Euler's Formula Motivated, Trigonometric Derivatives Eased Too

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Abstract

This article gives a way to quickly understand Euler's formula.

Introduction

Euler's formula, $e^{\pi i} = -1$ is arguably the most important and wonderful equation of mathematics. Most mathematicians want to see it as soon as possible in mathematics courses, even elementary algebra. But alas! It is mentioned in Blitzer's algebra textbook [2], but without any proof or argument for why it might be true. In calculus texts, such as Larson's [4], it is implied in a problem, but not stated directly. It is certainly possible to prove it using Taylor series for e^x, $\sin x$, and $\cos x$, but as complex numbers are thought of as off topic for elementary calculus, this isn't generally done – Larson in his calculus text doesn't do it, for example. Stewart [7] does give this Taylor series proof albeit in an appendix. An older introductory analysis (now termed pre-calculus) text does give such a proof in a regular chapter [3] and an older trigonometry book, [5] does give in some sense a more intuitive or deeper proof as part of its chapter on series. We reproduce these proofs below. Differential equations texts rely on complex numbers and in particular $e^{ix} = \cos x + i \sin x$, also called Euler's formula. In Zill's book on the subject [8], he gives the series proof in footnote form. Later in the mathematical curriculum, in real and complex analysis books, such at [1, 6], e^x is defined with a series and e^{ix} is used to define, with series, $\cos x$ and $\sin x$. We want a quick idea as to why it might be true. That's attempted here.

DeMoivre's Theorem

DeMoivre's theorem really does the trick. The theorem:

$$(\cos x + i \sin x)^n = \cos nx + i \sin nx$$

and more in particular its beginnings,

$$(\cos x + i \sin x)(\cos y + i \sin y) = \cos(x + y) + i \sin(x + y)$$

should immediately be likened to $(e^x)^n = e^{nx}$ and $e^x \cdot e^y = e^{x+y}$, respectively: exponential ideas. Granted there isn't any particular reason why the base should be e and not some other number, but the exponential identities are clear and the point to be emphasized.

The back story

The introduction of $\cos x + i \sin x$ is just a gloss on the plane. Typically a unit circle is used to develop the circular (as opposed to trigonometric) functions. The Cartesian versus the complex plane are the same plane with multiplication (complex) possible.

The *geometry* glitch solved

The derivation of the derivative of e^x is e^x is straight forward and unlike the derivative of the sin does not involve a geometric proof of ancillary limits.

We can use e^x to find two derivatives. If we assume

$$e^{ix} = \cos x + i \sin x$$

and

$$\frac{d}{dx} e^{ix} = i e^{ix} = -\sin x + i \cos x,$$

then we can conclude

$$\frac{d}{dx} \cos x = -\sin x \text{ and } \frac{d}{dx} \sin x = \cos x.$$

If it is desired to avoid the geometric proofs that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \text{ and } \lim_{x \to 0} \frac{1 - \cos x}{x} = 0, \tag{1}$$

then the question is how to prove Euler without appeals to series, appeals to Taylor that are dependent on derivatives of cos and sin. Dolciani gives the way.

Using approximations for Euler

Here is a reproduction of a proof that $e^{ix} = \cos x + i \sin x$ given in [5]. It is based on DeMoivre's theorem and the identity

$$\lim_{n\to\infty}(1+1/n)^n = e. \tag{2}$$

Blitzer [2] uses (2) to define e. Larson [4] proofs it in an appendix. One can start with the series for e and apply the binomial theorem to (2) and grind it out. Rudin [6] uses this method. Larson's a little quicker and more elegant than Rudin, but not as quick and elegant as Stewart [7].

Let $f(x) = \ln x$ then $f'(x) = 1/x$ and so $f'(1) = 1$. Using the definition of the derivative as a limit, this means

$$f'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{x\to 0}\ln(1+x)^{1/x} = 1.$$

But what must e be raised to in order to equal e^1? This gives (2).

Theorem 1.

$$e^{ix} = (\cos x + i \sin x) \tag{3}$$

Proof. We have, for large n,

$$\left(\cos\frac{x}{n} + i\sin\frac{x}{n}\right)^n \approx \left(1 + i\frac{x}{n}\right)^n,$$

using DeMoivre's gives

$$\cos x + i \sin x \approx \left(1 + i\frac{x}{n}\right)^n$$

and with some further algebraic manipulations this is $\cos x + i \sin x = e^{ix}$. $\square$

Euler regular

Once you have Taylor for cos and sin then the limits in (1) are trivial to prove. Just use

$$\cos x = 1 - \text{ terms with } x$$

for the first and

$$\sin x = x(1 - \text{ terms with } x \text{ to higher degrees than } 1)$$

for the second.

For Euler regular and assuming (3), just use

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

and substitute xi for x.

Conclusion

Motivation is the key for some mathematicians. The math given here seems more motivated and natural. The next stop is to cross reference trigonometric identities with solving integrals. It seems clear to me that solving integrals and differential equations piques juices more. All the major trigonometric identities are used in later math to good avail; perhaps modern students are ready to see this bigger picture sooner. It is the bigger picture that should be stressed – think of global warming. Just having obedience to and faith in authorities is silence and what is silence now?

References

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