Finding the Hamiltonian

Hung Tran

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Amsterdam, The Netherlands

Abstract

We first find a Hamiltonian $\hat{H}$ that has the Hurwitz zeta functions $\zeta(s,x)$ as eigenfunctions. Then we continue constructing an operator $\hat{G}$ that is self-adjoint, with appropriate boundary conditions. We will find that the $\zeta(s,x)$-functions do not meet these boundary conditions, except for the ones where s is a nontrivial zero of the Riemann zeta, with the real part of s being greater than $\frac{1}{2}$. Finally, we find that these exceptional functions cannot exist, proving the Riemann hypothesis, that all nontrivial zeros have real part equal to $\frac{1}{2}$.

1 Introduction

The Hurwitz zeta function $\zeta(s,x)$ can be represented by a Newton series representation [2].

$$\zeta(s,x) = \frac{1}{s-1}\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(x+k)^{1-s} \tag{1}$$

For $Re(s) > 1$, this function can be expressed as a single infinite series.

$$\zeta(s,x) = \sum_{n=0}^{\infty}\frac{1}{(n+x)^s} \tag{2}$$

The Riemann zeta function $\zeta(s,1)$ has trivial zeros at the negative even s-values. The Riemann hypothesis states that the nontrivial zeros of the Riemann zeta function are complex numbers with real part $\frac{1}{2}$. In this paper, we will prove for the first time that the Riemann hypothesis is true.

So now that we know what problem we are solving, lets go ahead and find a Hamiltonian!

2 Finding the Hamiltonian

Right now, I haven't even told you what a Hamiltonian is. But let me explain along the way, because we have a long journey ahead! First, lets expand the series in (2) and see what we are working with here.

$$\zeta(s,x) = \frac{1}{x^s} + \frac{1}{(1+x)^s} + \frac{1}{(2+x)^s} + \frac{1}{(3+x)^s} + ...$$

Now this seems like a nice pattern. When we shift x by one, we almost get the same series.

$$\zeta(s,x+1) = \frac{1}{(1+x)^s} + \frac{1}{(2+x)^s} + \frac{1}{(3+x)^s} + ...$$

The two series are almost the same, so subtracting might be a good idea here.

$$\zeta(s,x) - \zeta(s,x+1) = \frac{1}{x^s}$$

That leaves us with only one term, instead of infinite many. That's good progress so far. If we use the symbol Δ for these two operations combined, we'd be saving some ink and the environment.

$$\Delta\zeta(s,x) = \frac{1}{x^s} \tag{3}$$

This is known as the forward difference. Now what can we do next? Lets try differentiating.

$$\frac{\partial}{\partial x}\Delta\zeta(s,x) = -\frac{s}{x^{s+1}}$$

Every new operator that we put in front and on the left, acts on the function to the right of it.

We now have a different exponent in de denominator. We can fix this by multiplying by x.

$$x\frac{\partial}{\partial x}\Delta\zeta(s,x) = -\frac{s}{x^s}$$

This looks good. It looks similar to what we had before differentiating. We can even get back the infinite series by inverting the forward difference.

$$\Delta^{-1}x\frac{\partial}{\partial x}\Delta\zeta(s,x) = -s\left(\frac{1}{x^s} + \frac{1}{(1+x)^s} + ...\right) =$$
$$= -s\zeta(s,x)$$

After applying four operators, we got back our ζ-function, multiplied by $-s$. We can give this whole sequence of operations a name, $\hat{A} = \Delta^{-1}x\frac{\partial}{\partial x}\Delta$.

$$\hat{A}\zeta(s,x) = -s\zeta(s,x)$$

We call $\zeta(s,x)$ an eigenfunction of operator $\hat{A}$ with eigenvalue $-s$. For each value of s, we get a different eigenfunction with a different eigenvalue.

After the forward difference step, we could have also multiplied by x first, and then differentiate after. With

$\hat{B} = \Delta^{-1}\frac{\partial}{\partial x}x\Delta$, the ζ-functions turn out to be eigenfunctions of $\hat{B}$ as well, but with different eigenvalues $1-s$.

$$\hat{B}\zeta(s,x) = (1-s)\zeta(s,x)$$

We can also see that the order of operation can affect the result. Though some operators commute with each other. Differentiation commutes with the forward difference.

$$\frac{\partial}{\partial x}\Delta f(x) = f'(x) - f'(x+1) = \Delta\frac{\partial}{\partial x}f(x)$$

And differentiation also commutes with the backward difference ∇.

$$\frac{\partial}{\partial x}\nabla f(x) = f'(x) - f'(x-1) = \nabla\frac{\partial}{\partial x}f(x)$$

We can change the order for commuting operators. This will be important later on.

$$\frac{\partial}{\partial x}\Delta = \Delta\frac{\partial}{\partial x} \tag{4}$$

$$\frac{\partial}{\partial x}\nabla = \nabla\frac{\partial}{\partial x} \tag{5}$$

The sum of $\hat{A}$ and $\hat{B}$ looks very symmetrical and it has the nice property, that the ζ-functions are also eigenfunctions of $\hat{A}+\hat{B}$, with eigenvalues $1-2s$.

$$\hat{A}+\hat{B} = \Delta^{-1}\left(x\frac{\partial}{\partial x} + \frac{\partial}{\partial x}x\right)\Delta$$

We've now reached a point in our journey that several others [1] have reached before. Further along the road, there has been a lot of struggling. Right now, would be a good time to pause for a moment and look at some properties of the operators we have used so far.

In the next section we'll take some time to discuss the concept of inner products and adjoint operators. After that, we will continue finding the Hamiltonian and construct a new operator that is closely related to this Hamiltonian. The good thing about the new operator will be, that it is self-adjoint!

3 Inner product and adjoint operators

Now, you may already be familiar with the concept of an inner product or a dot product. For two complex functions f and g, we can use the folowing inner product.

$$\langle f|g\rangle = \int_{-\infty}^{\infty} dx\ \overline{f(x)}\ g(x) \tag{6}$$

It's good to notice that $\langle f|g\rangle$ is usually not the same as $\langle g|f\rangle$ for complex functions. Rather, they are each others complex conjugate.

$$\begin{aligned}\overline{\langle f|g\rangle} &= \overline{\int_{-\infty}^{\infty} dx\ \overline{f(x)}g(x)} = \\ &= \int_{-\infty}^{\infty} dx\ f(x)\ \overline{g(x)} = \\ &= \langle g|f\rangle\end{aligned}$$

Two functions are said to be orthogonal to each other if their inner product is zero. The inner product of a nonzero function with itself is always a positive real number. The norm of a function is the square root of this number.

$$\|f\| = \sqrt{\langle f|f\rangle} \tag{7}$$

If operator $\hat{V}$ is the adjoint of operator $\hat{W}$, then the following equality must hold by definition.

$$\left\langle \hat{V}f \middle| g\right\rangle = \left\langle f \middle| \hat{W}g\right\rangle \tag{8}$$

The position operator x is self-adjoint. Proof of this is very straightforward.

$$\begin{aligned}\langle f|xg\rangle &= \int_{-\infty}^{\infty} dx\ \overline{f(x)}\ xg(x) = \\ &= \int_{-\infty}^{\infty} dx\ \overline{xf(x)}\ g(x) = \\ &= \langle xf|g\rangle\end{aligned}$$

The differentiation operator $\frac{\partial}{\partial x}$ is not self-adjoint.

$$\begin{aligned}\left\langle f \middle| \frac{\partial}{\partial x}g\right\rangle &= \int_{-\infty}^{\infty} dx\ \overline{f(x)}\ g'(x) = \\ &= \left[\overline{f(x)}\ g(x)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dx\ \overline{f'(x)}\ g(x) = \\ &= -\left\langle \frac{\partial}{\partial x}f \middle| g\right\rangle \end{aligned} \tag{9}$$

We've used that $\left[\overline{f(x)}\ g(x)\right]_{-\infty}^{\infty} = 0$, which is valid under the following boundary conditions.

$$\lim_{x\to-\infty} \overline{f(x)}\ g(x) = 0 \tag{10}$$

$$\lim_{x\to\infty} \overline{f(x)}\ g(x) = 0 \tag{11}$$

We will want to replace the differentiation operator by the momentum operator p, which also includes differentiation.

$$p = -i\hbar\frac{\partial}{\partial x} \tag{12}$$

Because the momentum operator is self-adjoint, when the boundary conditions are met.

$$\left\langle f \middle| -i\hbar\frac{\partial}{\partial x}g\right\rangle = \left\langle -i\hbar\frac{\partial}{\partial x}f \middle| g\right\rangle \tag{13}$$

The momentum operator and the reduced Planck constant $\hbar$ play major roles in quantum physics.

The backward difference operator is the adjoint of the forward difference operator.

$$\begin{aligned}\langle f|\Delta g\rangle &= \int_{-\infty}^{\infty} dx\ \overline{f(x)}\,(g(x) - g(x+1)) = \\ &= \int_{-\infty}^{\infty} dx\ \overline{f(x)}\ g(x) - \int_{-\infty}^{\infty} dx\ \overline{f(x)}\ g(x+1) = \\ &= \int_{-\infty}^{\infty} dx\ \overline{f(x)}\ g(x) - \int_{-\infty}^{\infty} dy\ \overline{f(y-1)}\ g(y) = \\ &= \int_{-\infty}^{\infty} dx\ \overline{(f(x) - f(x-1))}g(x) = \\ &= \langle \nabla f|g\rangle\end{aligned} \tag{14}$$

Now we are ready to continue with finding the Hamiltonian and constructing the self-adjoint operator from it.

4 The Hamiltonian

At the end of section 2, we found that the ζ-functions were eigenfunctions of $\hat{A}+\hat{B}$, with eigenvalues $1-2s$.

$$\left(\hat{A}+\hat{B}\right)\zeta(s,x)=\Delta^{-1}\left(x\frac{\partial}{\partial x}+\frac{\partial}{\partial x}x\right)\Delta\ \zeta(s,x)= \\ =(1-2s)\zeta(s,x)$$

Now replacing the differentiation operator by the impulse operator, gives us a Hamiltonian $\hat{H}$.

$$\hat{H}=\Delta^{-1}\left(xp+px\right)\Delta \tag{15}$$

It has the same eigenfunctions, but the eigenvalues are now $-i\hbar(1-2s)$. In quantum mechanics, the eigenvalues of a Hamiltonian represent the stable energy levels of a system. The values for the energy levels are always real numbers. This has lead researchers to believe that studying this Hamiltonian, could lead to a proof of the Riemann hypothesis [1], because for $\hat{H}$ these eigenvalues can only be real when $Re(s)=\frac{1}{2}$.

One of the problems in moving forward from here, is that this Hamiltonian is not self-adjoint. If $\hat{H}$ were self-adjoint, then that would ensure the reality of the eigenvalues. If φ is a normalizable eigenfunction of a self-adjoint Hamiltonian $\hat{H}$ with corresponding eigenvalue λ, then the eigenvalue has to be real.

$$\lambda\left\langle\varphi|\varphi\right\rangle=\left\langle\varphi\middle|\hat{H}\varphi\right\rangle=\left\langle\hat{H}\varphi\middle|\varphi\right\rangle=\overline{\left\langle\varphi\middle|\hat{H}\varphi\right\rangle}=\overline{\lambda}\left\langle\varphi|\varphi\right\rangle$$

An eigenfunction φ is normalizable if the inner product $\langle\varphi|\varphi\rangle$ exists. This normalizability is exactly the second problem. The eigenfunctions $\zeta(s,x)$ have infinite norm for $s<1$.

We solve the first problem by constructing a new and self-adjoint operator $\hat{G}$. And by using this operater, we get around the second problem as well. The operator is be very similar to $\hat{H}$, only that Δ^{-1} has been replaced by the backward difference operator ∇.

$$\hat{G}=\nabla\left(xp+px\right)\Delta \tag{16}$$

But this substitution comes with a cost. The ζ-functions are not eigenfunctions of $\hat{G}$. But we do have a close relationship between $\hat{H}$ and $\hat{G}$.

$$\begin{aligned}\hat{G}&=\nabla\left(xp+px\right)\Delta= \\ &=\nabla\Delta\left(\Delta^{-1}\left(xp+px\right)\Delta\right)= \\ &=\nabla\Delta\hat{H}\end{aligned} \tag{17}$$

With this new operator, we can go forward and have $\hat{G}$ act on the ζ-functions.

$$\begin{aligned}\hat{G}\zeta(s,x)&=\nabla\Delta\hat{H}\zeta(s,x)= \\ &=-i\hbar(1-2s)\nabla\Delta\zeta(s,x)\end{aligned} \tag{18}$$

Finally, we take the following inner product.

$$\begin{aligned}\left\langle\zeta(s,x)\middle|\hat{G}\zeta(s,x)\right\rangle&=-i\hbar(1-2s)\left\langle\zeta(s,x)|\nabla\Delta\zeta(s,x)\right\rangle= \\ &=-i\hbar(1-2s)\left\langle\Delta\zeta(s,x)|\Delta\zeta(s,x)\right\rangle= \\ &=-i\hbar(1-2s)\left\langle x^{-s}\middle|x^{-s}\right\rangle\end{aligned} \tag{19}$$

So the eigenvalues λ_s of the Hamiltonian $\hat{H}$ can be expressed as follows.

$$\lambda_s=-i\hbar(1-2s)=\frac{\left\langle\zeta(s,x)\middle|\hat{G}\zeta(s,x)\right\rangle}{\left\langle x^{-s}|x^{-s}\right\rangle} \tag{20}$$

If inner product in the denominator in (20) does not diverge to infinity, it has to be a positive real number. And assuming that $\bar{G}$ is self-adjoint, the numerator in (20) must be real as well, as it would be equal to its complex conjugate.

$$\begin{aligned}\left\langle\zeta(s,x)\middle|\hat{G}\zeta(s,x)\right\rangle&=\left\langle\hat{G}\zeta(s,x)\middle|\zeta(s,x)\right\rangle= \\ &=\overline{\left\langle\zeta(s,x)\middle|\hat{G}\zeta(s,x)\right\rangle}\end{aligned} \tag{21}$$

Then we would have found proof that the eigenvalues of $\hat{H}$ are real for the values of s that meet the boundary conditions. But in general, the $\zeta(s,x)$-functions do not obey the boundary conditions.

In the next section we will find out what the boundary conditions are exactly and use it to prove the Riemann hypothesis. So lets move on to the endgame and determine boundary conditions for which $\hat{G}$ is self-adjoint.

5 Boundary conditions

We are at the final steps of proving the Riemann hypothesis. 160 years after the publishing of Riemann's Manuscript in 1859, the moment is here and now. So lets get onto it!

We need conditions on the ζ-functions for which $\hat{G}$ is self-adjoint. We first define another operator w. We will see why very soon.

$$w=\nabla x\Delta \tag{22}$$

The operator w is self-adjoint. We prove this by using (8) three times.

$$\begin{aligned}\langle f|wg\rangle&=\langle f|\nabla x\Delta g\rangle= \\ &=\langle\Delta f|x\Delta g\rangle= \\ &=\langle x\Delta f|\Delta g\rangle= \\ &=\langle\nabla x\Delta f|g\rangle= \\ &=\langle wf|g\rangle\end{aligned} \tag{23}$$

The operator $\hat{G}$ can be split into a sum of two operators.

$$\begin{aligned}\hat{G}&=\nabla\left(xp+px\right)\Delta= \\ &=\nabla xp\Delta+\nabla px\Delta= \\ &=\nabla x\Delta p+p\nabla x\Delta \\ &=wp+pw\end{aligned} \tag{24}$$

We let operator w act on the ζ-function.

$$\begin{aligned} w\zeta(s,x) &= \nabla x \Delta \zeta(s,x) = \\ &= \nabla x \left(x^{-s}\right) = \\ &= \nabla \left(x^{1-s}\right) = \\ &= x^{1-s} - (x-1)^{1-s} \end{aligned} \tag{25}$$

Now, lets find $pw\zeta(s,x)$. We will use it to determine the domain for the ζ-functions.

$$\begin{aligned} pw\zeta(s,x) &= -i\hbar \left(x^{1-s} - (x-1)^{1-s}\right) = \\ &= -i\hbar(1-s)\left(x^{-s} - (x-1)^{-s}\right) \end{aligned}$$

It's important to notice here, that $pw\zeta(s,x)$ is continuous on the domain $[1,\infty)$. If we start the domain at any value lower than 1, there would be a discontinuity at $x=1$ for $Re(s)>0$ and the formula for integration in (9) would not work.

Lets consider the following inner product.

$$\langle \zeta(s,x) | pw\zeta(s,x) \rangle \tag{26}$$

The same ζ-function appears on both sides of the inner product. If we used two different ζ-functions, it would not contribute at all to proving the reality of the eigenvalues.

The momentum operator p is self-adjoint, if the following condition applies at the left boundary of $x=1$ and for $x \to \infty$.

$$\overline{\zeta(s,x)}\ w\ \zeta(s,x) = 0$$

With (25), this becomes:

$$\overline{\zeta(s,x)}\left(x^{1-s} - (x-1)^{1-s}\right) = 0$$

The two boundary conditions are then as follows.

$$\overline{\zeta(s,1)} = 0 \tag{27}$$

$$\lim_{x\to\infty} \overline{\zeta(s,x)}\left(x^{1-s} - (x-1)^{1-s}\right) = 0 \tag{28}$$

The first boundary condition, requires that s is a zero of the Riemann zeta. Now for the second boundary condition, we need to find out how this expression

$$\overline{\zeta(s,x)}\left(x^{1-s} - (x-1)^{1-s}\right)$$

behaves for large values of x.

For the first factor $\overline{\zeta(s,x)}$, we can use $\zeta(s,x) - \zeta(s,x+1) = x^{-s}$. Because the forward difference of $\zeta(s,x)$ is x^{-s}, we can expect $\zeta(s,x)$ to grow like $x^{1-s}/(1-s)$. But this is an important part of the proof and we don't want to leave any doubts. So here are the more rigorous steps, using calculus.

$$\begin{aligned} -\zeta(s,1) &= 0 \\ \zeta(s,1) - \zeta(s,2) &= 1^{-s} \\ \zeta(s,2) - \zeta(s,3) &= 2^{-s} \\ \zeta(s,3) - \zeta(s,4) &= 3^{-s} \\ &\ldots \end{aligned}$$

If we use the first n equations above, take the sum of the left hand sides, then it must be equal to the sum of the right hand sides.

$$\begin{aligned} -\zeta(s,n) &= 1^{-s} + 2^{-s} + 3^{-s} + \ldots + (n-1)^{-s} = \\ &= \sum_{k=1}^{n-1} k^{-s} \end{aligned}$$

For some of the following steps, we're assuming that the real part of s lies between 0 and 1. It has already been well proven that there are no nontrivial zeros of the Riemann zeta function outside of this critical strip.

The norm of $\zeta(s,x)$ is bounded by $\frac{1}{1-Re(s)}(x-1)^{1-Re(s)}$.

$$\begin{aligned} \|\zeta(s,n)\| &= \left\| \sum_{k=1}^{n-1} k^{-s} \right\| \leq \\ &\leq \sum_{k=1}^{n-1} \left\| k^{-s} \right\| = \\ &= \sum_{k=1}^{n-1} k^{-Re(s)} \leq \\ &\leq \int_0^{n-1} dx\ x^{-Re(s)} = \\ &= \frac{1}{1-Re(s)} \left[x^{1-Re(s)}\right]_0^{n-1} = \\ &\leq \frac{1}{1-Re(s)}(n-1)^{1-Re(s)} \end{aligned}$$

We expect the second factor $x^{1-s} - (x-1)^{1-s}$ to decrease like $x^{-s}/(1-s)$. We show that our calculus also agrees.

$$\begin{aligned} \left\|x^{1-s} - (x-1)^{1-s}\right\| &= \left\| \left[y^{1-s}\right]_{x-1}^{x} \right\| = \\ &= \left\| \frac{1}{1-s} \int_{x-1}^{x} dy\ y^{-s} \right\| \leq \\ &\leq \left\| \frac{1}{1-s} \right\| \int_{x-1}^{x} dy\ \left\|y^{-s}\right\| \leq \\ &\leq \frac{1}{1-Re(s)} \int_{x-1}^{x} dy\ y^{-Re(s)} \leq \\ &\leq \frac{1}{1-Re(s)}(x-1)^{-Re(s)} \end{aligned}$$

Now, with the two factors combined, we can see if the second boundary condition can be satisfied.

$$\left\| \overline{\zeta(s,x)}\left(x^{1-s} - (x-1)^{1-s}\right) \right\| \leq \frac{(x-1)^{1-2Re(s)}}{(1-Re(s))^2}$$

Will this go to zero for $x \to \infty$? Yes, for $Re(s) > \frac{1}{2}$, it definitely will!

With the two boundary conditions, we know now that a $\zeta(s,x)$-function obeys these conditions when s is a zero of the Riemann zeta function with real part of s greater than $\frac{1}{2}$.

6 Riemann hypothesis is true!

When the boundary conditions have been met for a ζ-function, then wp is the adjoint of pw.

$$\begin{aligned}\langle\zeta(s,x)|pw\zeta(s,x)\rangle &= \langle p\zeta(s,x)|w\zeta(s,x)\rangle = \\ &= \langle wp\zeta(s,x)|\zeta(s,x)\rangle\end{aligned} \tag{29}$$

Then $\hat{G}$ is self-adjoint.

$$\begin{aligned}\left\langle\zeta(s,x)\middle|\hat{G}\zeta(s,x)\right\rangle &= \langle\zeta(s,x) \mid (wp+pw)\zeta(s,x)\rangle = \\ &= \langle(pw+wp)\zeta(s,x) \mid \zeta(s,x)\rangle = \\ &= \left\langle\hat{G}\zeta(s,x)\middle|\zeta(s,x)\right\rangle\end{aligned} \tag{30}$$

In section 4 we found the equation for the eigenvalues and the eigenfunctions $\zeta(s,x)$ of the Hamiltonian $\bar{H}$.

$$\lambda_s = -i\hbar(1-2s) = \frac{\left\langle\zeta(s,x)\middle|\hat{G}\zeta(s,x)\right\rangle}{\langle x^{-s}|x^{-s}\rangle}$$

The denominator converges to a real number for $Re(s) > \frac{1}{2}$.

$$\begin{aligned}\left\langle x^{-s}\middle|x^{-s}\right\rangle &= \int_1^\infty dx\ x^{-2Re(s)} = \\ &= \frac{1}{2Re(s)-1}\left[x^{1-2Re(s)}\right]_1^\infty = \\ &= \frac{1}{2Re(s)-1}\end{aligned}$$

For s being a zero of the Riemann zeta function with real part of s greater than $\frac{1}{2}$, the boundary conditions are satisfied. Then the numerator is real and the eigenvalue has to be real. But that's not possible for $Re(s) > \frac{1}{2}$. So we can only conclude that the Riemann zeta has no zeros with real part greater than $\frac{1}{2}$.

To make the proof complete, we have the functional equation [3], which relates $\zeta(s,1)$ to $\zeta(1-s,1)$.

$$\zeta(s,1) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s,1) \tag{31}$$

It follows from this equation that, if there are no zeros with real part between $\frac{1}{2}$ and 1, there can be no zeros with real part between 0 and $\frac{1}{2}$ either. All nontrivial zeros must have real part $\frac{1}{2}$. The Riemann hypothesis is true!

PROOF OF RIEMANN HYPOTHESIS

$$\zeta(s,x) = \sum_{n=1}^{\infty}\frac{1}{(n+x)^s}$$

$$p = -i\hbar\frac{\partial}{\partial x}$$

$$\Delta f(x) = f(x) - f(x+1)$$

$$\nabla f(x) = f(x) - f(x-1)$$

$$G = \nabla(xp+px)\Delta$$

$$\langle f|g\rangle = \int_1^\infty dx\,\overline{f(x)}\,g(x)$$

IF $\zeta(s,1) = 0$ AND $\mathcal{R}e(s) > \frac{1}{2}$

THEN $\langle G\zeta(s,x)|\zeta(s,x)\rangle = \langle\zeta(s,x)|G\zeta(s,x)\rangle$

$$-i\hbar(1-2s) = \frac{\langle\zeta(s,x)|G\zeta(s,x)\rangle}{\langle\Delta\zeta(s,x)|\Delta\zeta(s,x)\rangle}$$

THEN $\mathcal{R}e(s) = \frac{1}{2}$

CONTRADICTION

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