., $\big[\widehat{F}_1(\widehat{\mathbf{r}},\widehat{\mathbf{p}}),\, \widehat{F}_2(\widehat{\mathbf{r}},\widehat{\mathbf{p}})\big]$, *equals the second quantization of their single-particle commutator* $\big[F_1(\widehat{\mathbf{r}},\widehat{\mathbf{p}}),\, F_2(\widehat{\mathbf{r}},\widehat{\mathbf{p}})\big]$,

$$\begin{gathered}
\big[\widehat{F}_1(\widehat{\mathbf{r}},\widehat{\mathbf{p}}),\, \widehat{F}_2(\widehat{\mathbf{r}},\widehat{\mathbf{p}})\big] = \\
\big[\textstyle\int \widehat{\psi}^{\,\dagger}(\mathbf{p})\langle\mathbf{p}|F_1(\widehat{\mathbf{r}},\widehat{\mathbf{p}})|\mathbf{p}'\rangle\widehat{\psi}(\mathbf{p}')\, d^3\mathbf{p}' d^3\mathbf{p},\ \int \widehat{\psi}^{\,\dagger}(\mathbf{p}'')\langle\mathbf{p}''|F_2(\widehat{\mathbf{r}},\widehat{\mathbf{p}})|\mathbf{p}'''\rangle\widehat{\psi}(\mathbf{p}''')\, d^3\mathbf{p}''' d^3\mathbf{p}''\big] = \\
\textstyle\int d^3\mathbf{p}\, d^3\mathbf{p}' d^3\mathbf{p}'' d^3\mathbf{p}'''\langle\mathbf{p}|F_1(\widehat{\mathbf{r}},\widehat{\mathbf{p}})|\mathbf{p}'\rangle\langle\mathbf{p}''|F_2(\widehat{\mathbf{r}},\widehat{\mathbf{p}})|\mathbf{p}'''\rangle\big[\widehat{\psi}^{\,\dagger}(\mathbf{p})\widehat{\psi}(\mathbf{p}'),\, \widehat{\psi}^{\,\dagger}(\mathbf{p}'')\widehat{\psi}(\mathbf{p}''')\big] = \\
\textstyle\int d^3\mathbf{p}\, d^3\mathbf{p}' d^3\mathbf{p}'' d^3\mathbf{p}'''\langle\mathbf{p}|F_1(\widehat{\mathbf{r}},\widehat{\mathbf{p}})|\mathbf{p}'\rangle\langle\mathbf{p}''|F_2(\widehat{\mathbf{r}},\widehat{\mathbf{p}})|\mathbf{p}'''\rangle\times \\
\big(\widehat{\psi}^{\,\dagger}(\mathbf{p})\widehat{\psi}(\mathbf{p}''')\delta^{(3)}(\mathbf{p}'-\mathbf{p}'') - \widehat{\psi}^{\,\dagger}(\mathbf{p}'')\widehat{\psi}(\mathbf{p}')\delta^{(3)}(\mathbf{p}-\mathbf{p}''')\big) = \\
\big(\textstyle\int d^3\mathbf{p}\, d^3\mathbf{p}'''\widehat{\psi}^{\,\dagger}(\mathbf{p})\widehat{\psi}(\mathbf{p}''')\int d^3\mathbf{p}'\langle\mathbf{p}|F_1(\widehat{\mathbf{r}},\widehat{\mathbf{p}})|\mathbf{p}'\rangle\langle\mathbf{p}'|F_2(\widehat{\mathbf{r}},\widehat{\mathbf{p}})|\mathbf{p}'''\rangle - \\
\textstyle\int d^3\mathbf{p}''\, d^3\mathbf{p}'\widehat{\psi}^{\,\dagger}(\mathbf{p}'')\widehat{\psi}(\mathbf{p}')\int d^3\mathbf{p}\langle\mathbf{p}''|F_2(\widehat{\mathbf{r}},\widehat{\mathbf{p}})|\mathbf{p}\rangle\langle\mathbf{p}|F_1(\widehat{\mathbf{r}},\widehat{\mathbf{p}})|\mathbf{p}'\rangle\big) = \\
\textstyle\int d^3\mathbf{p}\, d^3\mathbf{p}'''\widehat{\psi}^{\,\dagger}(\mathbf{p})\widehat{\psi}(\mathbf{p}''')\langle\mathbf{p}|F_1(\widehat{\mathbf{r}},\widehat{\mathbf{p}})F_2(\widehat{\mathbf{r}},\widehat{\mathbf{p}})|\mathbf{p}'''\rangle - \int d^3\mathbf{p}''\, d^3\mathbf{p}'\widehat{\psi}^{\,\dagger}(\mathbf{p}'')\widehat{\psi}(\mathbf{p}')\langle\mathbf{p}''|F_2(\widehat{\mathbf{r}},\widehat{\mathbf{p}})F_1(\widehat{\mathbf{r}},\widehat{\mathbf{p}})|\mathbf{p}'\rangle = \\
\textstyle\int \widehat{\psi}^{\,\dagger}(\mathbf{p})\langle\mathbf{p}|\big(F_1(\widehat{\mathbf{r}},\widehat{\mathbf{p}})F_2(\widehat{\mathbf{r}},\widehat{\mathbf{p}}) - F_2(\widehat{\mathbf{r}},\widehat{\mathbf{p}})F_1(\widehat{\mathbf{r}},\widehat{\mathbf{p}})\big)|\mathbf{p}'\rangle\widehat{\psi}(\mathbf{p}')\, d^3\mathbf{p}' d^3\mathbf{p} = \\
\textstyle\int \widehat{\psi}^{\,\dagger}(\mathbf{p})\langle\mathbf{p}|\big[F_1(\widehat{\mathbf{r}},\widehat{\mathbf{p}}),\, F_2(\widehat{\mathbf{r}},\widehat{\mathbf{p}})\big]|\mathbf{p}'\rangle\widehat{\psi}(\mathbf{p}')\, d^3\mathbf{p}' d^3\mathbf{p},
\end{gathered} \tag{6a}$$

so the Eq. (4h) lemma *indeed* establishes that the commutator of the second quantizations of $F_1(\widehat{\mathbf{r}},\widehat{\mathbf{p}})$ and $F_2(\widehat{\mathbf{r}},\widehat{\mathbf{p}})$, i.e., $\big[\widehat{F}_1(\widehat{\mathbf{r}},\widehat{\mathbf{p}}),\, \widehat{F}_2(\widehat{\mathbf{r}},\widehat{\mathbf{p}})\big]$, equals the second quantization of these two functions' single-particle commutator $\big[F_1(\widehat{\mathbf{r}},\widehat{\mathbf{p}}),\, F_2(\widehat{\mathbf{r}},\widehat{\mathbf{p}})\big]$. *In particular*, for *the commutators of the components of* $\widehat{\widehat{\mathbf{r}}}$ *and* $\widehat{\widehat{\mathbf{p}}}$*, since*,

$$\big[(\widehat{\mathbf{r}})^i, (\widehat{\mathbf{p}})^j\big] = i\hbar\delta_{ij}, \tag{6b}$$

$\big($*a vector echo* of the Eq. (4a) field commutators $[\widehat{\chi}(\mathbf{p}), \widehat{\phi}(\mathbf{p}')] = i\hbar\delta^{(3)}(\mathbf{p}-\mathbf{p}')\big)$ the Eq. (6a) result *implies*,

$$\big[\big(\widehat{\widehat{\mathbf{r}}}\big)^i, \big(\widehat{\widehat{\mathbf{p}}}\big)^j\big] = i\hbar\delta_{ij}\widehat{1} = i\hbar\delta_{ij}\widehat{N}. \tag{6c}$$

Well-known *standard inequality theorems* applied to the Eq. (6c) commutation result *further imply*,

$$\big\langle\big(\big(\Delta\widehat{\widehat{\mathbf{r}}}\big)^i\big)^2\big\rangle^{\frac{1}{2}}\big\langle\big(\big(\Delta\widehat{\widehat{\mathbf{p}}}\big)^j\big)^2\big\rangle^{\frac{1}{2}} \geq (1/2)\hbar\delta_{ij}\langle\widehat{N}\rangle. \tag{6d}$$

Squaring both sides of Eq. (6d) produces,

$$\big\langle\big(\big(\Delta\widehat{\widehat{\mathbf{r}}}\big)^i\big)^2\big\rangle\big\langle\big(\big(\Delta\widehat{\widehat{\mathbf{p}}}\big)^j\big)^2\big\rangle \geq (1/4)\hbar^2\delta_{ij}\langle\widehat{N}\rangle^2. \tag{6e}$$

Summing both sides of Eq. (6e) over the three values of the index i yields,

$$\big\langle\big|\Delta\widehat{\widehat{\mathbf{r}}}\big|^2\big\rangle\big\langle\big(\big(\Delta\widehat{\widehat{\mathbf{p}}}\big)^j\big)^2\big\rangle \geq (1/4)\hbar^2\langle\widehat{N}\rangle^2, \tag{6f}$$

and then summing both sides of Eq. (6f) over the three values of the index j yields,

$$\big\langle\big|\Delta\widehat{\widehat{\mathbf{r}}}\big|^2\big\rangle\big\langle\big|\Delta\widehat{\widehat{\mathbf{p}}}\big|^2\big\rangle \geq (3/4)\hbar^2\langle\widehat{N}\rangle^2. \tag{6g}$$

Taking the square root of both sides of Eq. (6g) produces,

$$\big\langle\big|\Delta\widehat{\widehat{\mathbf{r}}}\big|^2\big\rangle^{\frac{1}{2}}\big\langle\big|\Delta\widehat{\widehat{\mathbf{p}}}\big|^2\big\rangle^{\frac{1}{2}} \geq \big(\sqrt{3}/2\big)\hbar\langle\widehat{N}\rangle. \tag{6h}$$

which is the *the multiparticle uncertainty relation* stated in Eq. (1a).