Form Invariant High Order Differentials

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*Han Xiao completed his PhD in 2011 at Institute of Automation, Chinese Academy of Sciences. This paper is about a research result Han Xiao achieved at the end of 2002 as an undergraduate student at Beihang University, Beijing.

Contents

Abstract

Traditionally, an infinitesimal is regarded as a variable that runs toward 0. Since a differential is a kind of infinitesimal, a differential is essentially a variable running toward 0 too. As a result, differentials are form invariant but not meaning invariant. This paper proposes a Number Field of Ordered Infinitesimals and Infinities (OII Number Field) which can be seen as a kind of extension of real number field. The terminus of a variable running toward 0 is no longer 0, but a point in the OII Number Field, with an Order and a Weight. In this way, the process of running is recorded in the destination, making infinitesimals a kind of number which can be compared and operated easily. On this basis, the differential of a variable is invariant not only in form, but also in meaning. As a differential becomes a variable on another number axis parallel to the real number axis in OII Number Field, a differential can generate differential too, thus giving rise to high order differentials which are also invariant both in form and in meaning.

Index Terms - High Order Differentials; Infinitesimals; Infinities; OII Number Field

1 Introduction

When Newton and Leibniz built the theory of calculus hundreds of years ago, the concept of infinitesimal was not explained clearly. Newton sometimes said infinitesimals are 0, sometimes said they are not, making people bewildered and causing many attacks on calculus. With the development of mathematics, infinitesimals were later defined rigorously, which was the ε-δ definition well known today.

The ε-δ definition is actually saying that an infinitesimal is a variable that runs toward 0. Since a differential is a kind of infinitesimal, it is also a kind of variable that runs toward 0. Suppose $y = f(x)$ and $x = h(t)$. When x is taken as independent variable, the differential of y is

$$\mathrm{d}y = f'(x)\mathrm{d}x\,. \tag{1.1}$$

When t is taken as independent variable, the differential of x is

$$\mathrm{d}x = h'(t)\mathrm{d}t\,, \tag{1.2}$$

and the differential of y is

$$\mathrm{d}y = f'(x)h'(t)\mathrm{d}t = f'(x)\mathrm{d}x\,. \tag{1.3}$$

The dx in Equation (1.1) represents the infinitesimal increment of x itself, but the dx in Equation (1.2) and (1.3) represents the differential of x caused by infinitesimal increment of t. Therefore, dx has two different meanings in the three equations. Similarly, dy also has two different meanings. The dy in Equation (1.1) represents the differential of y caused by infinitesimal increment of x, while dy in Equation (1.3) represents the differential of y caused by infinitesimal increment of t. Although the dy in Equation (1.1) and the dy in Equation (1.3) are both variables running toward 0, they are generally not equal during the process of running. However, the dy in Equation (1.1) and the dy in Equation (1.3) can both be expressed as $f'(x)\mathrm{d}x$, though with different meanings of dx. This phenomenon is called "the form invariance of differentials".

While the dy in Equation (1.1) and the dy in Equation (1.3) are generally not equal during the process of running toward 0, their ratio approaches 1. That is to say, they are equivalent infinitesimals. In calculus, it is the relationship at the limits that is concerned. So it will be much easier and clearer if the process of running can be skipped and the relationship between two infinitesimal variables can be studied directly at the limits. For this purpose, this paper proposes a Number Field of Ordered Infinitesimals and Infinities (OII Number Field) as the final destination of infinitesimals running toward 0. In other words, an infinitesimal, as a running variable, does not arrive at 0 in the end, but arrive at a number in the OII Number Field, with a certain Order and a certain Weight. In this way, the process of running is recorded in the destination. On this basis, differentials are invariant not only in form, but also in meaning. The differential dy, no matter induced by which variable, has the same Order and Weight in the OII Number Field, and is therefore equal to the infinitesimal increment of y itself at the limits.

Now that first order differentials have gained invariance in meaning, the research on high order differentials becomes much easier. Traditionally, the second order differential of $y = f(x)$ is defined as $\mathrm{d}^2 y = f''(x)\mathrm{d}x^2$ which is not invariant in form. If t is taken as the independent variable instead of x, then $x = h(t)$, $\mathrm{d}x = h'(t)\mathrm{d}t$, and as a result d^2y can no longer be expressed as $f''(x)\mathrm{d}x^2$. In fact, second order differential should be the differential of first order differential. Therefore, by taking differential on both sides of $\mathrm{d}y = f'(x)\mathrm{d}x$, the following equation is obtained:

$$\mathrm{d}^2 y = f''(x)\mathrm{d}x^2 + f'(x)\mathrm{d}^2 x\,. \tag{1.4}$$

It is the d^2y and d^2x in Equation (1.4) that are truly form invariant second order differentials. Unfortunately, it is very difficult to explain the meanings of d^2y and d^2x in Equation (1.4) based on the ε-δ definition of infinitesimals. With the help of OII Number Field, however, this difficulty is overcome. This paper will discuss the meanings and properties of high order differentials in depth.

The content of this paper was originally a research result of the author Han Xiao as an undergraduate student at Beihang University, with a Chinese manuscript [1] finished on Jan. 6th, 2003 and uploaded to Baidu Library on Jul. 21st, 2014. This paper is mostly the same as the original Chinese manuscript, but rather than directly translated into English, the content is reorganized to present the idea more logically. Besides, there are a few additions. For example, the recursion formulas of high order differentials are presented in this paper. Furthermore, two errors in the original Chinese manuscript have been corrected.

The paper is organized as follows. Section 2 discusses the definition and essence of infinitesimals, and proposes the concept of OII Number Field and the operational laws in it. Section 3 analyzes first order differentials from the perspective of OII Number Field, and discusses the three properties of first order differentials. Section 4 analyzes high order differentials from the perspective of OII Number Field, discusses the three properties of high order differentials, and demonstrates the geometrical meanings of high order differentials. Some further issues are discussed in Section 5, including differential equations directly built with high order differentials, and the application of OII Number Field in engineering problems. Additionally, two questions to be solved in the future are brought up. Section 6 summarizes the whole paper.

2 Definition of Infinitesimals

2.1 Traditional Definition of Infinitesimals

2.1.1 The ε-δ Definition of Infinitesimals

Definition 2.1 Suppose function $f(x)$ is defined in a deleted neighborhood of x_0 (or has definition when $|x|$ is greater than a certain positive number). If for any positive number ε (no matter how small it is), there exists a positive number δ (or positive number X) that for any x satisfying $0<|x-x_0|<\delta$ (or $|x|>X$) the corresponding function value $f(x)$ satisfies the inequation

$$\left|f(x)\right|<\varepsilon ,$$

then function $f(x)$ is called an infinitesimal when $x\to x_0$ (or $x\to\infty$).

From this definition it can been seen that an infinitesimal is essentially a variable running toward 0. The ε-δ definition is equivalent to stating that "$y=f(x)$ is an infinitesimal if y runs toward 0 when x runs toward x_0". However, this statement is redundant. As long as y runs toward 0, y is an infinitesimal. There is no need to resort to the functional relationship between y and other variables.

That is to say, if $x\to 0$, then x is an infinitesimal. Of course, x can also be seen as a function of itself, which satisfies the above ε-δ definition with the configuration of $f(x)=x$, and $x_0=0$.

When x is an infinitesimal, so are $5x$, x^2, and x^3, which all satisfy the above ε-δ definition. However, the ε-δ definition can actually be abandoned and replaced by a statement that reveals the essence, namely "because x, $5x$, x^2, and x^3 are all running toward 0, they are all infinitesimals".

Therefore, the essence of infinitesimals can be summarized in one simple statement "variables running toward 0". When variable y starts from a nonzero real number and changes gradually toward 0, it is called an infinitesimal. In calculus, it is this process of changing that is studied.

2.1.2 The Relative Sizes of Infinitesimals

Functional relationship can exist between variables. Since an infinitesimal is a kind of variable, so can functional relationship exist between infinitesimals. Suppose x and y are two infinitesimals, i.e. $x\to 0$ and $y\to 0$, and suppose there exists a functional relationship between x and y. Then during the process of x running toward 0, each value it takes corresponds to a value of y. For instance, when x = 0.1, y could be equal to 0.07, and when x = 0.01, y could be equal to 0.005, etc. As x approaches 0 infinitely, y also approaches 0 infinitely. Though both running toward 0, the speeds of x and y may not be equal. The functional relationship between x and y determines the relationship between their speeds, which can be classified into following cases.

1) If $x/y\to 0$, then x is a "higher order infinitesimal" relative to y, while y is a "lower order infinitesimal" relative to x. In this case, x runs faster than y. During their process of running toward 0, x becomes more and more insignificant in face of y. As an analogy, when y equals the size of a basketball, x equals the size of a table tennis ball; when y has shrunk to the size of a table tennis ball, x has shrunk to the size of a bacterium. In other words, the closer they are to 0, the greater their disparity in size is, and this disparity grows infinitely.

2) If $y/x\to 0$, then x is a "lower order infinitesimal" relative to y, while y is a "higher order infinitesimal" relative to x. In this case, y runs faster than x. During their process of running toward 0, y becomes more and more insignificant in face of x.

3) If $x/y\to A$, where A is a nonzero constant, then x and y are "infinitesimals of the same order". In this case, the gap in their speeds is not large. During their process of running toward 0, the ratio between their sizes approaches a constant.

4) If $x/y\to 1$, then x and y are "equivalent infinitesimals" which is a special case of "infinitesimals of the same order". In this case, the speeds of x and y can be deemed as the same. During their process of running toward 0, the ratio between their sizes approaches 1:1.

2.2 New Definition of Infinitesimals

2.2.1 Number Field of Ordered Infinitesimals and Infinities

Infinitesimals are a kind of variable running toward 0, and by investigating their processes of running, the ratio between the sizes of two infinitesimals with functional relationship can be studied. Most importantly, it is the limit of their size ratio that really matters. So, is it possible to skip the process of running toward 0 and compute the size ratio between infinitesimals directly at the limits? Apparently, it seems not, because the limits of different infinitesimals are all 0, which makes no difference. It seems that the size ratio between infinitesimals can only be studied by investigating their process of running toward 0. However, if a new kind of number field is defined, this problem can be solved.

As shown in Fig. 1, a Number Field of Ordered Infinitesimals and Infinities (OII Number Field) is defined. In this figure, the horizontal axis indicates the Weight, while the vertical axis indicates the Order. This figure is from infinitesimals' perspective, in which the Order of an infinitesimal is always positive, the Order of an infinity is always negative, while the Order of a finite real number is always 0. In other words, a finite real number can be seen as a 0th Order infinitesimal or a 0th Order infinity. If $x \to 0$, then x can be seen as unit first Order infinitesimal, namely a number whose Order and Weight are both 1, as marked with Point A in the figure. Accordingly, $2x$, $3x$ and $4.8x$ are all first Order infinitesimals, with Weights being 2, 3 and 4.8 respectively, as marked with Points B, C and D in the figure. Meanwhile, $3x^2$ and $-2x^2$ are second Order infinitesimals with Weights 3 and -2 respectively, as marked with Points E and F in the figure. Furthermore, $1/x$ is unit first Order infinity, while $3/x$ and $-2/x$ are first Order infinities with Weights 3 and -2 respectively, as marked with Points G, H and K in the figure.

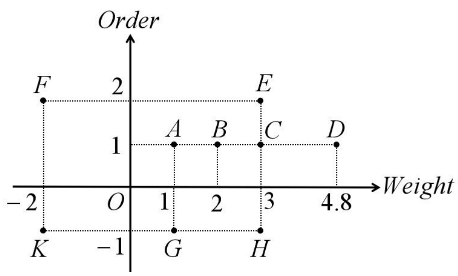

Fig. 1. OII Number Field

In this way, by defining OII Number Field, the processes of infinitesimals running toward 0 are recorded in the limits, making the relative sizes of infinitesimals clear and explicit in this new number field. The numbers in the OII Number Field are actually a kind of abstraction of the processes of running. When a running infinitesimal has not reached the destination yet, it is a variable with a shrinking value in the real number field; however, when it has reached its destination, its value is deemed to be a number in the OII Number Field, rather than 0. When x is taken as independent variable, the destination of $x \to 0$ is usually defined as a number with Order 1 and Weight 1 in the OII Number Field, namely unit first Order infinitesimal. On this basis, x can be used as a reference to deduce the Orders and Weights of other infinitesimal variables.

It is worth noting that infinities are included in this new number field too, and that infinities are ordered too. In this way, finite real numbers, different Orders of infinitesimals and different Orders of infinities are unified in one number field. Therefore, OII Number Field can be considered as a kind of extension of real number field.

2.2.2 Symbols and Geometric Meanings

In the OII Number Field shown with Fig. 1, all the numbers in the upper half plane are infinitesimals, while all the numbers in the lower half plane are infinities. A number in OII Number Field can be expressed as $a(0)^b$, in which a represents the Weight while b represents the Order. When b>0, $a(0)^b$ is an infinitesimal located in the upper half plane in the figure. When b<0, $a(0)^b$ is an infinity located in the lower half plane in the figure. A number in OII Number Field can also be expressed as $a(\infty)^b$, where a still represents the Weight while the Order from infinitesimals' perspective is $-b$ (or from infinities' perspective, the Order is b). When b>0, $a(\infty)^b$ is an infinity located in the lower half plane in Fig. 1. When b<0, $a(\infty)^b$ is an infinitesimal, located in the upper half plane in Fig. 1.

$a(0)^b$ is called "an infinitesimal with Weight a and Order b"; or equivalently, it can be seen as an infinity with Weight a and Order $-b$.

$a(\infty)^b$ is called "an infinity with Weight a and Order b"; or equivalently, it can be seen as an infinitesimal with Weight a and Order $-b$.

As shown in Fig. 2(a), if a cube, with the length of each edge being 1, is sliced evenly into infinite pieces, then the volume of each slice is unit first Order infinitesimal, namely $(0)^1$, while the total number of slices is unit first Order infinity, namely $(\infty)^1$. The sum of the volumes of the infinite slices is the volume of the cube. That is to say, unit first Order infinitesimal multiplied by unit first Order infinity equals 1, i.e.

$$(0)^1 \times (\infty)^1 = 1 .$$

As shown in Fig. 2(b), if each slice is cut evenly into infinite bars, which means a volume of $(0)^1$ is divided evenly into $(\infty)^1$ portions, then the volume of each bar is unit second Order infinitesimal, namely $(0)^2$. Since the total number of slices is $(\infty)^1$, and the number of bars in each slice is also $(\infty)^1$, the total number of bars in the cube is $(\infty)^1 \times (\infty)^1 = (\infty)^2$ which is a unit second Order infinity. The sum of the volumes of the infinite bars is the volume of the cube. That is to say, unit second Order infinitesimal multiplied by unit second Order infinity equals 1, i.e.

$$(0)^2 \times (\infty)^2 = 1 .$$

As shown in Fig. 2(c), each bar can be further cut evenly into infinite blocks, and the volume of each block is unit third Order infinitesimal, namely $(0)^3$, while the total number of blocks in the cube is unit third Order infinity $(\infty)^3$. The sum of volumes of the infinite blocks is the volume of the cube. That is to say, unit third Order infinitesimal multiplied by unit third Order infinity equals 1, i.e.

$$(0)^3 \times (\infty)^3 = 1 .$$

An infinitesimal has no effect when added to a finite number. For example, in Fig. 2(a), the volume of a slice added to the volume of the cube still amounts to the volume of the cube, i.e.,

$$1 + (0)^1 = 1 .$$

A finite number has no effect when added to an infinity. For example, there are $(\infty)^1$ slices in Fig. 2(a), and if one more slice is added, the total number of slices is still $(\infty)^1$, i.e.,

$$1+(\infty)^1=(\infty)^1 .$$

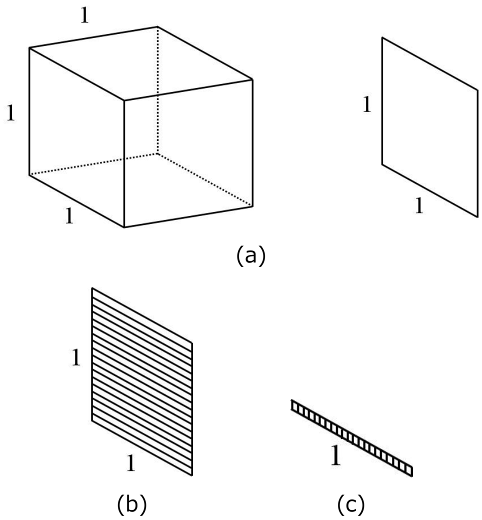

Fig. 2. Infinite Division of a Cube

A higher Order infinitesimal has no effect when added to a lower Order infinitesimal. For example, a slice in Fig. 2(b) is divided into $(\infty)^1$ bars, and since the volume of a slice is $(0)^1$, the volume of each bar is $(0)^2$. If one more bar is added to the slice, the volume of the slice will not change, i.e.,

$$(0)^1+(0)^2=(0)^1 .$$

An infinitesimal is different from real number 0, because 0s, even in such large number as a first Order infinity, when added together, is still 0, i.e.,

$$0\times(\infty)^1=0 ,$$

but first Order infinitesimals, if in a number as large as a first Order infinity, when added together, will produce a finite number, like

$$(0)^1\times(\infty)^1=1 .$$

In the OII Number Field depicted in Fig. 1, if points with the same Order are connected into number axes, infinitesimal number axes and infinity number axes of different Orders are obtained, as shown in Fig. 3. These number axes are parallel to the real number axis, and numbers within each axis obey operational laws similar to that of real numbers. The only difference is that each number axis has a different unit number. For example, the unit number of the first Order infinitesimal axis is $(0)^1$, the unit number of the second Order infinitesimal axis is $(0)^2$, the unit number of the first Order infinity axis is $(\infty)^1$, and the unit number of the real number axis is 1 which can also be written as 0th Order infinitesimal $(0)^0$ or 0th Order infinity $(\infty)^0$.

2.2.3 Operational Laws

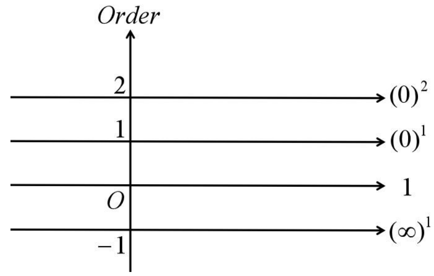

Fig. 3. Number Axes of Different Orders in OII Number Field

Generally, numbers in OII Number Field obey the following operational laws.

1) Reciprocal Law:

$$\frac{1}{a(0)^b}=\frac{1}{a}(0)^{-b}=\frac{1}{a}(\infty)^b \text{, where } a\neq 0 ,$$

$$\frac{1}{a(\infty)^b}=\frac{1}{a}(\infty)^{-b}=\frac{1}{a}(0)^b \text{, where } a\neq 0 ;$$

2) Multiplication Law:

$$a(0)^b\times c(0)^d=ac(0)^{b+d} ,$$

$$a(\infty)^b\times c(\infty)^d=ac(\infty)^{b+d} ,$$

$$a(0)^b\times c(\infty)^d=ac(0)^{b-d}=ac(\infty)^{-b+d} ;$$

3) Addition in the Same Order:

$$a(0)^n+c(0)^n=(a+c)(0)^n ,$$

$$a(\infty)^n+c(\infty)^n=(a+c)(\infty)^n ;$$

4) Addition across Different Orders:

$$a(0)^n+c(0)^m=a(0)^n \text{, where } n<m ,$$

$$a(\infty)^n+c(\infty)^m=c(\infty)^m \text{, where } n<m ;$$

5) Subtraction of Equivalent Numbers:

$$a(0)^n-a(0)^n=c(0)^m \text{, where } n<m ,$$

$$a(\infty)^m-a(\infty)^m=c(\infty)^n \text{, where } n<m .$$

Among these operational laws, "Addition across Different Orders" and "Subtraction of Equivalent Numbers" are worth noting.

According to the rule of "Addition across Different Orders", a higher Order infinitesimal has no effect when added to a lower Order infinitesimal, and a lower Order infinity has no effect when added to a higher Order infinity. Infinitesimals can be considered as infinities of negative Orders, while infinities can be considered as infinitesimals of negative Orders. Therefore, in the above formulas, n and m can both take negative values.

A higher Order infinitesimal has no effect when added to a lower Order infinitesimal. However, having no effect is different from nonexistence. It is because the disparity in size is infinitely large that the “effect” does not show up. When an infinitesimal is subtracted from an equivalent one, the lowest Order components are canceled out, and the effect of higher Order components pops up. This is how the rule of “Subtraction of Equivalent Numbers” works, which can be demonstrated with the following example.

When $x \to 0$, x can be specified as unit first Order infinitesimal, i.e. $x = (0)^1$. Then, $\frac{1}{2}x^2$ is now a second Order infinitesimal with Weight 1/2, i.e. $\frac{1}{2}x^2 = \frac{1}{2}(0)^2$. According to Taylor’s formula,

$$1-\cos x = \frac{1}{2}x^2 - \frac{x^4}{4!} + o(x^4).$$

Therefore, $1-\cos x$ is also a second Order infinitesimal with Weight 1/2, but it contains components of a fourth Order infinitesimal and even higher Order infinitesimals. According to the operational laws in OII Number Field,

$$1-\cos x = \frac{1}{2}(0)^2 - \frac{(0)^4}{4!} + o\big((0)^4\big) = \frac{1}{2}(0)^2.$$

Therefore, $1-\cos x$ and $\frac{1}{2}x^2$ are equivalent infinitesimals which are equal to each other in the OII Number Field and both equal to $\frac{1}{2}(0)^2$. However, when subtracting $\frac{1}{2}x^2$ from $1-\cos x$, the result is not 0. This is because subtraction only cancels out second Order components in the two infinitesimals, and higher Order components will pop up when lower Order components are absent. In fact,

$$1-\cos x - \frac{1}{2}x^2 = -\frac{(0)^4}{4!} + o\big((0)^4\big) = -\frac{1}{24}(0)^4,$$

which means that $1-\cos x - \frac{1}{2}x^2$ is a fourth Order infinitesimal with Weight $-\frac{1}{24}$.

2.2.4 Interpretation of Definite Integral

As shown in Fig. 4, one can compute the area of Rectangle *ABCD* by dividing it into many vertical bars and adding up the areas of all the bars. The shadowed area in the figure is one of the vertical bars. When the rectangle is divided evenly into infinite bars, the width of each bar can be specified as $(0)^1$, and the total number of bars will be $3(\infty)^1$, where 3 is the difference between the horizontal coordinates of *B* and *A*. As the height of each bar is 2, the area of each bar is $2(0)^1$, and therefore the sum of the areas of all the bars is $2(0)^1 \times 3(\infty)^1 = 6$ which is exactly the area of the rectangle. Since first Order infinitesimals in such large number as a first Order infinity are added up here, it is actually a definite integral of the simplest form.

As depicted in Fig. 5, if the side *CD* of Rectangle *ABCD* is replaced with a curve, one can still compute the area under Curve *CD* by dividing it into many vertical bars and adding up the areas of all the bars. The width of each bar can still be specified as $(0)^1$, and as a result, the total number of bars is still $3(\infty)^1$. However, since the height of each bar is different now, the area of each bar is different. The areas of the bars are first Order infinitesimals with different Weights. In this circumstance, adding up the areas of all the vertical bars is still computing the summation of first Order infinitesimals in such large number as a first Order infinity, but the Weight of each infinitesimal is different. This is a definite integral of the general form.

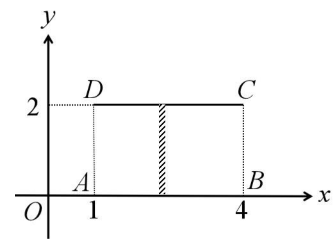

Fig. 4. Computing the Area of a Rectangle

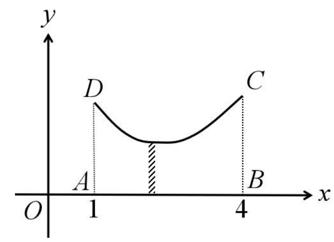

Fig. 5. Computing the Area under the Curve

Similarly, it is easy to know that a double integral is the summation of second Order infinitesimals in such large number as a second Order infinity, with the Weight of each infinitesimal being different, and that a triple integral is the summation of third Order infinitesimals in such large number as a third Order infinity, with the Weight of each infinitesimal being different.

It is worth noting that, when the area under Curve *CD* in Fig. 5 is divided into a finite number of vertical bars, the shape of each bar is not a rectangle, but a trapezoid with a curve side, like the one in Fig. 6. It is obvious that, when the width *EF* of the curve-sided trapezoid approaches 0, the lengths of the two vertical sides *HE* and *KF* tend to be equal, i.e., the length of *KG* approaches 0. Therefore, when the area under Curve *CD* is divided evenly into infinite bars, with the width of each bar specified as unit first Order infinitesimal $(0)^1$, the width *EF* of the curve-sided trapezoid is unit first Order infinitesimal $(0)^1$, and the length of *KG* is also an infinitesimal (of first Order or higher Order). As a result, the area of Rectangle *EFGH*, or S_{EFGH}, is a first Order infinitesimal, while the area of Curve-sided Triangle *HGK*, or S_{HGK}, is a higher Order infinitesimal.

The area of Curve-sided Trapezoid *EFKH* is the summation of S_{EFGH} and S_{HGK}. Since a higher Order infinitesimal has no effect when added to a lower Order infinitesimal, in OII Number Field, the area of Curve-sided Trapezoid *EFKH* is exactly equal to the area of Rectangle *EFGH* in that both their Weights and Orders are equal to each other. Now that they are equal, they can substitute for each other. Therefore, in OII Number Field, the area of Curve-sided Trapezoid *EFKH* can be substituted with the area of Rectangle *EFGH*. Finally, the area under Curve *CD* is obtained exactly by adding up the areas of rectangles in such large number as a first Order infinity. This principle can be called "**Equivalent Infinitesimals Have the Same Effect in Definite Integral**". It is the most important principle for writing out definite integral expressions in applications.

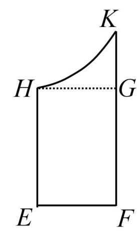

Fig. 6. Trapezoid with a Curve Side

Similarly, another example can be given. As shown in Fig. 7, one can compute the volume of a cone by slicing it evenly into infinite pieces and adding up the volumes of the infinite slices. Each slice is a circular truncated cone whose thickness can be specified as unit first Order infinitesimal $(0)^1$, and the difference between the radius of the lower surface and the radius of the upper surface is a first Order infinitesimal with Weight $\tan\theta$, where θ is half of the apex angle of the cone. It is easy to see that the volume of the inside cylinder with the same upper surface is a first Order infinitesimal, and that the difference between the volume of the circular truncated cone

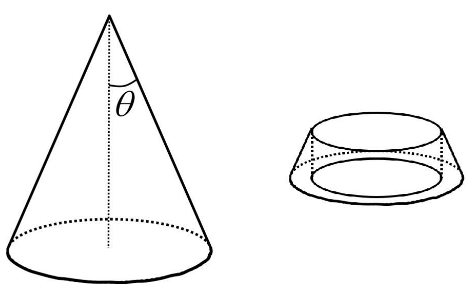

Fig. 7. Computing the Volume of a Cone

and the volume of the cylinder is a higher Order infinitesimal which has no effect when added to a first Order infinitesimal. Therefore, in OII Number Field, the volume of the circular truncated cone is exactly equal to the volume of the cylinder in that their Weights and Orders are both equal. Therefore, the volume of the circular truncated cone can be substituted with the volume of the cylinder. Finally, the summation of volumes of the cylinders in such large number as a first Order infinity is exactly the volume of the cone.

3 First Order Derivative and First Order Differential

3.1 First Order Derivative

Suppose Variable y and Variable x have functional relationship $y = f(x)$ whose curve is shown in Fig. 8. The horizontal coordinate of Point *A* on the curve is x_A, while the vertical coordinate is $f(x_A)$. If x has an increment Δx at x_A, y will have a corresponding increment Δy, and the point on the curve will move to Point *B*. Ratio $\frac{\Delta y}{\Delta x}$ represents the slope of Line *AB*. When $\Delta x \to 0$, Δy also approaches 0, and consequently, Point *B* approaches Point *A*. In this circumstance, the limit of $\frac{\Delta y}{\Delta x}$ is defined as the derivative of y with respect to x at Point *A*, namely $f'(x_A)$, and its geometrical meaning is the slope of the tangent line at Point *A*, as shown in Fig. 9.

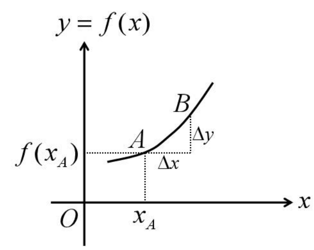

Fig. 8. The Increments of Independent and Dependent Variables

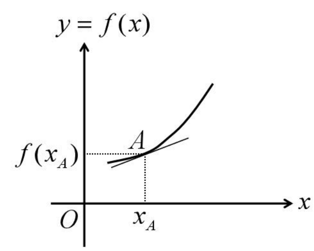

Fig. 9. A Derivative Represents the Slope of a Tangent Line

Traditionally, the derivative is defined in this way by investigating the processes of Δx and Δy running toward 0. Now that OII Number Field is defined, with the processes of variables running toward 0 directly recorded in the limits, the derivative of y with respect to x at Point *A* can be explained to be the ratio between two infinitesimals Δy and Δx in OII Number Field. If the Weight of Δx is specified to be 1, i.e. $\Delta x = (0)^1$, then first Order infinitesimal Δy can be obtained by $\Delta y = f(x_A + (0)^1) - f(x_A)$, and the derivative can be obtained by directly computing the ratio.

3.2 First Order Differential

In Fig. 10, the length of the line segment AC is the increment Δx at Point A, while the length of the line segment BC is the increment Δy at Point A. When Δy and Δx both take finite values, their ratio is not equal to $f'(x_A)$, the derivative of y with respect to x at Point A. Traditionally, Δx is defined as the differential of x at Point A, i.e. $\mathrm{d}x \equiv \Delta x$, while $f'(x_A)\mathrm{d}x$ is defined as the differential of y at Point A, i.e. $\mathrm{d}y \equiv f'(x_A)\mathrm{d}x$. It can be seen from Fig. 10 that, dy is the length of the line segment CD which is not equal to Δy. When Δx runs toward 0, Δy, dx and dy all runs toward 0, and during their processes of running, dx is always equal to Δx, while dy is always not equal to Δy. That is to say, traditionally, only the differential of an independent variable is equal to the increment of the variable itself, the differential of a dependent variable is generally not equal to the increment of the variable itself.

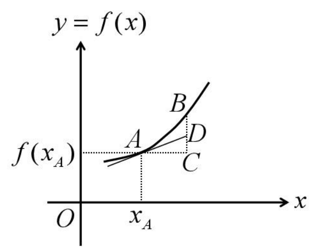

Fig. 10. Traditional Geometric Meaning of a Differential

When $\Delta x \to 0$, $\frac{\Delta y}{\Delta x} \to f'(x_A)$, and if $f'(x_A) \neq 0$, then $\frac{\Delta y}{\Delta x \cdot f'(x_A)} \to 1$. Taking the definitions of dx and dy into account, namely $\mathrm{d}x \equiv \Delta x$ and $\mathrm{d}y \equiv f'(x_A)\mathrm{d}x$, then $\frac{\Delta y}{\Delta x \cdot f'(x_A)}$ can be reduced to $\frac{\Delta y}{\mathrm{d}x \cdot f'(x_A)} = \frac{\Delta y}{\mathrm{d}y}$. Therefore, when $\Delta x \to 0$, $\frac{\Delta y}{\mathrm{d}y} \to 1$. That is to say, Δy and dy are equivalent infinitesimals which are exactly equal to each other in OII Number Field, with the same Order and Weight.

If Δx is specified to be $(0)^1$, then $\mathrm{d}x = (0)^1$ and $\mathrm{d}y = \Delta y = f'(x_A)(0)^1$. Therefore, in OII Number Field, the differential of any variable is equal to the increment of the variable itself, no matter whether the variable is independent or not.

3.3 Form Invariance of First Order Differential

Suppose $y = f(x)$ and $x = h(t)$. From traditional point of view, only the differential of an independent variable is equal to the increment of the variable itself. As a result, when x is independent variable, $\mathrm{d}y = f'(x_A)\mathrm{d}x$, where $\mathrm{d}x = \Delta x$; when t is independent variable, $\mathrm{d}y = f'(x_A)h'(t_A)\mathrm{d}t = f'(x_A)\mathrm{d}x$, where $\mathrm{d}x = h'(t_A)\mathrm{d}t \neq \Delta x$. That is to say, when the independent variable has changed from x to t, the meanings of dx and dy have both changed, but dx and dy still satisfy the equation $\mathrm{d}y = f'(x_A)\mathrm{d}x$. Traditionally, this phenomenon is called "the Form Invariance of Differentials" which implies that the meanings of the differentials have changed under invariant forms.

Although different choice of independent variable changes the meanings of dx and dy, it does not change the fact that dx and dy are running toward 0. In other words, whichever variable is chosen as independent variable, dx and dy are always running toward 0. What really matters is the limit of the relationship between their sizes during their processes of running. With the definition of OII Number Field, the running processes of dx and dy are recorded in the limits. In other words, they no longer arrive at 0, but arrive at numbers in OII Number Field. On this basis, the relative sizes of dx and dy can be directly studied at the limits in OII Number Field.

After OII Number Field is defined, the meaning of a differential no longer change with different choice of independent variable, because the finite real numbers that dx and dy take during their processes of running toward 0 are no longer cared about, instead, dx and dy are directly studied in their destination of running, namely OII Number Field. Whichever variable is chosen as independent variable, the meaning of Differential dy is invariant, because in OII Number Field, Differential dy is always equal to the increment of y itself, namely Δy.

Therefore, a differential can be redefined as **the value of infinitesimal increment of a variable itself in OII Number Field**.

It is worth noting that the value of a differential in OII Number Field has some freedom. For example, when x is taken as independent variable, the value dx can be specified as $(0)^1$, and then $\mathrm{d}y = f'(x_A)(0)^1$; by contrast, when t is taken as independent variable, the value dt can be specified as $(0)^1$, and as a result, $\mathrm{d}y = f'(x_A)h'(t_A)(0)^1$. Obviously, the value of dy in OII Number Field is different under these two specifications. However, the meaning of dy has not changed, and dy always equals the increment of y itself in OII Number Field, namely Δy.

Therefore, $\mathrm{d}y = f'(x_A)h'(t_A)\mathrm{d}t = f'(x_A)\mathrm{d}x$ can be called "the constraint between differentials", for these equations describes the relationships between the infinitesimal increments of different variables in OII Number Field. When $f'(x_A) \neq 0$ and $h'(x_A) \neq 0$, dy, dt and dx are infinitesimals of the same Order, and as long as one of their Weights is specified, the other two Weights are determined.

Therefore, **after OII Number Field is defined, infinitesimals are not only invariant in form, but also invariant in meaning**. In other words, the meaning of an infinitesimal does not change with different choice of independent variable either.

Hence, it can be summarized that differentials have three properties.

1) Meaning Invariance.

A differential always represents the value of infinitesimal increment of the variable itself in OII Number Field.

2) Form Invariance.

The differential relationship between the two variables x and y satisfying functional relationship $y = f(x)$ is always $\mathrm{d}y = f'(x)\mathrm{d}x$, which is not affected by any other variable. In fact, it is more appropriate to call this property "the constraint between differentials".

3) Some Freedom.

Suppose $y = f(x)$, where x varies on Interval $[a,b]$. Then, the Weight of Differential dx at each point in Interval $[a,b]$ can actually be specified arbitrarily. In other words, the curve in Fig. 11 can have arbitrary shape. Drawing a curve is equivalent to specifying the Weight of Differential dx at each point, and the Weight of dy at each point is thus determined by the Equation $\mathrm{d}y = f'(x)\mathrm{d}x$.

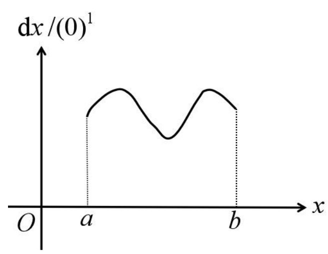

Fig. 11. Specifying the Weight of a Differential at Each Point

4 High Order Derivatives and High Order Differentials

4.1 Two Lemmas

Lemma 4.1 Suppose A and B are variables in real number field or infinitesimal number fields, i.e. the unit of A and B are $(0)^m$ and $(0)^n$ respectively, where $m \geqslant 0$ and $n \geqslant 0$. Then,

when $A\mathrm{d}B + B\mathrm{d}A \neq 0$, $\mathrm{d}(AB) = A\mathrm{d}B + B\mathrm{d}A$;

when $A\mathrm{d}B + B\mathrm{d}A = 0$, $\mathrm{d}(AB) = \mathrm{d}A\mathrm{d}B$.

Proof

$$\begin{aligned}\mathrm{d}(AB) &= (A+\mathrm{d}A)(B+\mathrm{d}B) - AB \\ &= AB + A\mathrm{d}B + B\mathrm{d}A + \mathrm{d}A\mathrm{d}B - AB \\ &= A\mathrm{d}B + B\mathrm{d}A + \mathrm{d}A\mathrm{d}B \qquad (4.1)\end{aligned}$$

Since A and B are an m-th and n-th Order infinitesimals respectively, AdB and BdA are both $(m+n+1)$th Order infinitesimals, while dAdB is a $(m+n+2)$th Order infinitesimal.

When $A\mathrm{d}B + B\mathrm{d}A \neq 0$, according to the rule of "Addition across Different Orders" in Section 2.2.3, which is also described as "higher Order infinitesimals have no effect when added to lower Order infinitesimals", $\mathrm{d}(AB) = A\mathrm{d}B + B\mathrm{d}A$. In this case, d$(AB)$ is a $(m+n+1)$th Order infinitesimal.

When $A\mathrm{d}B + B\mathrm{d}A = 0$, Equation (4.1) becomes $\mathrm{d}(AB) = \mathrm{d}A\mathrm{d}B$. In this case, d$(AB)$ is a $(m+n+2)$th Order infinitesimal. That is to say, when the Weight of lower Order component is 0, higher Order component pops up.

End of Proof

An example can be given to demonstrate Lemma 4.1. Suppose A and B are both equal to x, it is easy to know that d(AB) = d(x^2), AdB+BdA = xdx+xdx = 2xdx, and dAdB = dx^2. Obviously, when $x \neq 0$, d(x^2) = 2xdx, i.e. d(AB) = AdB+BdA; when $x = 0$, d(x^2) = dx^2, i.e. d(AB) = dAdB.

Lemma 4.2 Suppose A and B are variables in real number field or infinitesimal number fields, i.e. the unit of A and B are $(0)^m$ and $(0)^n$ respectively, where $m \geqslant 0$, $n \geqslant 0$ and $B \neq 0$. Then,

$$d\left(\frac{A}{B}\right) = \frac{B\mathrm{d}A - A\mathrm{d}B}{B^2}.$$

Proof

$$d\left(\frac{A}{B}\right) = \frac{A+\mathrm{d}A}{B+\mathrm{d}B} - \frac{A}{B} = \frac{(A+\mathrm{d}A)B - (B+\mathrm{d}B)A}{(B+\mathrm{d}B)B} = \frac{B\mathrm{d}A - A\mathrm{d}B}{B^2 + B\mathrm{d}B}$$

Since B is an n-th Order infinitesimal, B^2 is a 2n-th Order infinitesimal, while BdB is a $(2n+1)$th Order infinitesimal. Since $B \neq 0$, according to the rule of "Addition across Different Orders" in Section 2.2.3, B^2+BdB = B^2. As a result,

$$d\left(\frac{A}{B}\right) = \frac{B\mathrm{d}A - A\mathrm{d}B}{B^2}.$$

End of Proof

4.2 How does High Order Differentials Arise

Suppose Variables x and y have monotonic functional relationship on their respective intervals, i.e. $y = f(x)$ and $x = g(y)$. In addition, suppose x and y are both monotonic functions of t on their respective intervals, i.e. $x = x(t)$, $y = y(t)$, $t = h(x)$ and $t = s(y)$.

Taking differential on both sides of $y = f(x)$ results in $\mathrm{d}y = f'(x)\mathrm{d}x$.

Taking differential on both sides of $x = x(t)$ results in $\mathrm{d}x = x'(t)\mathrm{d}t$.

Taking differential on both sides of $y = y(t)$ results in $\mathrm{d}y = y'(t)\mathrm{d}t$.

If t is taken as independent variable, and is supposed to generate equal differentials at each point, i.e. dt = $(0)^1$ is specified, then $\mathrm{d}x = x'(t)(0)^1$ and $\mathrm{d}y = y'(t)(0)^1$. Obviously, in this case, dx is not equal at different points, and dy is not equal at different points either. In this circumstance, dx, dy and $f'(x)$ in the Equation $\mathrm{d}y = f'(x)\mathrm{d}x$ are all functions of t, with dx and dy varying on the first Order infinitesimal number axis in Fig. 3 while $f'(x)$ varying on the real number axis.

Taking differential again on both sides of $\mathrm{d}y = f'(x)\mathrm{d}x$, recording d(d$y$) as d^2y, recording $\mathrm{d}x \cdot \mathrm{d}x$ as $\mathrm{d}x^2$, recording d(dx) as d^2x, and according to Lemma 4.1, the following equation is obtained:

$$\mathrm{d}^2 y = \mathrm{d}(f'(x))\mathrm{d}x + f'(x)\mathrm{d}(\mathrm{d}x) = f''(x)\mathrm{d}x^2 + f'(x)\mathrm{d}^2 x$$

i.e.

$$\mathrm{d}^2 y = f''(x)\mathrm{d}x^2 + f'(x)\mathrm{d}^2 x \tag{4.2}$$

where, $\mathrm{d}^2 y = \mathrm{d}(y'(t)(0)^1) = y''(t)\mathrm{d}t(0)^1 = y''(t)(0)^2$,

$\mathrm{d}x^2 = (x'(t)(0)^1)^2 = x'^2(t)(0)^2$,

$\mathrm{d}^2 x = \mathrm{d}(x'(t)(0)^1) = x''(t)\mathrm{d}t(0)^1 = x''(t)(0)^2$.

This means that d^2y, $\mathrm{d}x^2$ and d^2x are all variables on the second Order infinitesimal axis, and that they are all functions of t.

d^2y and d^2x are second order differentials of y and x respectively. By reviewing the process of their birth, it can be found that an independent variable t has been used, and that t has generated differentials twice at each point. When t generates the first Differential dt, x and y generate corresponding differentials dx and dy at each point. As dt at each point is specified to be $(0)^1$, dt becomes a constant, while dx and dy become functions of t. Next, t generates Differential dt again at each point. This time, not only x and y generate corresponding differentials dx and dy, the previously generated Differential dx and Differential dy now also generate corresponding differentials d^2x and d^2y respectively at each point. Hence it can be seen that second order differentials are rooted in the fact that first order differentials dx and dy have become variables on the first Order infinitesimal number axis. As variables, dx and dy can also generate increments in themselves, and their infinitesimal increments are naturally located on the second Order infinitesimal number axis.

4.3 Form Invariance of High Order Differentials

In Section 3.3, the first order differential is redefined with OII Number Field, and three properties of first order differentials are summarized: 1) Meaning Invariance, 2) Form Invariance, and 3) Some Freedom.

In Section 4.2, dt at each point is specified to be $(0)^1$. However, as differentials have some freedom, the Weights of dt at different points can actually be specified arbitrarily. As shown in Fig. 12, suppose the value range of t is Interval $[t_a, t_b]$, then an arbitrary curve can be drawn to specify the Weights of dt at different points.

Once the curve in Fig. 12 has been drawn, the functional relationship between dt and t has been specified, and the

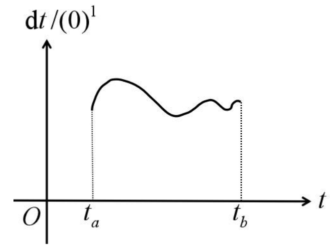

Fig. 12. The Weights of Differential dt at Different Points

Weights of dt at different points have been determined. According to $\mathrm{d}x = x'(t)\mathrm{d}t$ and $\mathrm{d}y = y'(t)\mathrm{d}t$, the Weights of d$x$ and dy at different points have been determined too. In other words, dx and dy are both functions of t. Since x is supposed to be a monotonic function of t, for each value of x, there is a corresponding value of t, and consequently, there is a corresponding value of dx. Therefore, dx is a function of x. Similarly, dy is a function of y. The functional relationship between dx and x corresponds to the curve in Fig. 13. The functional relationship between dy and y corresponds to the curve in Fig. 14. As long as any one of the three curves in Fig. 12, Fig. 13 and Fig. 14 is drawn, the other two curves are determined thereafter.

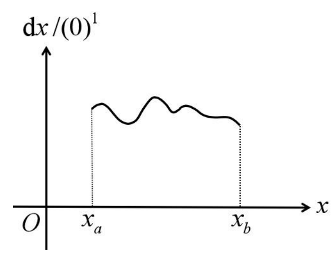

Fig. 13. The Weights of Differential dx at Different Points

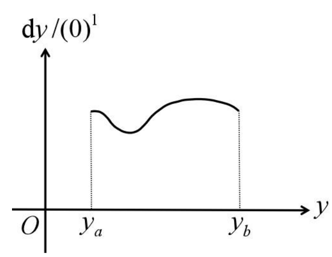

Fig. 14. The Weights of Differential dy at Different Points

That is to say, Variable t is actually not needed at all. The functional relationship between dx and x can be specified arbitrarily, and once it has been specified, the functional relationship between dy and y is also determined. Suppose dx = $u(x)(0)^1$, then according to $\mathrm{d}y = f'(x)\mathrm{d}x$ and $x = g(y)$, the functional relationship between dy and y can be obtained as $\mathrm{d}y = f'(g(y))u(g(y))(0)^1$.

Since x and y are supposed to have monotonic functional relationship in their respective intervals, the dy, $f'(x)$ and dx in Equation $\mathrm{d}y = f'(x)\mathrm{d}x$ all have monotonic functional relationship both with respect to x and with respect to y. In order to obtain second order differentials, x has to generate Differential dx again at each point, and this time the Weights of dx at different points can be specified arbitrarily again. In other words, the functional relationship between dx and x is to be specified arbitrarily again. In order to differentiate from the previously generated dx, the dx generated for the first time can be recorded as dx_1, while dx generated for the second time can be recorded as dx_2. Correspondingly, dy generated for the first time is recorded as dy_1. Therefore,

$$\mathrm{d}y_1 = f'(x)\mathrm{d}x_1 . \tag{4.3.1}$$

Taking differential again on both sides, and according to Lemma 4.1, the following equation is obtained:

$$\begin{aligned}\mathrm{d}(\mathrm{d}y_1) &= \mathrm{d}(f'(x))\mathrm{d}x_1 + f'(x)\mathrm{d}(\mathrm{d}x_1) \\ &= f''(x)\mathrm{d}x_2\mathrm{d}x_1 + f'(x)\mathrm{d}(\mathrm{d}x_1).\end{aligned}$$

Since the d(dy_1) and d(dx_1) in this equation are both induced by dx_2, they can be recorded as d^2y_{12} and d^2x_{12}, and as a result, the above equation can be written as

$$\mathrm{d}^2y_{12} = f''(x)\mathrm{d}x_2\mathrm{d}x_1 + f'(x)\mathrm{d}^2x_{12} . \tag{4.3.2}$$

Differentials can be taken once more on both sides, and with Lemma 4.1 referenced repeatedly, the following equation can be obtained:

$$\begin{aligned}\mathrm{d}(\mathrm{d}^2y_{12}) = f^{(3)}(x)\mathrm{d}x_3\mathrm{d}x_2\mathrm{d}x_1 &+ f''(x)\mathrm{d}(\mathrm{d}x_2)\mathrm{d}x_1 \\ &+ f''(x)\mathrm{d}x_2\mathrm{d}(\mathrm{d}x_1) + f''(x)\mathrm{d}x_3\mathrm{d}^2x_{12} \\ &+ f'(x)\mathrm{d}(\mathrm{d}^2x_{12}),\end{aligned}$$

where d(d^2y_{12}), d(dx_2), d(dx_1) and d(d^2x_{12}) are all caused by dx_3, and can be recorded as d^3y_{123}, d^2x_{23}, d^2x_{13} and d^3x_{123}, resulting in the following equation

$$\begin{aligned}\mathrm{d}^3y_{123} = f^{(3)}(x)\mathrm{d}x_3\mathrm{d}x_2\mathrm{d}x_1 &+ f''(x)\mathrm{d}^2x_{23}\mathrm{d}x_1 \\ &+ f''(x)\mathrm{d}x_2\mathrm{d}^2x_{13} + f''(x)\mathrm{d}x_3\mathrm{d}^2x_{12} \\ &+ f'(x)\mathrm{d}^3x_{123}.\end{aligned} \tag{4.3.3}$$

Three equations have been obtained. Equation (4.3.1) describes the relationship between first order differentials of x and y. Equation (4.3.2) describes the relationship between second order differentials of x and y. Equation (4.3.3) describes the relationship between third order differentials of x and y. First order differentials have one degree of freedom, i.e., the Weights of dx at different points can be specified arbitrarily, and the Weights of dy at different points are determined thereafter. Second order differentials have two degrees of freedom, i.e., the dx generated for the first time and the dx generated for the second time can both have their Weights specified arbitrarily at different points. The dx generated for the first time is recorded as dx_1, while the dx generated for the second time is recorded as dx_2, and this means that dx_1 and dx_2 are two independent functions of x. Similarly, third order differentials have three degrees of freedom, and the dx generated for each time is recorded as dx_1, dx_2 and dx_3 respectively, which are all functions of x that can be defined arbitrarily.

Similarly, differentials can be taken on both sides of Equation (4.3.3) continuously, resulting in fourth order differentials, fifth order differentials and so on. It is easy to know that n-th order differentials have n degrees of freedom. That is to say, dx has to be generated n times and recorded as dx_1, dx_2, ..., and dx_n which are all variables on the first Order infinitesimal number axis. The Weights of the n infinitesimal variables are all functions of x, and these n functions can be defined arbitrarily.

Since Equation (4.3.1) to (4.3.3) can be obtained without resorting to Variable t, the relationships that high order differentials of x and y satisfy have nothing to do with other variables. Therefore, similar to first order differentials, high order differentials are invariant in form.

As differentials of different orders are all variables in OII Number Field, high order differentials are also invariant in meaning, just as first order differentials. For example, d^2x_{23} represents the infinitesimal increment of dx_2 induced by dx_3, and is located on the second Order infinitesimal number axis. As this fact is irrelevant to any other variable, d^2x_{23} have an invariant meaning.

Therefore, high order differentials have three properties similar to those of first order differentials: 1) Meaning Invariance, 2) Form Invariance, 3) Some Freedom.

4.4 High Order Differentials with Single Degree of Freedom

It has already been discovered in the previous section that n-th order differentials have n degrees of freedom. In other words, dx has to be generated n times and recorded as dx_1, dx_2, ..., and dx_n which are all variables on the first Order infinitesimal number axis. The Weights of the n infinitesimal variables are all functions of x, and these n functions can be defined arbitrarily. If these n functions are all defined to be the same function, for instance, all defined with the curve in Fig. 13, then the degree of freedom of n-th order differentials is reduced to 1. In this circumstance, although dx_1, dx_2, ..., and dx_n are generated at different times, their values at a given point are the same, and therefore they can all be written as dx. Similarly, the values of d^2x_{23}, d^2x_{12} and d^2x_{13} at a given point are the same, and therefore they can all be written as d^2x. According to the constraint between first order differentials, dy_k is equal to $f'(x)\mathrm{d}x_k$, where k=1,2,...,n. This indicates that the values of dy_1, dy_2, ..., and dy_n at a given point are no longer different, and therefore they can all be written as dy. As a result, the constraint between first order differentials of x and y can be written in a unified form:

$$\mathrm{d}y = f'(x)\mathrm{d}x . \tag{4.4.1}$$

Originally, taking differential twice on both sides of $\mathrm{d}x_1 = g'(y)\mathrm{d}y_1$ will result in an equation similar to Equation (4.3.3), i.e., on the right side of the equation will appear three types of second order differential: d^2y_{23}, d^2y_{12} and d^2y_{13} which represent the values of infinitesimal increments of dy_2, dy_1 and dy_1 caused by dx_3, dx_2 and dx_3 respectively. However, now that dx_1, dx_2, ..., and dx_n are specified with the same function, the values of d^2y_{23}, d^2y_{12} and d^2y_{13} at a given point are no longer different, and can all be written as d^2y. As a result, Equation (4.3.2) becomes

$$d^2y = f''(x)dx^2 + f'(x)d^2x . \qquad (4.4.2)$$

Similarly, Equation (4.3.3) becomes

$$d^3y = f^{(3)}(x)dx^3 + 3f''(x)dxd^2x + f'(x)d^3x . \qquad (4.4.3)$$

Therefore, when the Weights of dx at different points are always specified with the same curve, the degrees of freedom of high order differentials are all reduced to 1, resulting in concise equations of high order differentials. High order differentials of this type still satisfy the aforementioned three properties: 1) Meaning Invariance, 2) Form Invariance, 3) Some Freedom.

For example, d^2y is a variable on the second Order infinitesimal number axis in OII Number Field, and it always represents the infinitesimal increment in the infinitesimal increment of y itself, namely the infinitesimal increment in dy, with dy located on the first Order infinitesimal number axis. The Weight of dy is a function of y, and this function can be defined arbitrarily. Therefore, d^2y is irrelevant to any other variable, and is thus invariant in meaning.

Of course, once the functional relationship between dy and y is specified, the functional relationship between dx and x is determined. In turn, once the functional relationship between dx and x is specified, the functional relationship between dy and y is determined. Nevertheless, this will not affect the meaning invariance of d^2y and d^2x. The second order differentials d^2y and d^2x always represent the infinitesimal increment in the infinitesimal increment of y and x respectively, while they always satisfy the constraint described in Equation (4.4.2). Since this constraint equation is irrelevant to any other variable, d^2y and d^2x are invariant in form too. In fact, the so called "Form Invariance" is actually saying that, whether or not x is independent variable, or in other words, no matter how the functional relationship between dx and x is specified, d^2y can always be expressed in the form of $f''(x)dx^2 + f'(x)d^2x$.

It is notable that, when the Weights of dx at different points are specified to be equal, dx becomes a constant, and therefore $d^2x = 0$. In this case, Equation (4.4.2) becomes

$$d^2y = f''(x)dx^2 . \qquad (4.4.4)$$

If Equation (4.4.4) is used as the definition of d^2y, it is not invariant in form. This is because Equation (4.4.4) is only satisfied when the Weights of dx at different points are specified to be equal, and is not satisfied in general.

To simplify the research, only high order differentials with single degree of freedom are discussed in rest of the paper, and $y = f(x)$ is always discussed on one of its monotonic intervals. As the Weight of dx at a given point is specified to be always the same each time dx is generated, the Weight of dy at a given point is always the same too, each time dy is generated. Therefore, all one needs to do is specify the functional relationship between dx and x once, or specify the functional relationship between dy and y once, and the Weight of every order of differential of every variable at every point is thus determined.

Equation (4.4.1) to (4.4.3) are the equations of first, second and third order differentials derived from $y = f(x)$. Similarly, equations of first, second and third order differentials can also be derived from $x = g(y)$, and they are equivalent to Equations (4.4.1) to (4.4.3) respectively. Equations of even higher order differentials can be derived by taking differentials continuously on both sides of Equation (4.4.3).

4.5 Geometric Meanings of High Order Differentials

As depicted in the left of Fig. 15, the height of a square prism is h, while the length of each side of the basal square is x. Therefore, the volume of the prism is $y = x^2h$. When the length of each side of the basal square generates an increment dx, the volume of the square generates an increment dy. It can be seen from the figure that the geometric meaning of dy is like a hinge of a door which is shown again separately in the right of the figure for clarity. It is easy to know that $dy = (x+dx)hdx + xhdx = 2xhdx + hdx^2 = 2xhdx$. Since hdx^2 is a higher Order infinitesimal compared to $2xhdx$, it has no effect when added to $2xhdx$.

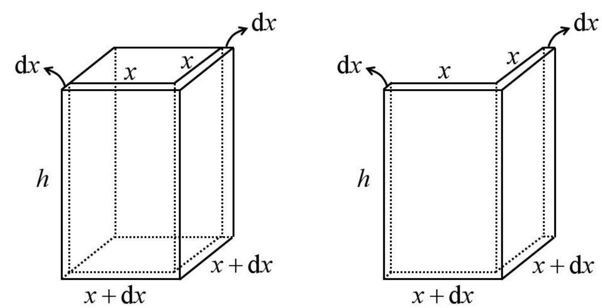

Fig. 15. The Geometric Meaning of a First Order Differential

As shown in Fig. 16, if x generates an increment again, recorded as Δx , then two prisms are obtained. The length of each side of the basal square of one of the prisms is x, while the length of each side of the basal square of the other prism is $x + \Delta x$. The volume increment caused by the dx that happens in the first prism is $dy = 2xhdx$, and its geometric meaning is a small hinge. By contrast, the volume increment caused by the dx that happens in the second prism is $dy = 2(x + \Delta x)hdx$, and its geometric meaning is a large hinge. For clarity, these two hinges are shown separately in the right of the figure again. If dx is specified to be equal at different points, then the thickness of the two hinges are the same, both equal to dx. Now, the two hinges can be put together overlapping with each other, and the difference between their volumes are the shaded areas in the left of Fig. 17. This volume difference represents the $\Delta(\mathrm{d}y)$ caused by Δx .

If $\Delta x \to 0$, then Δx becomes dx generated for the second time, and $\Delta(\mathrm{d}y)$ becomes d^2y. According to the stipulation of single freedom, the dx generated at a given point each time is always the same. Therefore, as shown in the right of Fig. 17, the shaded areas in the left of Fig. 17 become two infinitesimally thin square prisms which are the geometric meaning of the second order differential d^2y in this problem.

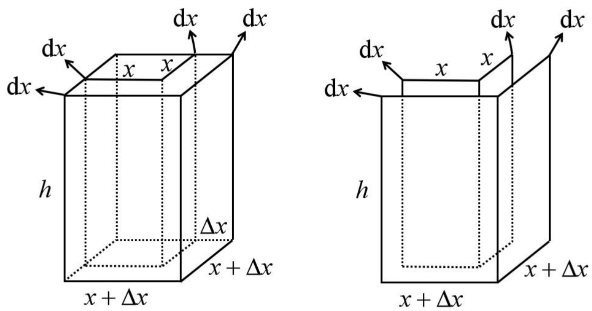

Fig. 16. First Order Differentials at Two Different Positions

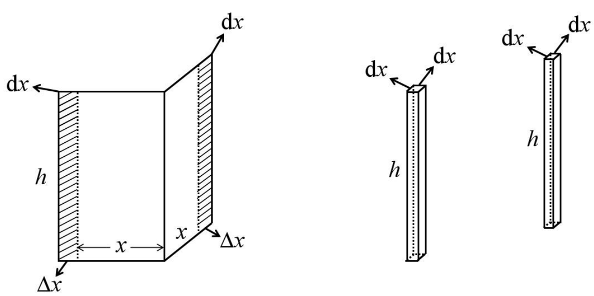

Fig. 17. The Geometric Meaning of a Second Order Differential

This shows that, when the original function $y = x^2h$ represents the volume of a three dimensional object, the first order differential $\mathrm{d}y = 2xh\mathrm{d}x$ represents the volume of some surfaces, while the second order differential $\mathrm{d}^2y = 2h\mathrm{d}x^2$ represents the volume of some lines. It is worth noting that the $\mathrm{d}^2y = 2h\mathrm{d}x^2$ here is obtained on the assumption that the Weight of dx is the same at different points. Therefore, this is a second order differential of a special case. In general, the Weight of dx is not the same at different points, and consequently the second order differential of y should be expressed in its full form, i.e. $\mathrm{d}^2y = 2h\mathrm{d}x^2 + 2xh\mathrm{d}^2x$. In this case, the geometric meaning of d^2y not only includes the two infinitesimally thin square prisms in the right of Fig. 17, but also includes the product of d^2x and $2xh$, where d^2x represents the difference between the thicknesses of the two hinges while $2xh$ represents the area of one of the hinges.

4.6 Expressing High Order Derivatives with High Order Differentials

Suppose $y = f(x)$ is a monotonic function of x on a certain interval, and the functional relationship can also be written as $x = g(y)$. Starting from $y = f(x)$, the constraints between first, second and third order differentials of x and y can be deduced successively, resulting in Equation (4.4.1) to (4.4.3) which are displayed again as follows.

$$\mathrm{d}y = f'(x)\mathrm{d}x \quad (4.4.1)$$

$$\mathrm{d}^2y = f''(x)\mathrm{d}x^2 + f'(x)\mathrm{d}^2x \quad (4.4.2)$$

$$\mathrm{d}^3y = f^{(3)}(x)\mathrm{d}x^3 + 3f''(x)\mathrm{d}x\mathrm{d}^2x + f'(x)\mathrm{d}^3x \quad (4.4.3)$$

Starting from $x = g(y)$, the constraints between first, second and third order differentials of x and y can be deduced again, resulting in the following three equations.

$$\mathrm{d}x = g'(y)\mathrm{d}y \quad (4.6.1)$$

$$\mathrm{d}^2x = g''(y)\mathrm{d}y^2 + g'(y)\mathrm{d}^2y \quad (4.6.2)$$

$$\mathrm{d}^3x = g^{(3)}(y)\mathrm{d}y^3 + 3g''(y)\mathrm{d}y\mathrm{d}^2y + g'(y)\mathrm{d}^3y \quad (4.6.3)$$

Equation (4.4.1) is equivalent to Equation (4.6.1). Equation (4.4.2) is equivalent to Equation (4.6.2). Equation (4.4.3) is equivalent to Equation (4.6.3). These equations can be collectively called "the constraints between high order differentials", with first order differential deemed as a special case of high order differentials. The high order differentials studied here are all the single freedom high order differentials described in Section 4.4.

Equation (4.4.1) can be transformed into

$$f'(x) = \frac{\mathrm{d}y}{\mathrm{d}x} . \quad (4.6.4)$$

So the first order derivative of y with respect to x, namely $f'(x)$, is expressed with first order differentials dy and dx. Taking derivative again on both sides of Equation (4.6.4), and using Lemma 4.2, results in

$$f''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{1}{\mathrm{d}x}\frac{\mathrm{d}x\mathrm{d}^2y - \mathrm{d}y\mathrm{d}^2x}{\mathrm{d}x^2} = \frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2x}{\mathrm{d}x^2} ,$$

namely

$$f''(x) = \frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2x}{\mathrm{d}x^2} . \quad (4.6.5)$$

So the second order derivative of y with respect to x, namely $f''(x)$, is expressed with second order differentials d^2y and d^2x as well as first order differentials dy and dx. Equation (4.6.5) is equivalent to Equation (4.4.2).

Similarly, Equation (4.6.1) can be transformed into

$$g'(y) = \frac{\mathrm{d}x}{\mathrm{d}y} . \quad (4.6.6)$$

Taking derivative again on both sides of the equation with respect to y, results in

$$g''(y)=\frac{\mathrm{d}^2x}{\mathrm{d}y^2}-\frac{\mathrm{d}x}{\mathrm{d}y}\frac{\mathrm{d}^2y}{\mathrm{d}y^2}\,. \quad (4.6.7)$$

In this way, the first order and second order derivatives of x with respect to y are expressed with differentials of different orders. Equation (4.6.7) is equivalent to Equation (4.6.2), and is also equivalent to Equation (4.6.5).

Starting from Equation (4.6.5) or Equation (4.6.7), taking derivatives continuously with respect to x or with respect to y on both sides of the equal sign, and using Lemma 4.1 and Lemma 4.2, results in derivatives of even higher orders expressed with differentials of different orders.

It is easy to know from Equation (4.6.5) that, if the values of dx at different points are specified to be the same, then $\mathrm{d}^2x = 0$. Only in this special case is $f''(x)$ equal to the quotient of d^2y and $\mathrm{d}x^2$, namely $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$. In general, $f''(x)$ is not equal to the quotient of d^2y and $\mathrm{d}x^2$. However, traditionally, people are accustomed to using $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ to represent the second order derivative of y with respect to x. Therefore, in this traditional way of expression, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ has to be treated as an indivisible whole. If one wants to treat $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ as divisible, then he has to express the second order derivative $f''(x)$ with Equation (4.6.5).

5 Further Discussion

5.1 Is There a General Equation for n-th Order Differentials?

From Equation (4.4.1) to Equation (4.4.3), the formulas of first, second and third order differentials are deduced in sequence. And this deduction process can be continued. The higher the order is, the more complex the formula is. Then, is there a general formula for n-th order differentials? This is a question to be answered in the future.

Similarly, from Equation (4.6.4) to Equation (4.6.5), the formulas expressing first and second order derivatives with differentials of corresponding orders are deduced successively, and this deduction process can be continued to obtain the expressions of even higher order derivatives. Then, for a general n-th order derivative, is there a universal formula to express it with high order differentials? In fact, this is essentially the same problem as the former one, because it can be discovered by observing Equation (4.4.2) and (4.4.3) with Lemma 4.1 that, for the general n-th order differential $\mathrm{d}^n y$, the first and last item in its expression must be $f^{(n)}(x)\mathrm{d}x^n$ and $f'(x)\mathrm{d}^n x$ respectively, while the intermediate items include $f^{(k)}(x)$ in sequence, where $k=n-1,n-2,\ldots,2$. Therefore, the high order differential expression of $f^{(n)}(x)$ can be obtained by subtracting all the items on the right side of the equation except $f^{(n)}(x)\mathrm{d}x^n$ and then dividing both sides of the equation with $\mathrm{d}x^n$.

Though it seems difficult to obtain the explicit form of the general formula of n-th order differentials, it is quite easy to deduce the recursion forms of the general formula, with the help of Leibniz's Formula.

Theorem 5.1 (Leibniz's Formula) If functions u and v have derivatives of any order, then

$$(u\cdot v)^{(n)}=\sum_{k=0}^{n}C_n^k u^{(n-k)}v^{(k)}\,, \quad (5.1.1)$$

where $C_n^k=\frac{n!}{k!(n-k)!}$, $u^{(0)}=u$ and $v^{(0)}=v$.

The derivative and the differential of the product of two functions obey similar operational laws. The derivative obeys

$$(u\cdot v)'=u'\cdot v+u\cdot v'\,,$$

while the differential obeys

$$\mathrm{d}(u\cdot v)=\mathrm{d}u\cdot v+u\cdot \mathrm{d}v\,.$$

Therefore, Leibniz's Formula can be adapted to high order differentials.

Theorem 5.2 If functions u and v have derivatives of any order, then

$$\mathrm{d}^n(u\cdot v)=\sum_{k=0}^{n}C_n^k \mathrm{d}^{n-k}u\,\mathrm{d}^k v\,, \quad (5.1.2)$$

where $C_n^k=\frac{n!}{k!(n-k)!}$, $\mathrm{d}^0u=u$ and $\mathrm{d}^0v=v$.

Suppose $y=f(x)$, then $\mathrm{d}y=f'(x)\mathrm{d}x$. Taking $(n-1)$th order differential on both sides of the equation, results in

$$\mathrm{d}^n y=\mathrm{d}^{n-1}(\mathrm{d}y)=\mathrm{d}^{n-1}(f'(x)\mathrm{d}x)\,. \quad (5.1.3)$$

Replacing the n, u and v in Equation (5.1.2) with $n-1$, $f'(x)$ and dx respectively, results in

$$\mathrm{d}^{n-1}(f'(x)\mathrm{d}x)=\sum_{k=0}^{n-1}C_{n-1}^k \mathrm{d}^{n-1-k}f'(x)\mathrm{d}^{k+1}x\,. \quad (5.1.4)$$

Taking the equation $y=f(x)$ into account, the following formula can be deduced from Equation (5.1.3) and (5.1.4):

$$\mathrm{d}^n f(x)=\sum_{k=0}^{n-1}C_{n-1}^k \mathrm{d}^{n-1-k}f'(x)\mathrm{d}^{k+1}x\,, \quad (5.1.5)$$

where $n \geqslant 2$.

Replacing the n, u and v in Equation (5.1.2) with $n-1$, dx and $f'(x)$ respectively, results in

$$\mathrm{d}^{n-1}(f'(x)\mathrm{d}x) = \sum_{k=0}^{n-1} C_{n-1}^{k} \mathrm{d}^{k} f'(x) \mathrm{d}^{n-k} x \ . \tag{5.1.6}$$

Taking the equation $y = f(x)$ into account, the following formula can be deduced from Equation (5.1.3) and (5.1.6):

$$\mathrm{d}^{n} f(x) = \sum_{k=0}^{n-1} C_{n-1}^{k} \mathrm{d}^{k} f'(x) \mathrm{d}^{n-k} x \ , \tag{5.1.7}$$

where $n \geqslant 2$.

Equation (5.1.5) and (5.1.7) are equivalent. They are both recursion formulas, expressing the n-th order differential $\mathrm{d}^{n} f(x)$ with the 0th to $(n-1)$th order differentials of the derivative function $f'(x)$ and the 1st to n-th order differentials of x.

5.2 Literal "High Order Differential" Equation

Traditional "high order differential equations" are actually "high order derivative equations" composed of derivatives of different orders, like

$$y'' + xy' + y = 0 \ .$$

High order differentials defined on the basis of OII Number Field are invariant both in form and in meaning. Therefore, it is now possible to study equations directly composed of high order differentials, making the phrase "differential equation" worth its name. To facilitate discussion, only high order differentials with single degree of freedom as described in Section 4.4 are studied here.

Equations directly built with high order differentials can be further classified into two categories: those equivalent to high order derivative equations, and those not.

For instance, $\mathrm{d}^2 y \mathrm{d}x - \mathrm{d}y \mathrm{d}^2 x - \mathrm{d}y \mathrm{d}x^2 = x \mathrm{d}x^3$ can be transformed to a high order derivative equation. In fact, dividing both sides of the equation with dx^3, results in

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2 x}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} = x \ .$$

According to Equation (4.6.4) and (4.6.5), it is equivalent to

$$y'' - y' = x \ .$$

Now that it can be transformed to a high order derivative equation, it can be solved as a traditional high order derivative equation.

If an equation composed of high order differentials cannot be transformed to a high order derivative equation, then it is generally an "indefinite differential equation". For example,

$$\mathrm{d}^2 y + \mathrm{d}x\mathrm{d}y + y\mathrm{d}x^2 = 0 \tag{5.2.1}$$

is an indefinite differential equation. In fact, according to Equation (4.4.2), namely

$$\mathrm{d}^2 y = y''\mathrm{d}x^2 + y'\mathrm{d}^2 x \ ,$$

and by substituting for d^2y in Equation (5.2.1), the following equation is obtained

$$y''\mathrm{d}x^2 + y'\mathrm{d}^2 x + \mathrm{d}x\mathrm{d}y + y\mathrm{d}x^2 = 0 \ . \tag{5.2.2}$$

Dividing both sides of the equation with dx^2, results in

$$y'' + y'\frac{\mathrm{d}^2 x}{\mathrm{d}x^2} + y' + y = 0 \ . \tag{5.2.3}$$

High order differentials have one degree of freedom, which means that the functional relationship between dx and x can be defined arbitrarily. Suppose $\mathrm{d}x = h(x)(0)^1$, then

$$\mathrm{d}^2 x = \mathrm{d}(\mathrm{d}x) = h'(x)(0)^1 \mathrm{d}x \ ,$$

and it follows that

$$\frac{\mathrm{d}^2 x}{\mathrm{d}x^2} = \frac{h'(x)(0)^1 \mathrm{d}x}{\mathrm{d}x^2} = \frac{h'(x)(0)^1}{\mathrm{d}x} = \frac{h'(x)(0)^1}{h(x)(0)^1} = \frac{h'(x)}{h(x)} \ . \tag{5.2.4}$$

So Equation (5.2.3) can be transformed to

$$y'' + y'\frac{h'(x)}{h(x)} + y' + y = 0 \ , \tag{5.2.5}$$

and can be further transformed to

$$\frac{h'(x)}{h(x)} = -\frac{y'' + y' + y}{y'} \ . \tag{5.2.6}$$

For any function $y = f(x)$, the expression on the right side of Equation (5.2.6) can be represented with $w(x)$, and the equation can be therefore rewritten as

$$\frac{\mathrm{d}h}{\mathrm{d}x}\frac{1}{h} = w(x)$$

which can be transformed to

$$\frac{\mathrm{d}h}{h} = w(x)\mathrm{d}x \ .$$

Taking indefinite integral on both sides of the equation, results in

$$\ln|h| = \int w(x)\mathrm{d}x + C_1 \ ,$$

where C_1 is an arbitrary constant. This equation can be further transformed to

$$h = \pm e^{C_1} e^{\int w(x)\mathrm{d}x} = Ce^{\int w(x)\mathrm{d}x} \ ,$$

where C is an arbitrary nonzero constant. Since $h(x)$ can be defined arbitrarily, this means that any function $y = f(x)$ can satisfy Equation (5.2.5), with $h(x)$ defined as

$$h(x) = Ce^{\int w(x)\mathrm{d}x} \ ,$$

where $w(x) = -\dfrac{f''(x) + f'(x) + f(x)}{f'(x)}$.

In other words, by specifying $\mathrm{d}x = h(x)(0)^1 = Ce^{\int w(x)\mathrm{d}x}(0)^1$, any function $y = f(x)$ can satisfy Equation (5.2.3). This means that Equation (5.2.3) is an indefinite differential equation. As

Equation (5.2.3) and Equation (5.2.1) are equivalent, Equation (5.2.1) is also an indefinite differential equation.

Multiple indefinite differential equations may constitute a "definite differential equation set". For instance, the equation set

$$\begin{cases} d^2y + dxdy + ydx^2 = 0 \\ (1+x)d^2y + d^2x = 0 \end{cases} \tag{5.2.7}$$

is composed of two indefinite differential equations. The first equation has already been analyzed above. It is also easy to prove that the second equation is an indefinite differential equation. According to Equation (4.4.2), namely $d^2y = y''dx^2 + y'd^2x$, and by substituting for the d^2y in the second equation, the following equation can be obtained

$$(1+x)y'' + (1+y'+xy')\frac{d^2x}{dx^2} = 0 . \tag{5.2.8}$$

Similar to the analysis of the first equation, it can be deduced that any function $y = f(x)$ can satisfy Equation (5.2.8), making it an indefinite differential equation. The first equation in Equation Set (5.2.7) is equivalent to Equation (5.2.3), while the second equation is equivalent to Equation (5.2.8). As $\frac{d^2x}{dx^2}$ is contained both in Equation (5.2.3) and Equation (5.2.8), $\frac{d^2x}{dx^2}$ can be canceled out, resulting in

$$(1+x)y''y' - (1+y'+xy')(y''+y'+y) = 0$$

which can be further reduced to

$$y'' + (1+y'+xy')(y'+y) = 0 . \tag{5.2.9}$$

Equation (5.2.9) is a second order derivative equation. Therefore, the two indefinite differential equations in Equation Set (5.2.7) form a "definite differential equation set" which can finally be reduced to a high order derivative equation.

In some application problems, it might be possible to write out differential equation (set) directly with high order differentials. There are two situations. One is the case when a single differential equation which can be transformed into a derivative equation is obtained. The other is the case when multiple "indefinite differential equations" are obtained, constituting a "definite differential equation set". The latter situation might be more common. As a result, high order differentials might facilitate the processes of building mathematical models for complex application problems.

5.3 Application of OII Number Field

As the theoretical basis of high order differentials, OII Number Field is very helpful in many fields. Problems related to infinitesimals and infinities are often complicated and not easy to explain. However, from the perspective of OII Number Field, those problems become very clear and explicit. Here is an example to demonstrate this.

Example Please explain the impulse response of a second order circuit.

Answer

In the second order circuit as shown in Fig. 18, the property of the capacitor is described by

$$i = C\frac{du_C}{dt} , \tag{5.3.1}$$

while the property of the inductor is described by

$$u_L = L\frac{di}{dt} . \tag{5.3.2}$$

Therefore,

$$u_L = L\frac{d}{dt}\left(C\frac{du_C}{dt}\right) = LC\frac{d}{dt}\left(\frac{du_C}{dt}\right) . \tag{5.3.3}$$

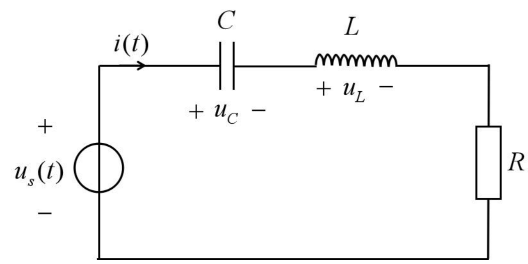

Fig. 18. A Second Order Circuit

According to Kirchhoff's Voltage Law,

$$u_L + iR + u_C = u_s . \tag{5.3.4}$$

From Equation (5.3.1), (5.3.3) and (5.3.4), the following equation is obtained

$$LC\frac{d}{dt}\left(\frac{du_C}{dt}\right) + RC\frac{du_C}{dt} + u_C = u_s . \tag{5.3.5}$$

When $u_s = \delta(t)$ which is a unit impulse function, the response $u_C(t)$ is to be determined from this equation.

The definition of $\delta(t)$ is: $\int_{0-}^{0+} \delta(t)dt = 1$, and when $t \neq 0$, $\delta(t) = 0$.

From the perspective of OII Number Field, $0-$ and $0+$ are two points on the left and right side of 0 respectively on the real number axis, with their distances from 0 being first Order infinitesimals. The dt at $t = 0$ is the time interval from $t = 0-$ to $t = 0+$, and can be recorded as $dt|_{0-<t<0+}$ which is a first Order infinitesimal too. If its Weight is specified to be 1, i.e. $dt|_{0-<t<0+} = (0)^1$, and if $\delta(t)$ is considered to be a rectangular impulse, then $\delta(0)$ is unit first Order infinity, i.e. $\delta(0) = (\infty)^1$. In this case, $\int_{0-}^{0+} \delta(t)dt$ represents the product of $dt|_{0-<t<0+}$ and $\delta(0)$, i.e. $(0)^1 \times (\infty)^1$, and the result is naturally 1.

That is to say, when the time interval dt from $t = 0-$ to $t = 0+$ is specified to be unit first Order infinitesimal $(0)^1$, and

$\delta(t)$ is considered to be a rectangular impulse, then the function $\delta(t)$ can be defined as

$$\delta(t)=\begin{cases}0\ , & t\le 0-\ or\ t\ge 0+\\ (\infty)^1\ , & 0-<t<0+\end{cases}. \qquad (5.3.6)$$

In this case, $\delta(t)$ is a rectangular impulse with its width being unit first Order infinitesimal $(0)^1$ and its height being unit first Order infinity $(\infty)^1$, as depicted in Fig. 19. Since first Order infinity appears in the function values of $\delta(t)$, $\delta(t)$ is different from conventional functions whose values are always finite numbers.

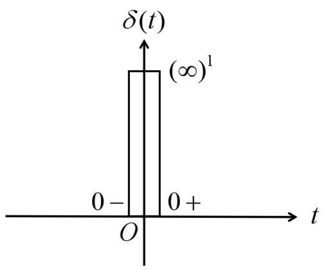

Fig. 19. Impulse Function $\delta(t)$

Now let us analyze the differential equation (5.3.5).

First of all, let us prove with reductio ad absurdum that u_C cannot jump at $t=0$.

If u_C is hypothesized to jump from $t=0-$ to $t=0+$, then $\mathrm{d}u_C$ across $t=0$ is a finite number rather than a first Order infinitesimal. Suppose $u_C(0-)=0$ and $u_C(0+)=k$, then the increment of u_C from $t=0-$ to $t=0+$ is $\mathrm{d}u_C\,|_{0-<t<0+}=k$, and therefore the derivative of u_C at t=0 is

$$\left.\frac{\mathrm{d}u_C}{\mathrm{d}t}\right|_{t=0}=\frac{\mathrm{d}u_C|_{0-<t<0+}}{\mathrm{d}t|_{0-<t<0+}}=\frac{k}{(0)^1}=k(\infty)^1\ ,$$

which means that $\left.\frac{\mathrm{d}u_C}{\mathrm{d}t}\right|_{t=0}$ is a first Order infinity with Weight k. Because the time interval $\mathrm{d}t\,|_{0-<t<0+}$ multiplied by the derivative $\left.\frac{\mathrm{d}u_C}{\mathrm{d}t}\right|_{t=0}$ equals the increment of u_C from $t=0-$ to $t=0+$, the derivative of u_C has to be constantly equal to $\left.\frac{\mathrm{d}u_C}{\mathrm{d}t}\right|_{t=0}$ at every point in the time interval from $t=0-$ to $t=0+$. This means that $\frac{\mathrm{d}u_C}{\mathrm{d}t}$ is a rectangular impulse with its width being unit first Order infinitesimal $(0)^1$ and its height being a first Order infinity $k(\infty)^1$, as depicted in Fig. 20. To make things clear, the value of $\frac{\mathrm{d}u_C}{\mathrm{d}t}$ in the time interval from $t=0-$ to $t=0+$ is recorded as $\left.\frac{\mathrm{d}u_C}{\mathrm{d}t}\right|_{0-<t<0+}$, and it is easy to know that

$$\left.\frac{\mathrm{d}u_C}{\mathrm{d}t}\right|_{0-<t<0+}=k(\infty)^1\ . \qquad (5.3.7)$$

In order to demonstrate the difference between a rectangular impulse and a non-rectangular impulse, Fig. 21 shows a triangular impulse with its width being unit first Order infinitesimal $(0)^1$ and its height being the same first Order infinity $k(\infty)^1$. Then the area of this triangular impulse is

$$\frac{1}{2}(0)^1\times k(\infty)^1=\frac{k}{2}\ ,$$

while the area of the rectangular impulse in Fig. 20 is k. This shows that a rectangular impulse is different from a triangular impulse even though their widths are the same infinitesimal and their heights are the same infinity.

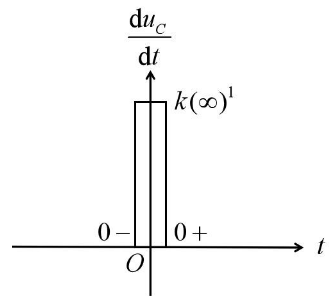

Fig. 20. A Rectangular Impulse

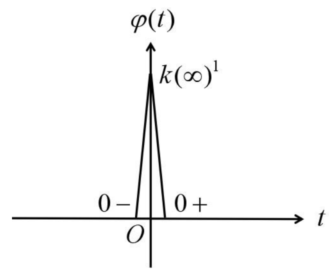

Fig. 21. A Triangular Impulse

It has been discussed previously in Section 3.3, Section 4.3 and Section 4.4 that the Weight of a differential has some freedom, or in other words, the Weight of a differential can be specified arbitrarily as long as it satisfies the constraints. The infinitesimal time interval from $t=0-$ to $t=0+$ is recorded as $\mathrm{d}t\,|_{0-<t<0+}$, and is specified as unit first Order infinitesimal $(0)^1$. The time interval from $t=0-$ to $t=0$ can be recorded as $\mathrm{d}t\,|_{0-<t<0}$, while the time interval from $t=0$ to $t=0+$ can be recorded as $\mathrm{d}t\,|_{0<t<0+}$. As long as $\mathrm{d}t\,|_{0-<t<0}$ and $\mathrm{d}t\,|_{0<t<0+}$ are infinitesimals with positive Weights satisfying the constraint condition $\mathrm{d}t\,|_{0-<t<0}+\mathrm{d}t\,|_{0<t<0+}=\mathrm{d}t\,|_{0-<t<0+}$, their Weights can actually be specified arbitrarily. Suppose their Weights are specified as a and b respectively, i.e. $\mathrm{d}t\,|_{0-<t<0}=a(0)^1$ and $\mathrm{d}t\,|_{0<t<0+}=b(0)^1$, where a and b are constants greater than 0 and $a+b=1$.

As $\frac{\mathrm{d}u_C}{\mathrm{d}t}$ is a rectangular impulse, it has a vertical rising edge at $t = 0-$, rising from 0 to $k(\infty)^1$. Because the width of this rectangular impulse is a first Order infinitesimal, the corresponding dt of the rising edge has to be a higher order infinitesimal. Let us deem it as a second Order infinitesimal with Weight c, then $\mathrm{d}t|_{U(0-)} = c(0)^2$, where $U(0-)$ represents an interval at $t = 0-$ with a width of a second Order infinitesimal. $U(0-)$ is the interval in which the rising edge of $\frac{\mathrm{d}u_C}{\mathrm{d}t}$ occurs. So the gradient of $\frac{\mathrm{d}u_C}{\mathrm{d}t}$ in the interval $U(0-)$ can be computed as

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}u_C}{\mathrm{d}t}\right)\right|_{U(0-)} = \frac{k(\infty)^1 - 0}{c(0)^2} = \frac{k}{c}(\infty)^3, \qquad (5.3.8)$$

which indicates that the $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}u_C}{\mathrm{d}t}\right)$ in Differential Equation (5.3.5) is a third Order infinity in the interval $U(0-)$.

It is easy to know that, in the interval $U(0-)$, $\frac{\mathrm{d}u_C}{\mathrm{d}t}$ rises from 0 to a first Order infinity $k(\infty)^1$, u_C remains 0, while $u_s = \delta(t)$ rises from 0 to a first Order infinity $(\infty)^1$.

When inspecting Differential Equation (5.3.5) in the interval $U(0-)$, it can be found that the summation result of the left side of the equation is a third Order infinity while the right side of the equation is a first Order infinity. Because the two infinities are different in Order, the left side and the right side of the equation are not equal, i.e. Differential Equation (5.3.5) is not satisfied.

Similarly, it can be proved that in the interval $U(0+)$, which is an interval at $t = 0+$ with a width of a second Order infinitesimal, Differential Equation (5.3.5) is not satisfied.

However, the differential equation obtained with Kirchhoff's Voltage Law should be satisfied at any moment. Therefore, the hypothesis that u_C jumps from $t = 0-$ to $t = 0+$ is wrong. This means that u_C is continuous at $t = 0$.

To make the differential equation satisfied, $\frac{\mathrm{d}u_C}{\mathrm{d}t}$, instead of u_C, should be assumed to jump at $t = 0$ with a finite increment in value. Only under this assumption are both sides of the differential equation first Order infinities which are possible to be equal. In this case, in the time interval from $t = 0-$ to $t = 0+$, the left side of Differential Equation (5.3.5) is the summation of a first Order infinity $LC\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}u_C}{\mathrm{d}t}\right)$, a finite number $RC\frac{\mathrm{d}u_C}{\mathrm{d}t}$, and u_C which remains 0. The result of this summation is the first Order infinity $LC\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}u_C}{\mathrm{d}t}\right)$.

Meanwhile, the right side of the equation is the impulse function $\delta(t)$, whose value is unit first Order infinity $(\infty)^1$. Therefore,

$$LC\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}u_C}{\mathrm{d}t}\right) = \delta(t), \text{ where } 0- < t < 0+.$$

Multiplying both sides with dt, results in

$$LC\cdot\mathrm{d}\left(\frac{\mathrm{d}u_C}{\mathrm{d}t}\right) = \delta(t)\mathrm{d}t, \text{ where } 0- < t < 0+.$$

This equation can also be written as

$$LC\cdot\left(\left.\frac{\mathrm{d}u_C}{\mathrm{d}t}\right|_{t=0+} - \left.\frac{\mathrm{d}u_C}{\mathrm{d}t}\right|_{t=0-}\right) = (\infty)^1 \times (0)^1$$

which is reduced to

$$LC\cdot\left(\left.\frac{\mathrm{d}u_C}{\mathrm{d}t}\right|_{t=0+} - 0\right) = 1,$$

i.e.

$$LC\left.\frac{\mathrm{d}u_C}{\mathrm{d}t}\right|_{t=0+} = 1.$$

Taking Equation (5.3.1) into account, results in

$$Li(0+) = 1.$$

Therefore,

$$i(0+) = \frac{1}{L},$$

which means that Current i jumps from 0 to $\frac{1}{L}$ at $t = 0$. This conclusion is the same as that in a circuit textbook.

5.4 Fractional Order Differentials

Fractional order calculus has a history of hundreds of years, and developed in parallel with integer order calculus. The theory of fractional order calculus gives fractional order derivatives their definition, such as the 0.5th order and 1.6th order derivatives of y with respect to x. Now that fractional order derivatives exist, it is natural to ask whether there are fractional order differentials.

From the perspective of OII Number Field, since integer order differentials are infinitesimals of integer Orders, fractional order differentials should naturally be infinitesimals of fractional Orders. As shown in Fig. 22, a 0.5th order differential should be located on the infinitesimal number axis of Order 0.5, and a 1.6th order differential should be located on the infinitesimal number axis of Order 1.6.

Then the key problem is how to define fractional order differentials, or equivalently, what is the relationship between fractional order differentials and fractional order derivatives.

If simply defining the 1.6th order differential of $y = f(x)$ as $\mathrm{d}^{1.6}y = f^{(1.6)}(x)\mathrm{d}x^{1.6}$, namely the 1.6th power of dx times the 1.6th order derivative of y with respect to x, then it might not be a good definition. As mentioned previously in Section 4.4, if

d^2y is defined as $d^2y = f''(x)dx^2$, then it is not invariant in form, because this equation is only satisfied when the Weights of dx at different points are specified to be equal, and in general this equation is not satisfied. Instead, the generally satisfied equation is Equation (4.4.2) which is displayed again here:

$$d^2y = f''(x)dx^2 + f'(x)d^2x\,. \qquad (4.4.2)$$

Then, for fractional order differentials, is there a form invariant general equation like Equation (4.4.2)? This is a problem to be solved in the future.

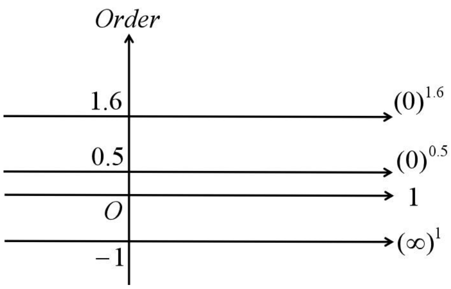

Fig. 22. Number Axes of Fractional Order Infinitesimals

6 Summary

From the traditional definition of infinitesimals, it can be discovered that an infinitesimal is essentially a variable running toward 0. By defining OII Number Field, the processes of infinitesimals running toward 0 are condensed onto the limits. The final foothold of a running infinitesimal variable is no longer 0, but a number in the OII Number Field, with a certain Order and a certain Weight. In this way, the relative sizes of different infinitesimals with functional relationship are clear and explicit.

The OII Number Field can be seen as an extension of the real number field. Different infinitesimals and infinities are all endowed with Orders and Weights, turned into a kind of number which can be operated. On this basis, definite integral is explained as “the summation of first Order infinitesimals with different Weights in such large number as a first Order infinity”.

With OII Number Field, the differential of a variable at a point can be treated as a number. In this way, differentials are not only invariant in form, but also invariant in meaning. At a given point, no matter dy is caused by which variable, it has the same Order and Weight in OII Number Field. Therefore, dy is always equal to the infinitesimal increment of variable y itself. The properties of differentials are summarized into three items:

1) Meaning invariance,

2) Form invariance, i.e. the relationship between the differentials of the two variables x and y in the univariate function $y = f(x)$ is not affected by any other variable, and

3) Some freedom, i.e. the Weight can be specified arbitrarily on condition that the constraint is satisfied.

Now that a differential at a given point becomes a number, the same differential at different points then form a variable which can generate a differential again, thus giving rise to high order differentials. It turns out that high order differentials also satisfy the above three properties. To simplify the research, the degree of freedom of any high order differential is restricted to 1. After that, constraint equations satisfied by high order differentials of two variables with functional relationship are deduced, with each order corresponding to one constraint equation.

Thus it can be seen that the research of high order differentials cannot go without OII Number Field. Traditionally, without OII Number Field, infinitesimals can only be studied by investigating the processes of variables running toward 0, thus leading to the consequence that differentials are only invariant in form but not invariant in meaning. Under this circumstance, the research of high order differentials is extremely difficult. Thanks to OII Number Field, all about infinitesimals and infinities becomes concise and explicit, making high order differentials exist naturally.

On this basis, differential equations directly built with high order differentials are discussed. Some problems in physics and engineering might benefit from this method, with mathematical models directly built with high order differentials, resulting in literal “high order differential” equations. This might give rise to more ways and ideas in research work.

After that, the impulse response of a second order circuit, which is an engineering problem, is analyzed from the perspective of OII Number Field. This example demonstrates the fact that OII Number Field not only facilitates the research of high order differentials, but also brings convenience to the analysis of problems in physics and engineering disciplines.

In addition, two problems to be solved are presented. The first one is about the existence of a general formula describing the relationship between n-th order differentials and n-th order derivatives. The second one is about the definition of fractional order differentials and their relationship with fractional order derivatives.

Reference

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