Solving the $n_1 \times n_2 \times n_3$ Points Problem for $n_3 < 6$

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Abstract: In this paper, we show enhanced upper bounds of the nontrivial n_1 × n_2 × n_3 points problem for every n_1 ≤ n_2 ≤ n_3 < 6. We present new patterns that drastically improve the previously known algorithms for finding minimum-link covering paths, solving completely a few cases (e.g., n_1 = n_2 = 3 and n_3 = 4).

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1 Introduction

The $n_1 \times n_2 \times n_3$ points problem [12] is a three-dimensional extension of the classic *nine dots problem* appeared in Samuel Loyd's *Cyclopedia of Puzzles* [1-9], and it is related to the well known NP-hard traveling salesman problem, minimizing the number of turns in the tour instead of the total distance traveled [1-15].
Given $n_1 \cdot n_2 \cdot n_3$ points in $\mathbb{R}^3$, our goal is to visit all of them (at least once) with a polygonal path that has the minimum number of line segments connected at their end-points (links or generically *lines*), the so called Minimum-link Covering Path [3-4-5-8]. In particular, we are interested in the best solutions for the nontrivial $n_1 \times n_2 \times n_3$ dots problem, where (by definition) $1 \leq n_1 \leq n_2 \leq n_3$ and $n_3 < 6$.
Let $h_l(n_1, n_2, n_3) \leq h(n_1, n_2, n_3) \leq h_u(n_1, n_2, n_3)$ be the length of the covering path with the minimum number of links for the $n_1 \times n_2 \times n_3$ points problem, we define the best known upper bound as $h_u(n_1, n_2, n_3) \geq h(n_1, n_2, n_3)$ and we denote as $h_l(n_1, n_2, n_3) \leq h(n_1, n_2, n_3)$ the current proved lower bound [12].
For the simplest cases, the same problem has already been solved [3-12]. Let $n_1 = 1$ and $n_2 < n_3$, we have that $h(n_1, n_2, n_3) = h(n_2) = 2 \cdot n_2 - 1$, while $h(n_1 = 1,\, n_2 = n_3 \geq 3) = 2 \cdot n_2 - 2$ [6]. Hence, for $n_1 = 2$, it can be easily proved that

$$h(2, n_2, n_3) = 2 \cdot h(1, n_2, n_3) + 1 = \begin{cases} 4 \cdot n_2 - 1 & \textbf{\textit{iff}} \quad n_2 < n_3 \\ 4 \cdot n_2 - 3 & \textbf{\textit{iff}} \quad n_2 = n_3 \end{cases} \tag{1}$$

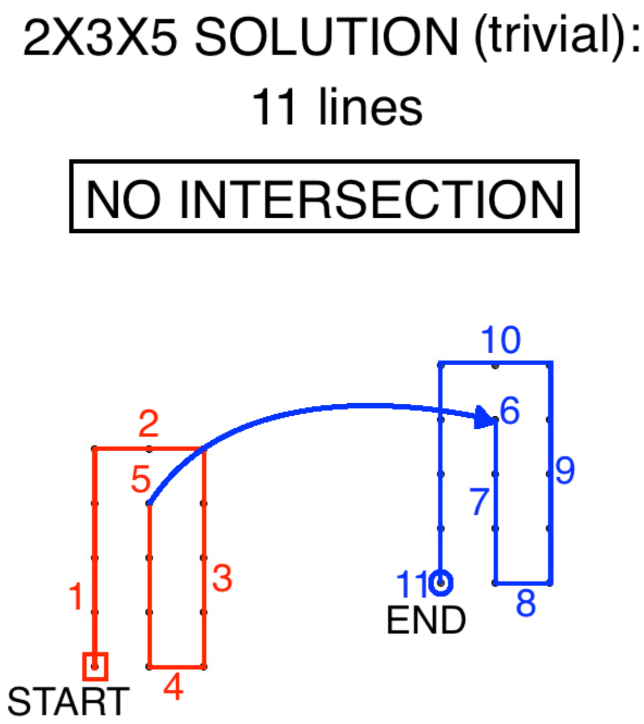

Figure 1. A trivial pattern that completely solves the 2×3×5 points puzzle.

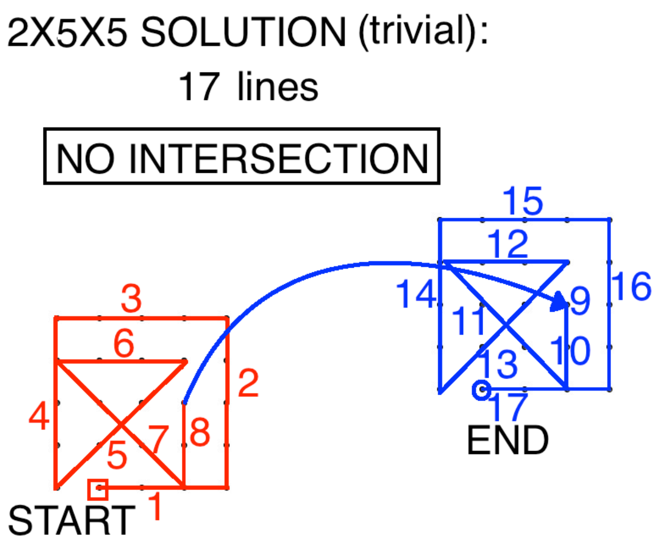

Figure 2. Another example of a trivial case: the 2×5×5 points puzzle.

Therefore, the aim of the present paper is to solve the ten aforementioned nontrivial cases where the current upper bound does not match the proved lower bound.

2 Improving the solution of the $n_1 \times n_2 \times n_3$ points problem for $n_3 < 6$

In this complex brain challenge we need to stretch our pattern recognition [7-10] in order to find a plastic strategy that improves the known upper bounds [3-13] for the most interesting cases (such as the nontrivial $n_1 \times n_2 \times n_2$ points problem and the $n_1 \times n_1 \times (n_1 + 1)$ set of puzzles), avoiding those standardized methods which are based on fixed patterns that lead to suboptimal covering paths, as the approaches presented in [2-8-11].

Let $3 \leq n_1 \leq n_2 \leq n_3 \leq 5$, a lower bound of the $n_1 \times n_2 \times n_3$ problem is given by [12]

$$h_l(n_1, n_2, n_3) = \left\lceil \frac{n_1 \cdot (2 \cdot n_2 \cdot (n_3 + 1) - n_1 - 1) - 2}{n_3 + n_2 - 2} \right\rceil - 1 \qquad (2)$$

The current best results are listed in Table 1, and a direct proof follows for each nontrivial upper bound shown below.

$\mathbf{n_1}$	$\mathbf{n_2}$	$\mathbf{n_3}$	**Best Lower Bound (h_l)**	**Best Upper Bound (h_u)**	**Discovered by**	**Gap ($h_u - h_l$)**
2	2	3	7	**7**	trivial	0
2	3	3	9	**9**	trivial	0
3	3	3	14	**14**	Marco Ripà (proved in 2013 [14])	0
2	2	4	7	**7**	trivial	0
2	3	4	11	**11**	trivial	0
2	4	4	13	**13**	trivial	0
3	3	4	15	**15**	Marco Ripà (proved on Jun. 27, 2019 [v1])	0

3	4	4	17	19	Marco Ripà (ibid.)	2
4	4	4	22	23	Marco Ripà (NNTDM [13])	1
2	2	5	7	**<u>7</u>**	trivial	0
2	3	5	11	**<u>11</u>**	trivial	0
2	4	5	15	**<u>15</u>**	trivial	0
2	5	5	17	**<u>17</u>**	trivial	0
3	3	5	15	16	Marco Ripà (proved on Jun. 27, 2019 [v1])	1
3	4	5	18	20	Marco Ripà (ibid.)	2
3	5	5	20	24	Marco Ripà (ibid.)	4
4	4	5	24	26	Marco Ripà (ibid.)	2
4	5	5	27	31	Marco Ripà (ibid.)	4
5	5	5	33	36	Marco Ripà (proved on Jul. 8, 2019 [v3])	3

Table 1: Current solutions for the $n_1\times n_2\times n_3$ points problem, where $n_1 \leq n_2 \leq n_3 \leq 5$.

Figures 3 to 12 show the patterns used to solve the $n_1 \times n_2 \times n_3$ puzzle (case by case). In particular, by combining the (2) with the original result shown in figure 4, we obtain a formal proof for the 3×3×4 points problem.

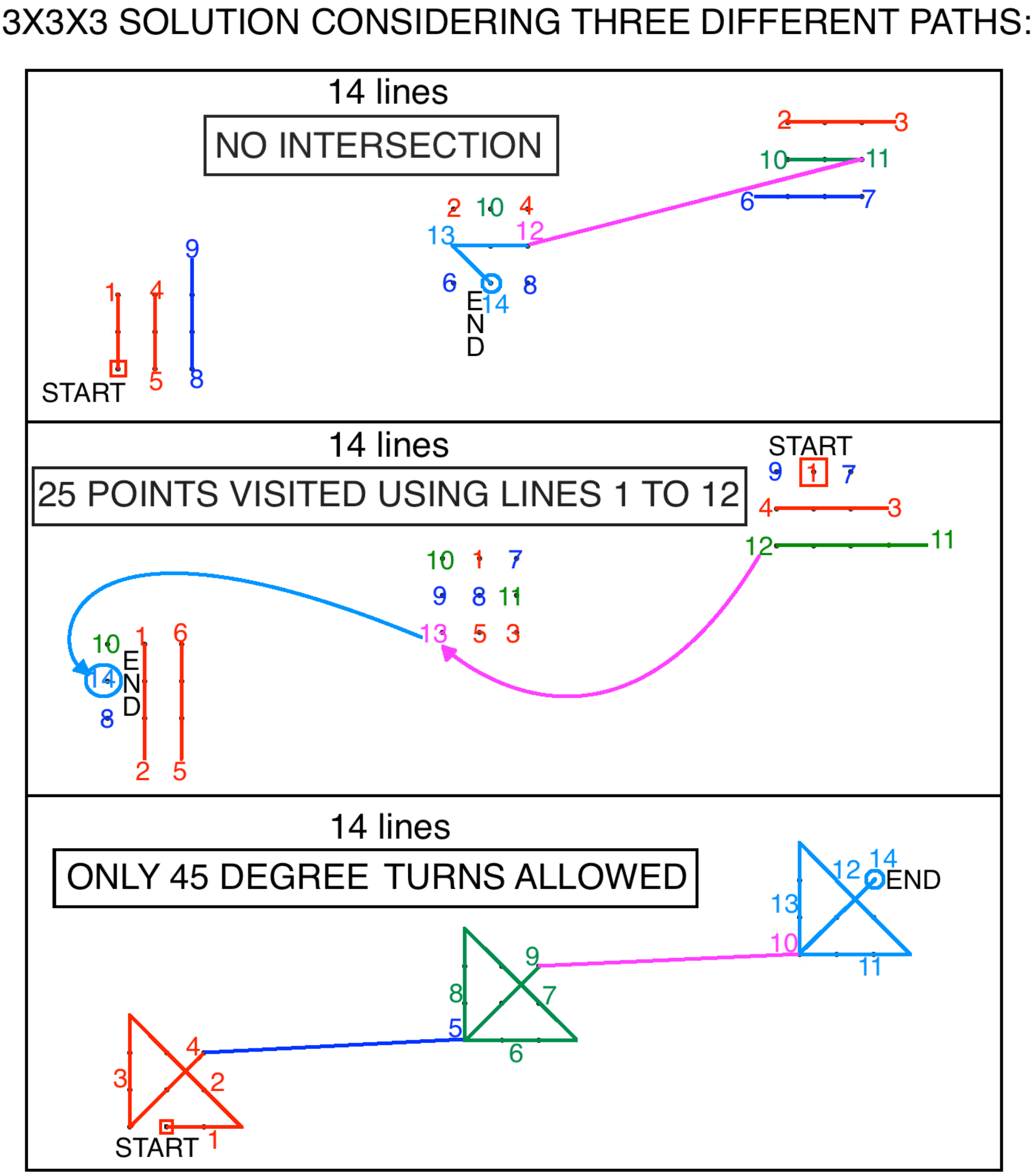

Figure 3. $h_u(3,3,3) = h_l(3,3,3) = 14$. This solution has been proved to be optimal [12-13].

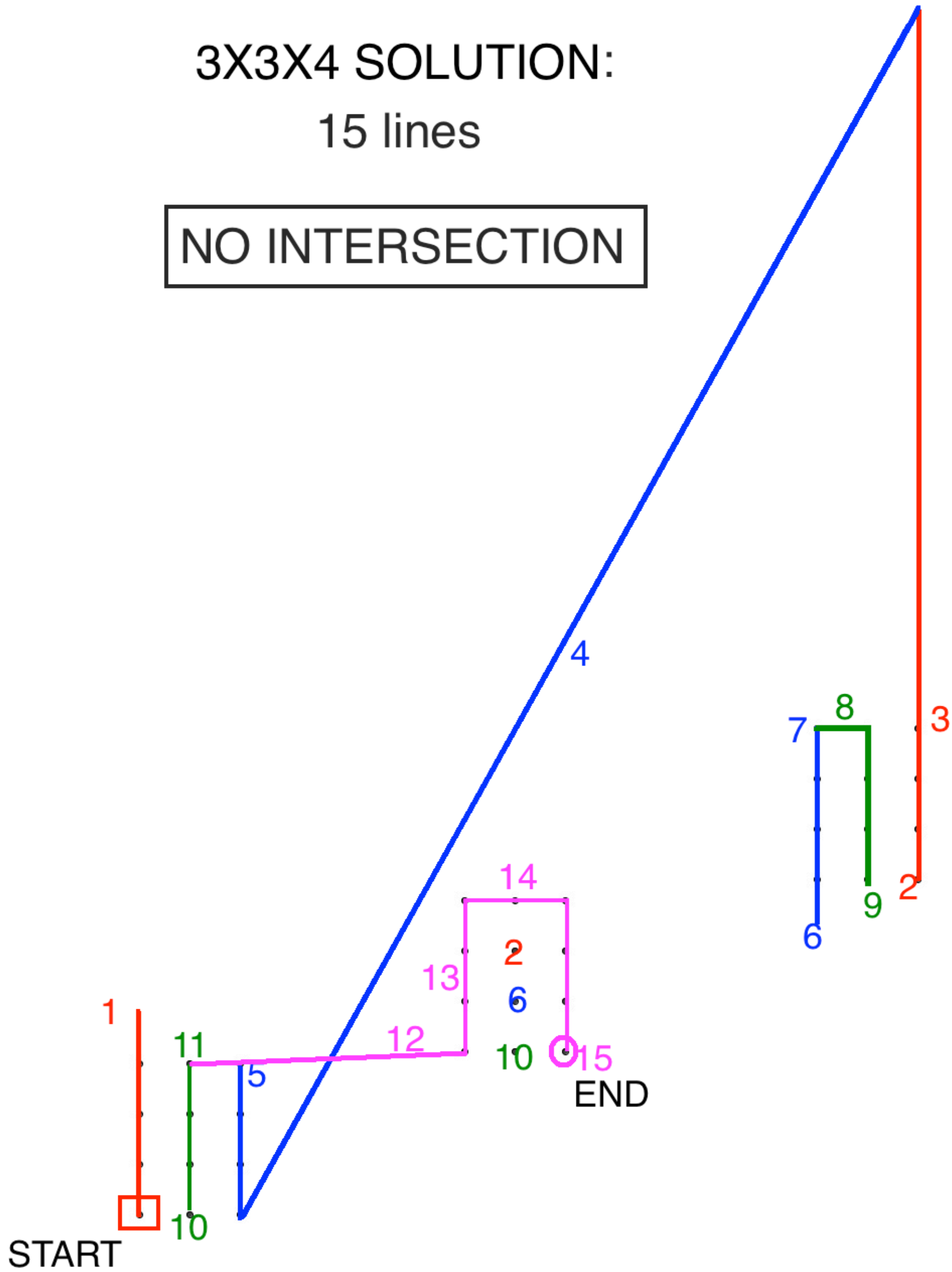

Figure 4. The 3×3×4 puzzle has finally been solved. $h_u = h_l = 15$ and no crossing lines.

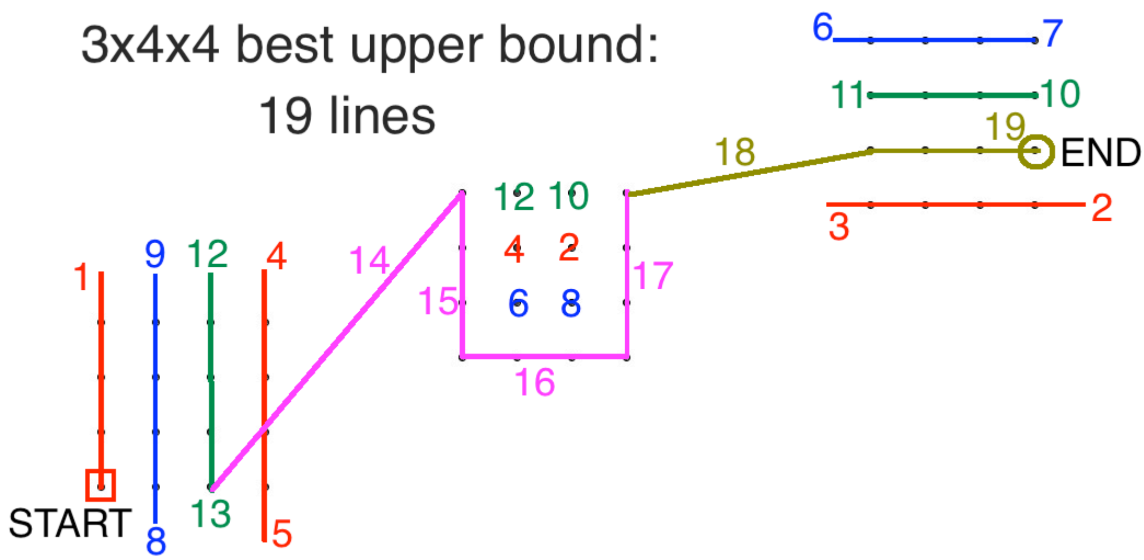

Figure 5. Best known upper bound of the 3×4×4 puzzle. $19 = h_u = h_l + 2$.

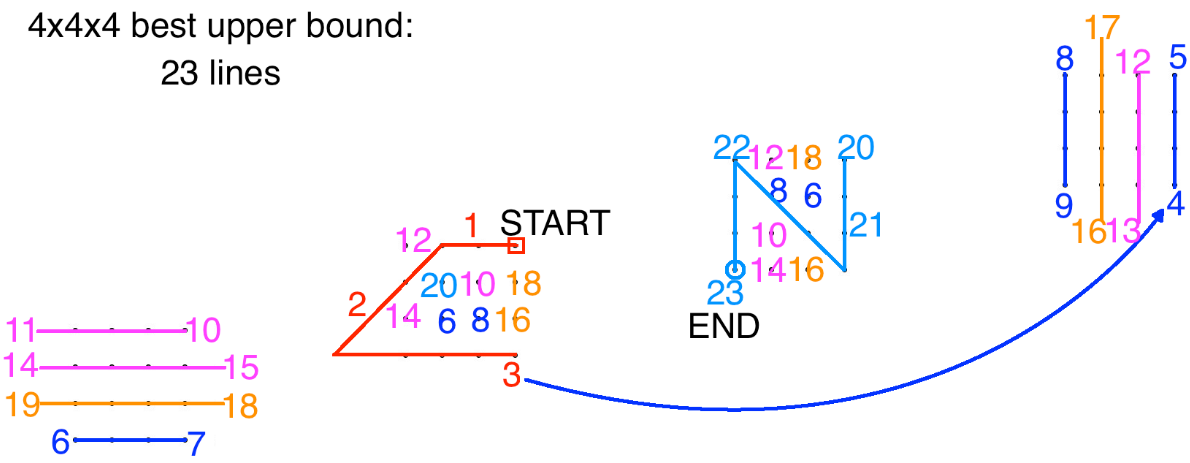

Figure 6. An original pattern for the 4×4×4 puzzle. $23 = h_u = h_l + 1$ [13].

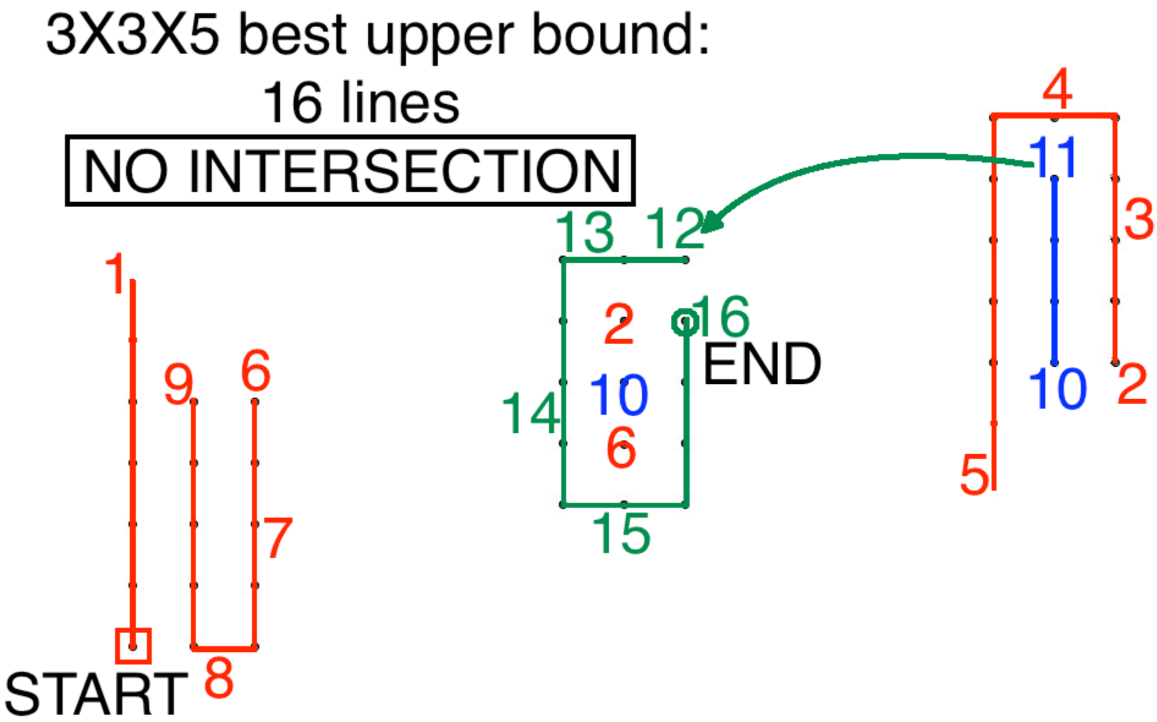

Figure 7. Best known upper bound of the 3×3×5 puzzle. $16 = h_u = h_l + 1$.

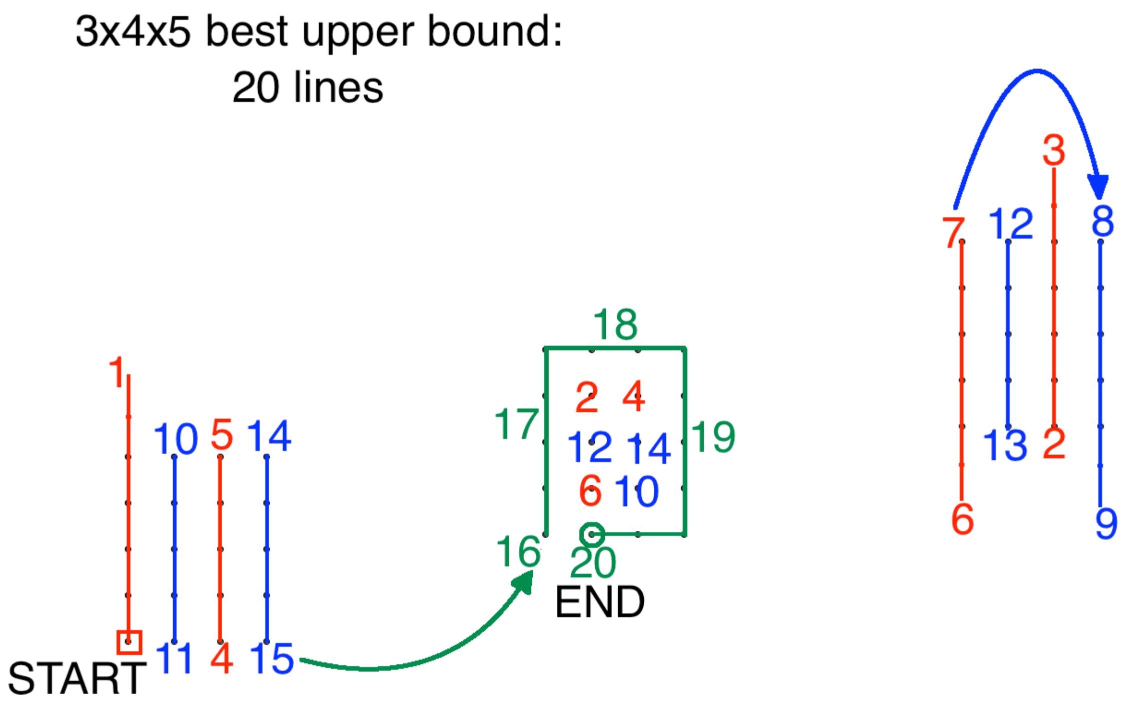

Figure 8. Best known upper bound of the 3×4×5 puzzle. $20 = h_u = h_l + 2$.

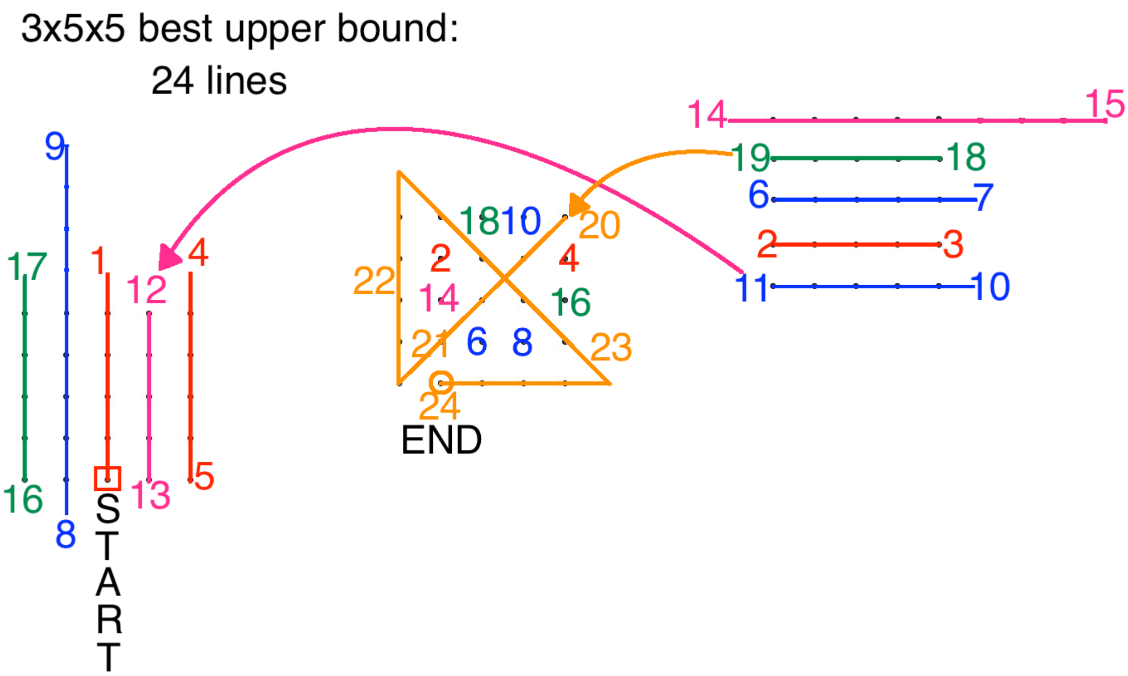

Figure 9. Best known upper bound of the 3×5×5 puzzle. $24 = h_u = h_l + 4$.

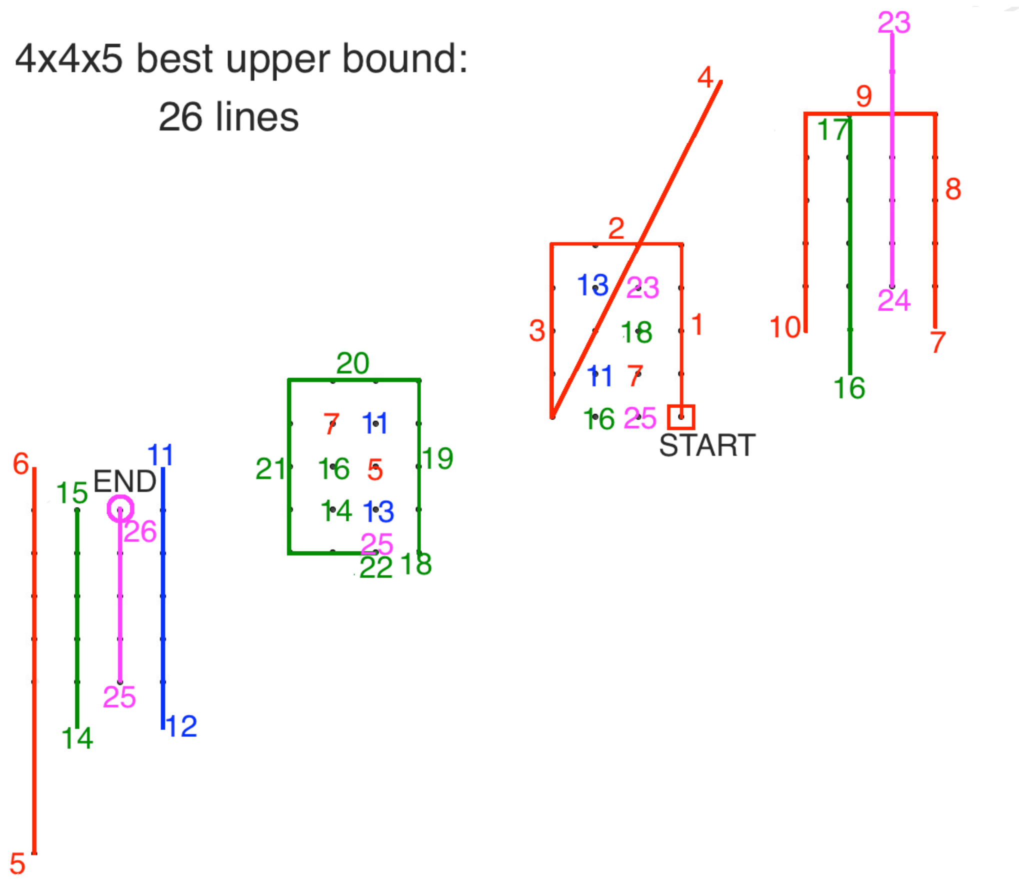

Figure 10. Best known upper bound of the 4×4×5 puzzle. $26 = h_u = h_l + 2$.

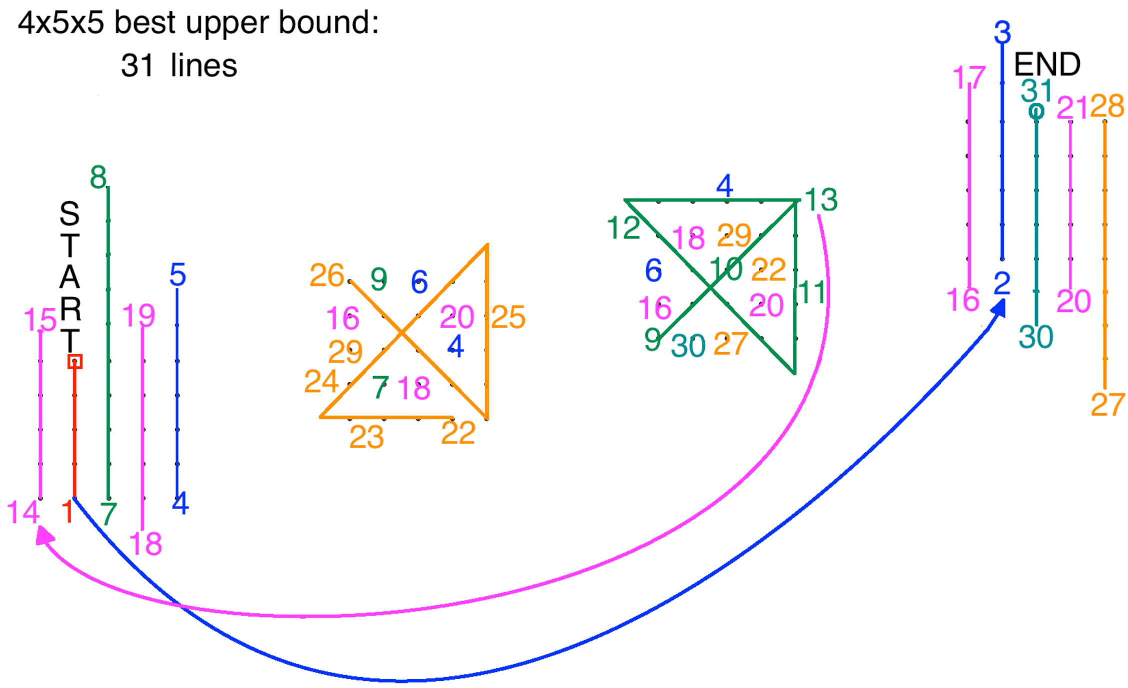

Figure 11. Best known upper bound of the 4×5×5 puzzle. $31 = h_u = h_l + 4$.

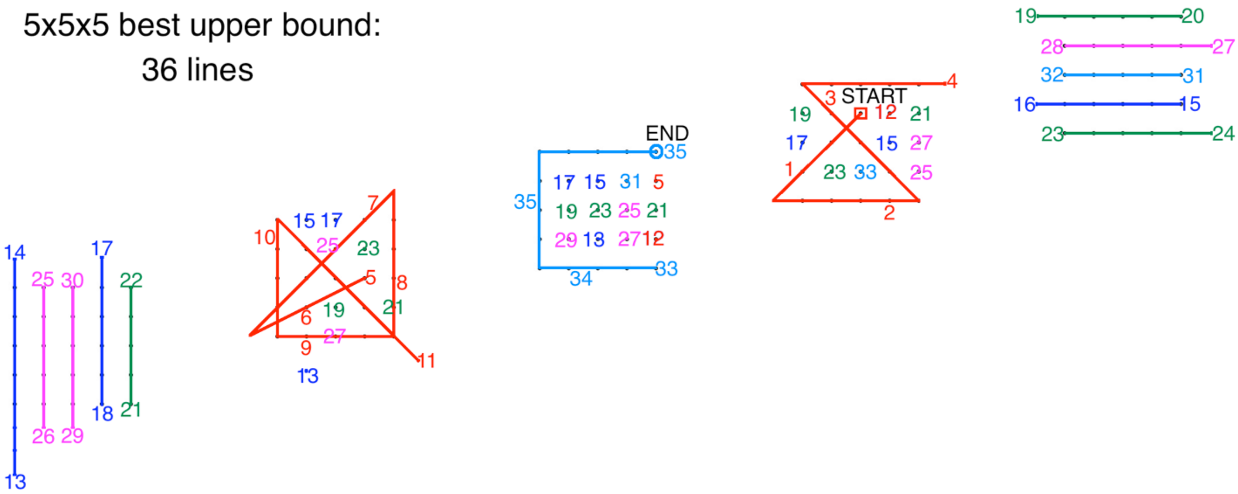

Figure 12. Best known upper bound of the 5×5×5 puzzle. $37 = h_u = h_l + 4$ [13].

Finally, it is interesting to note that the improved $h_u(n_1, n_2, n_3)$ can lower down the upper bound of the generalized k-dimensional puzzle too. As an example, we can apply the aforementioned 3D patterns to the generalized $n_1 \times n_2 \times \ldots \times n_k$ points problem using the simple method described in [12].

Let $k \geq 4$, given $n_k \leq n_{k-1} \leq \cdots \leq n_4 \leq n_1 \leq n_2 \leq n_3$, we can conclude that

$$h_u(n_1, n_2, n_3, \ldots, n_k) = (h_u(n_1, n_2, n_3) + 1) \cdot \prod_{j=4}^{k} n_j - 1 \qquad (3)$$

3 Conclusion

In the present paper we have drastically reduced the gap $h_u(n_1, n_2, n_3) - h_l(n_1, n_2, n_3)$ for every previously unsolved puzzle such that $n_3 < 6$. Moreover, we can easily disprove Bencini's claim that $h_u(3,3,4) = 17 = h_l(3,3,4)$ (see [2], page 7, lines 2-3), since $h_u(3,3,4) = 15 = h_l(3,3,4)$, as shown by combining (2) with the upper bound from figure 4. We do not know if any of the patterns shown in figures 5 to 12 represent optimal solutions, since (by definition) $h_l(n_1, n_2, n_3) \leq h(n_1, n_2, n_3)$. Therefore, some open questions about the $n_1 \times n_2 \times n_3$ points problem remain to be answered, and the research in order to cancel the gap $h_u(n_1, n_2, n_3) - h_l(n_1, n_2, n_3)$, at least for every $n_3 \leq 5$, is not over yet.

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