Solution of the central Problem of Fluid Turbulence

Rolf Warnemünde

Rolf Warnemünde
E-Mail: rolf.warnemuende@t-online.de

Abstract

The theory consists of:
I. a clear formulation of the turbulence problem by

1. *definition of a fluid continuum,*
2. *definition of a turbulent fluid continuum,*
3. *derivation, that Navier-Stokes-like equations cannot describe a turbulent fluid continuum*

II. solution of the turbulence problem by establishing the link between the theory of deterministic fluctuating vector fields and stochastic vector fields in the sense of an ensemble theory as a counterpart:

1. *derivation of a deterministic equation system of coupled vector vortex and curvature vector fields*
2. *derivation of a complete equation set for turbulent fluid movements*

The formulation of geometrodynamics of turbulence does not need an existent local thermodynamic equilibrium.

In the case of fluid turbulence there is no requirement for establishing chaos theories.

Contents

1. Introduction

Feynman[4]: "*Nobody in physics has really been able to analyze it mathematically satisfactorily in spite of its importance to the sister sciences. It is the analysis of circulating or turbulent fluids.*"

The description of turbulent movements within the framework of continuum mechanics turned out to be difficult since more than 165 years. However, laminar fluid movements can be calculated by the known basic equations successfully confirmed in experiments: equation of continuity, Navier-Stokes-Equations and energy equation. The efforts, treating movements of turbulence in a similar way, must be considered as failures. There are substantial reasons for believing, that the above equations describing turbulent collective movements of **non-homogenously distributed molecular matter** are inadequate. This was the situation that inspired the idea, to explain the phenomenon of turbulence by stochastic methods. In that context, particularly approaches of Kolmogorov are to be mentioned, which lead to spectral energy distributions, assuming highly hypothetically, that turbulence is statistically isotropical und homogeneous. Between them there is a wide range of models with physically not well founded hypotheses. Overall, this leads to the statement of Feynman cited at the beginning, whereupon not much has changed since then.

This situation is characterized in recent treatises as for example by Trinh, Khanh Tuoc [6] in the following way:

" *the study of turbulence is immediately hampered by the surprising lack of a clear and concise definition of the physical process. Tsinober (2001) has published a long list of attempts at a definition by some of the most noted researchers in turbulence. The most common descriptions are vague: "a motion in which an irregular fluctuation (mixing, or eddying motion) is superimposed on the main stream" (Schlichting 1960), "a fluid motion of complex and irregular character" (Bayly, Orszag, Herbert, 1988)* *or negative as in the breakdown of laminar flow (Reynolds' experiment 1883). Some of the definitions are quite controversial like Saffman's (1981) "One of the best definition of turbulence is that it is a field of random chaotic vorticity" because the words random and chaotic would imply that a formal mathematical solution, which is necessarily deterministic, does not exist. Perhaps the most accurate definition can be attributed to Bradshaw (1971) "The only short but satisfactory answer to the question "what is turbulence" is that it is the general-solution of the Navier-Stokes equation". This definition cannot be argued with but it is singularly unhelpful since no general solution of the NS yet exists 160 years after they were formulated.*"

A definition of a continuum of such fluctuation elements is mathematically a prerequisite deducing equations of motion in form of partial differential equations. In physics one is often happy having numerical results which are approved for special situations. This is the status of turbulence research assuming the turbulent fluid as a continuum. The known fluiddynamic equations are experimentally approved in the limiting case of laminar fluiddynamics but failed for the general turbulent case, because there is no one to one mapping of a continuum and an associated fluidelement set. The used equations are based on a hypothesis.

Fluctuation elements of the presented theory always form a dense point set. The fluidelement movements are described by interacting vortex- and vector-curvature fields. This is the consequence of the local movement of single turbulent fluid elements composed to a turbulent fluid continuum.

The interrelations of the deterministic and an associated stochastic ensemble theory of an unlimited number of in parallel existent deterministic systems enable a complete equation set of turbulently moved continua. The formulation of stochastically fluctuating processes of continua within the meaning of an ensemble theory is innovative for physics and mathematics. The known Navier-Stokes-Equations are not integrated in the complete equation system of turbulent moved fluids. The inclosed accelleration field $\frac{d\vec{v}}{dt}$ of the associated momentum equation is not sufficiently described by the usual Navier-Stokes equations and such the known energy equation, a composition of Navier-Stokes euations and continuum equation, proves to be not correct.

The complete system of equations consists of 12 equations with 12 unknowns and contains only variables of motion in form of the vector fields: velocity, vortex, curvature and an accelleration field. So the developed theory of turbulence proves to be a geometrodynamics in a 3+1 dimensional Euclidian Space. Thermodynamics and matter distribution do not occur explicitly. These quantities depend on the initial and boundary conditions, alone and are over time uniquely linked to the motion quantities. This theory of variables of motion is principally exact and is valid too, if no local thermodynamic balance is existent. A smeared distribution of matter over Space-Time results by evalution of the calculated velocity vector field and the equation of continuity.

Part I.

Formulation of the Turbulence Problem

2. A fluid can allways be represented by a continuum

At every time, space points ($\vec{x}$) are assigned to fluid elements in a unique correspondence. As this applies to every space point ($\vec{\mathbf{x}}$) of the fluid field, the set of fluid elements is seen as a continuum. A Continuum of fluid element points (simply called fluid elements) is considered, where a fluid environment of non infinitesimal size is uniquely allocated to every fluid element point. Two infinitesimally neighboring fluid elements differ apart from their distance by their velocities and not quite identical material distributions of their neighborhoods. The neighborhoods of two nearby fluid elements overlap. A fluid element is shifted moving the material of its neighborhood. Though the material of such a fluid element may have changed marginally after an infinitesimal time interval t_ε, it can be identified principally by its prior material status. As every molecule possesses its own identity, there has to be at least an infinitesimally greater difference of material distribution to the neighborhoods of other fluid elements.

The neighborhoods exchange material with neighborhoods of adjacent fluid elements and vary their thermodynamic state (a local thermodynamic state does not necessarily exist). Their size is not infinitesimal, because a local thermodynamic state (if physically existent) has to be detectable at least in thought experiment. The open neighborhoods have equally sized spherical shapes, generally. Near a solid border they are descibed by parts of spheres. Infinitesimally adjacent fluid elements possess overlapping neighborhoods. In an ε-surrounding they move in parallel. So one obtains a fluid, which is assumed to be a dense fluctuating point set, though there is no continuous matter distribution in Space-Time. That means it is possible to follow theoretically the history of every fluid element, though it has exchanged a lot of its initial material altering its local thermodynamic state.

Recapitulated:

Every space point ($\vec{\mathbf{x}}$) of the open point set of a considered fluid area is at every time in unique correspondence to a fluid element. The fluid is an abstract, dense set of fluctuating fluid elements, which do not generally correspond to material points. A continuum of moved fluid elements is considered each uniquely assigned to a neighborhood and a velocity.

$$\vec{\mathbf{v}}_{t_\epsilon} = \frac{\vec{\mathbf{x}}_2 - \vec{\mathbf{x}}_1}{t_\epsilon} \tag{2.1}$$

The fluidelement first determined in space point $\vec{\mathbf{x}}_1$ and t_ϵ-time later detected at $\vec{\mathbf{x}}_2$ is identified having at time $t_0 + t_\epsilon$ in $\vec{\mathbf{x}}_2$ the most similar material to that of $\vec{\mathbf{x}}_1$. In this connection it is remarqued, that parts of the individual particles or molecules may be identified, too.

The accuracies of the considered motion quantities are determined by t_ε-measurement processes t_ε characterising the accuracy. After a limiting process $\lim t_\epsilon \to 0$ the fluid elements move along with sufficiently often continuously differentiable trajectories and a velocity continuum is constituted. The whole of the velocities create a velocity vector field having $\boldsymbol{rot}(\vec{\mathbf{v}}) \neq \mathbf{0}$ generally.[1] Though $\boldsymbol{rot}(\vec{\mathbf{v}})$ has dimension $[\mathbf{1/sec}]$, it does not refer to a rotation of laminar flow.

2.1. The orthogonality of $\boldsymbol{rot}(\vec{\mathbf{v}})$ and $\vec{\mathbf{v}}$ is a consequence of the fluid continuum

A fluid continuum is characterized by

1. *continuously differentiable velocities*
2. *parallel velocities in an $\epsilon-$ surrounding of a space point $\vec{\mathbf{x}}$*

Considering without loss of generality a fluid movement of velocity $\vec{\mathbf{v}}(\vec{\mathbf{x}}_0) = (v_x, 0, 0)$ in a space point $\vec{\mathbf{x}}_0$ in cartesian coordinates, the velocity is described in an ϵ-neighborhood and parallel to the x-coordinate as follows:

$$\vec{\mathbf{v}}(\vec{\mathbf{x}}) = \begin{pmatrix} \mathbf{v}_x(\vec{\mathbf{x}}) \\ \mathbf{v}_y(\vec{\mathbf{x}}) \\ \mathbf{v}_z(\vec{\mathbf{x}}) \end{pmatrix} = \begin{pmatrix} \mathbf{v}_x(\vec{\mathbf{x}}_0) + & \frac{\partial \mathbf{v}_x}{\partial x}\Big|_{\vec{\mathbf{x}}_0} \cdot \Delta x + \frac{\partial \mathbf{v}_x}{\partial y}\Big|_{\vec{\mathbf{x}}_0} \cdot \Delta y + \frac{\partial \mathbf{v}_x}{\partial z}\Big|_{\vec{\mathbf{x}}_0} \cdot \Delta z + ... \\ & \frac{\partial \mathbf{v}_y}{\partial x}\Big|_{\vec{\mathbf{x}}_0} \cdot \Delta x + \frac{\partial \mathbf{v}_y}{\partial y}\Big|_{\vec{\mathbf{x}}_0} \cdot \Delta y + \frac{\partial \mathbf{v}_y}{\partial z}\Big|_{\vec{\mathbf{x}}_0} \cdot \Delta z + ... \\ & \frac{\partial \mathbf{v}_z}{\partial x}\Big|_{\vec{\mathbf{x}}_0} \cdot \Delta x + \frac{\partial \mathbf{v}_z}{\partial y}\Big|_{\vec{\mathbf{x}}_0} \cdot \Delta y + \frac{\partial \mathbf{v}_z}{\partial z}\Big|_{\vec{\mathbf{x}}_0} \cdot \Delta z + ... \end{pmatrix}$$

The velocity components $\mathbf{v}_y(\vec{\mathbf{x}})$ und $\mathbf{v}_z(\vec{\mathbf{x}})$ **osculate** at the velocity $\vec{\mathbf{v}}(\vec{\mathbf{x}}_0) = (v_x, 0, 0)$ spatially approaching (constant time t_0),

$$\begin{array}{c} \mathbf{v}_y(x_0, y, z_0) \longrightarrow \mathbf{v}_y(x_0, y_0, z_0) = \mathbf{0} \\ \mathbf{v}_z(x_0, y_0, z) \longrightarrow \mathbf{v}_z(x_0, y_0, z_0) = \mathbf{0} \end{array}.$$

[1] in english literature $\boldsymbol{curl}(\vec{\mathbf{v}}) \neq \mathbf{0}$ is used but in turbulence the name $\boldsymbol{rot}$ is more adapted as will be seen

That means especially, that all the partial derivations by y- or z-coordinate of 1. order of $\mathbf{v}_y(\vec{\mathbf{x}})$ and $\mathbf{v}_z(\vec{\mathbf{x}})$ disappear in the point (x_0, y_0, z_0).

$$\lim_{\boldsymbol{z}\to\boldsymbol{z}_0} \frac{\Delta \mathbf{v}_y}{\Delta z}\Big|_{\vec{\mathbf{x}}_0} = \lim_{\boldsymbol{y}\to\boldsymbol{y}_0} \frac{\Delta \mathbf{v}_z}{\Delta y}\Big|_{\vec{\mathbf{x}}_0} = \mathbf{0} \quad . \qquad \vec{\mathbf{x}}_0 = (x_0, y_0, z_0) \tag{2.2}$$

Applying the differential quotients in the $\vec{\boldsymbol{\nabla}}\times$ -operator expresssed in cartesian coordinates gives for the fluid velocity

$$(\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{v}})_{|\vec{\mathbf{x}}_0} = \begin{pmatrix} 0 \\ \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \\ \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \end{pmatrix}_{|\vec{\mathbf{x}}_0} , \vec{\mathbf{v}}(\vec{\mathbf{x}}_0) = (v_x, 0, 0) \tag{2.3}$$

The orthogonality of $\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{v}} \perp \vec{\mathbf{v}}$ is a fundamental quality [2][3] and a necessary condition for continuous fluid flow.

In this orthogonality velocity vector fields differ from deformation vector fields.

[2] this relation can not be found in literature.

[3] This is one reason why the known millenium prize question does not lead to a solution of the turbulence problem. However the validity problem of the Navier-Stokes-equations is more fatal.

3. Definition of a turbulent fluid

Trying to identify the state of movement of a fluid element in turbulent fluids by a velocity $\vec{\mathbf{v}}_{t_\epsilon}$ it should be recognized, that the state of movement is not yet determined, as the path in every space point (except in turning points) is uniquely adapted by an infinitesimal circle segment. In the infinitesimal neighborhood of a path point the velocity is identified by an instantaneous axis of rotation $\vec{\boldsymbol{\omega}}_{t_\epsilon}$ and a radius vector $\vec{\mathbf{r}}_{t_\epsilon}$.[1]

$$\boxed{\vec{\mathbf{v}}_{\boldsymbol{t_\epsilon}} = \vec{\boldsymbol{\omega}}_{\boldsymbol{t_\epsilon}} \times \vec{\mathbf{r}}_{\boldsymbol{t_\epsilon}}} \tag{3.1}$$

In a turbulent moved fluid the fluid elements move on curved trajectories in some space time points having turning points with $\vec{\boldsymbol{\omega}}_{\boldsymbol{t_\epsilon}} = \mathbf{0}$ and $\vec{\mathbf{b}}_{\boldsymbol{t_\epsilon}} = \mathbf{0}$. The considered vectorial motion quantities $\vec{\boldsymbol{\omega}}_{\boldsymbol{t_\epsilon}}$ und $\vec{\mathbf{r}}_{\boldsymbol{t_\epsilon}}$ are determined by t_ε-measurement processes, which are calculated later on by a limes process $\lim \boldsymbol{t_\epsilon} \to 0$. A fluid element originating from the point $\vec{\mathbf{x}}_0$ crossing $\vec{\mathbf{x}}_1$ after the time t_ε reaches $\vec{\mathbf{x}}_2$ after a further time t_ε.

$$\vec{\mathbf{x}}_0 \xrightarrow{\mathbf{t}_\epsilon} \vec{\mathbf{x}}_1 \xrightarrow{\mathbf{t}_\epsilon} \vec{\mathbf{x}}_2$$

By these 3 points a circle segment is uniquely drawn crossing point $\vec{\mathbf{x}}_1$ with radius vector $\vec{\mathbf{r}}_{t_\varepsilon}$ and speed of rotation $\vec{\boldsymbol{\omega}}_{t_\varepsilon}$. The local state of motion can not be described by velocity only, neither statistically nor deterministically. [2]

Thus the fluid element in the space-time-point $(\vec{\mathbf{x}}, t)$ is identified principally by the contents of the matter of its neighborhood and state of movement expressed by $\vec{\boldsymbol{\omega}}_{t_\epsilon}$ and $\vec{\mathbf{r}}_{t_\epsilon}$. In that way defined fluid elements move on sufficiently often continuously differentiable trajectories. They lead considering a continuum of fluctuating fluid elements to multiply continuously differentiable vector fields of motion. The continuum of moved fluid elements represent the turbulently collectiv movement of a discontinuously spaced Matter.

The field of turbulence is described by the two vector fields $\vec{\boldsymbol{\omega}}_{t_\epsilon}$ and $\vec{\mathbf{b}}_{t_\epsilon}$,

$$\vec{\mathbf{b}}_{t_\epsilon} = \vec{\mathbf{r}}_{t_\epsilon} / r^2_{t_\epsilon} \qquad \text{-curvature vector field.} \tag{3.2}$$

In addition, the results show that

$$\vec{\boldsymbol{\omega}}_{t_\epsilon} = \frac{1}{2}\mathbf{rot}(\vec{\mathbf{v}}_{t_\epsilon}). \tag{3.3}$$

[1] That is why turbulence can not be uniquely identified by experiments of local velocity statistics.

[2] This statement contadicts that of [11]

$\mathbf{rot}(\vec{\mathbf{v}})$ has the meaning of a local rotation in the frame of turbulence. An infinitesimal disturbance of stationary pipe flow leads to an change of the significance of $\mathbf{rot}(\vec{\mathbf{v}})$, where $\mathbf{rot}(\vec{\mathbf{v}})$ does not correspond to a rotation initially. Whether starting motions of turbulence are suppressed, depends on an existent viscosity. These decelerations are generally weak. The beginning of turbulent movements avoid Newtonian friction as well as pressure gradients by means of hereto orthogonal motions.

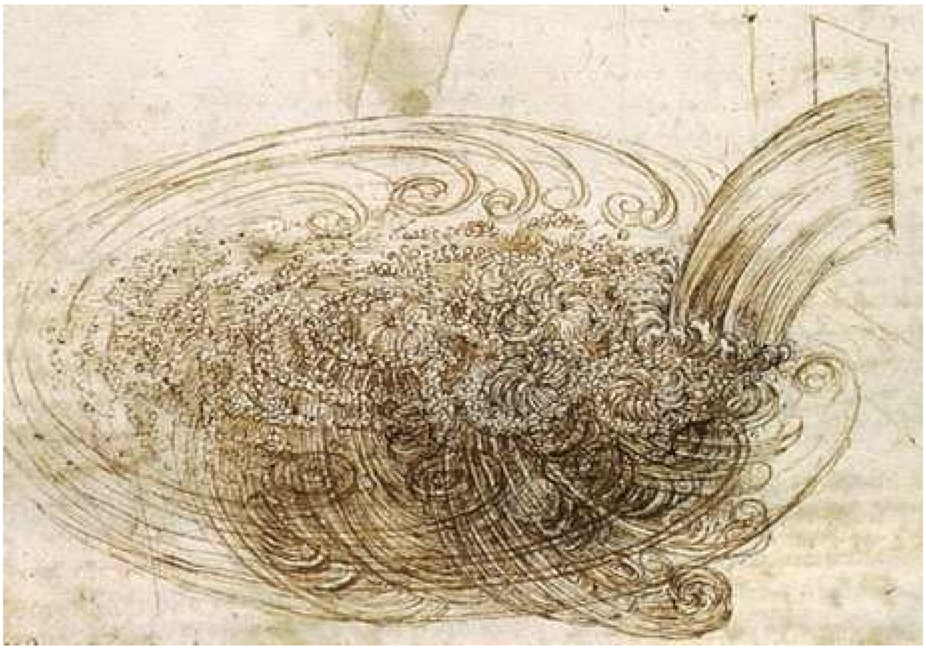

Figure 3.1.: Turbulences understood by Leonardo da Vinci

Vortex fields in turbulence (local rotation fields will be identified with vortex fields) and radius fields may have turning points $(\vec{\boldsymbol{x}}, t)$ along the paths of the fluid elements, which means $\vec{\boldsymbol{\omega}} = \mathbf{0}$ und $\vec{\boldsymbol{r}} = \boldsymbol{\infty}$. [3] In this case the velocities are to be calculated by interpolation or extrapolation of the neighborhood, for example. In the theory a further method will be shown. The fluid elements are accompanied by a moving frame of $\vec{\boldsymbol{\omega}}$,$\vec{\boldsymbol{b}}$ and $\vec{\boldsymbol{v}}$ along their paths.
In the following it is outlined, how locally Lagrangian and Eulerian formulations of fluid dynamics are reassembled in the turbulence theory. So deterministic considerations are found via stochastic descriptions, which could be designated as Lagrangian. Nevertheless, Lagrangian paths are calculated only after the deterministic turbulence field is determined. These relations will become clear in later chapters.

[3]The temporal and spatial neighborhood of a turning point does not have such singular properties.

4. Why Navier-Stokes equations cannot describe turbulence

This problem is best shown by the numerical time integration of the respectively used momentum equation (Navier-Stokes equations or simplified versions). The situation is characterized as follows:
for a single time step i one has calculated $\frac{\partial \vec{\mathbf{v}}}{\partial t}|_i$ and tries the time integration by the discrete difference sceme

$$\vec{\mathbf{v}}(\vec{\mathbf{x}}, t_{i+1}) = \frac{\partial \vec{\mathbf{v}}(\vec{\mathbf{x}}, t_i)}{\partial t}|\Delta t_i + \vec{\mathbf{v}}(\vec{\mathbf{x}}, t_i). \tag{4.1}$$

for every used space point. Usually numerical time-integrations via $\Delta \vec{\mathbf{v}} = \frac{\partial \vec{\mathbf{v}}}{\partial t} \cdot \Delta t$ lead in relation to turbulence calculations to errors not be compensated by which ever refined time steps are used. But in reality the velocity is composed by the two vector fields vorticity $\vec{\boldsymbol{\omega}}$ and curvature vector field $\vec{\mathbf{b}}$ with

$$\vec{\mathbf{v}}(\vec{\mathbf{x}}, t) = \vec{\boldsymbol{\omega}}(\vec{\mathbf{x}}, t) \times \frac{\vec{\mathbf{b}}(\vec{\mathbf{x}}, t)}{b^2}, \qquad \text{radius vector} \qquad \vec{\mathbf{r}} = \frac{\vec{\mathbf{b}}(\vec{\mathbf{x}}, t)}{b^2}. \tag{4.2}$$

This relation is a decisive reason for weather forcast problems of meteorology, too, which cannot be solved by computer systems, regardless of their efficency. This difficulty does not exist regarding laminar fluid dynamics. The explanation is as follows:
The partial diffetrentiation $\frac{\partial \vec{\mathbf{v}}}{\partial t}$ is written

$$\frac{\partial \vec{\mathbf{v}}}{\partial t} = \frac{\partial \vec{\boldsymbol{\omega}}}{\partial t} \times \vec{\mathbf{r}} + \vec{\boldsymbol{\omega}} \times \frac{\partial \vec{\mathbf{r}}}{\partial t}.$$

The numerical time evolution of $\vec{\mathbf{v}}_{\boldsymbol{i}} \Longrightarrow \vec{\mathbf{v}}_{\boldsymbol{i+1}}$ arises calculating $\vec{\mathbf{v}}_{\boldsymbol{i}} = \vec{\boldsymbol{\omega}}_i \times \vec{\mathbf{r}}_i$ by means of

$$\vec{\boldsymbol{\omega}}_{i+1} = \frac{\partial \vec{\boldsymbol{\omega}}}{\partial t}|_i \cdot \Delta t_i + \vec{\boldsymbol{\omega}}_i + ...$$

and

$$\vec{\mathbf{r}}_{i+1} = \frac{\partial \vec{\mathbf{r}}}{\partial t}|_i \cdot \Delta t_i + \vec{\mathbf{r}}_i + ...$$

to

$$\vec{\mathbf{v}}_{\boldsymbol{i+1}} = \left(\frac{\partial \vec{\boldsymbol{\omega}}}{\partial t}|_i \cdot \Delta t_i + \vec{\boldsymbol{\omega}}_i \right) \times \left(\frac{\partial \vec{\mathbf{r}}}{\partial t}|_i \cdot \Delta t_i + \vec{\mathbf{r}}_i \right) + ...$$

i.e.

$$\vec{\mathbf{v}}_{\boldsymbol{i+1}} = \left(\vec{\boldsymbol{\omega}}_i \times \vec{\mathbf{r}}_i \right) + \left(\frac{\partial \vec{\boldsymbol{\omega}}}{\partial t}|_i \cdot \Delta t_i \times \vec{\mathbf{r}}_i + \vec{\boldsymbol{\omega}}_i \times \frac{\partial \vec{\mathbf{r}}}{\partial t}|_i \cdot \Delta t_i \right) + \left(\frac{\partial \vec{\boldsymbol{\omega}}}{\partial t}|_i \cdot \Delta t_i \times \frac{\partial \vec{\mathbf{r}}}{\partial t}|_i \cdot \Delta t_i \right) + ...$$

respectively

$$\boxed{\vec{\mathbf{v}}_{\boldsymbol{i+1}} = \vec{\mathbf{v}}_i + \frac{\partial \vec{\mathbf{v}}}{\partial t}|_i \cdot \Delta t_i + \left(\frac{\partial \vec{\boldsymbol{\omega}}}{\partial t}|_i \times \frac{\partial \vec{\mathbf{r}}}{\partial t}|_i \right) \cdot (\Delta t_i)^2 + \boldsymbol{...}} \tag{4.3}$$

$\frac{\partial \vec{\mathbf{r}}}{\partial t}$ is derived as follows:

$$\vec{\mathbf{b}} = \vec{\mathbf{r}} \cdot (\vec{\mathbf{b}} \cdot \vec{\mathbf{b}}) \tag{4.4}$$

$\Longrightarrow$

$$\frac{\partial \vec{\mathbf{b}}}{\partial t} = \mathbf{b^2} \frac{\partial \vec{\mathbf{r}}}{\partial t} + 2\vec{\mathbf{r}} \left(\frac{\partial \vec{\mathbf{b}}}{\partial t} \cdot \vec{\mathbf{b}} \right)$$

$\Longrightarrow$

$$\frac{\partial \vec{\mathbf{r}}}{\partial t} = \left[\frac{\partial \vec{\mathbf{b}}}{\partial t} - 2 \frac{\vec{\mathbf{b}}}{b^2} \left(\frac{\partial \vec{\mathbf{b}}}{\partial t} \cdot \vec{\mathbf{b}} \right) \right] / b^2.$$

In particular space-time points $(\vec{\mathbf{x}}, t)$ fluid elements may be in the proximity or direct in a turning point, in which $\vec{\boldsymbol{\omega}}(\vec{\mathbf{x}}, t) = \mathbf{0}$ as well as $\vec{\mathbf{b}}(\vec{\mathbf{x}}, t) = \mathbf{0}$ and such $\vec{\mathbf{r}}(\vec{\mathbf{x}}, t) = \vec{\mathbf{b}}/b^2 = \boldsymbol{\infty}$ holds. This situation corresponds to an amendable singularity and the velocity has to be calculated by interpolation or extrapolation of the near space-time surrounding. So the temporal evolution term of 2nd order is vital for turbulence calculations not becoming available with the known fluiddynamic equation system. The with (4.1) mentioned velocity integration is not expedient. Considering a complete turbulence equation system including the curvature vector field $\vec{\mathbf{b}}(\vec{\mathbf{x}}, t)$ the temporal velocity integration results in the desired order. An according complete equation system is derived in the following Part II.

Part II.

Solution of the Turbulence Problem

5. Deterministic turbulent mass-transport and its stochastic formulation

$$f_{t_\epsilon}(t,\vec{\mathbf{x}},\vec{\boldsymbol{\omega}},\vec{\mathbf{r}}) = \int\limits_{\vec{\boldsymbol{\omega}}'}\int\limits_{\vec{\boldsymbol{r}}'} W_{t_\epsilon}(t,\vec{\mathbf{x}},\vec{\boldsymbol{\omega}},\vec{\mathbf{r}},\vec{\boldsymbol{\omega}'},\vec{\boldsymbol{r}'}) \cdot f_{t_\epsilon}(\vec{\mathbf{x}}-\Delta\vec{\mathbf{x}},t-t_\epsilon,\vec{\boldsymbol{\omega}}',\vec{\mathbf{r}'})d\vec{\boldsymbol{\omega}}'d\vec{\mathbf{r}'}$$

$$\Updownarrow$$

$$\frac{\partial}{\partial t}\vec{\boldsymbol{\omega}} - \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{a}} - \frac{1}{2}\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{q}} = 0$$

$$\frac{\partial}{\partial t}\vec{\mathbf{b}} + \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{\omega}} - \frac{1}{2}\vec{\mathbf{b}}\left[\frac{\vec{\boldsymbol{\omega}}}{\omega^2} \cdot \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{q}}\right] = 0$$

5.1. Introduction

A stochastical ensemble-consideration of deterministic fields is understood as the examination of an unlimited number of comparable, parallelly existent systems. In this case turbulently moved one phase fluids are examined considering statistical deliberations and its deterministic counterparts. That a linking of deterministic and stochastic theory may be available and further more that out of this connection additionally important (sometimes otherwise not known) relations arise for deterministic formulations, is shown in the following. This is discussed for a turbulent mass transport.

5.2. The transition: stochastic theory $\longleftrightarrow$ deterministic theory

Every space-time-point $(\vec{\boldsymbol{x}}, t)$ a continuously differentiable fluid element distribution over the motion quantities $\vec{\omega}_{t_\epsilon}$ and $\vec{\mathbf{r}}_{t_\epsilon}$ is assigned according to

$$f_{\boldsymbol{t_\epsilon}} = f_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\omega}}, \vec{\mathbf{r}}). \tag{5.1}$$

Indexing functions with $\boldsymbol{t_\epsilon}$ it is automatically assumed that the included motion quantities $(\vec{\boldsymbol{\omega}}, \vec{\mathbf{r}})$ are assigned to a $\boldsymbol{t_\epsilon}$-measurement accuracy. The indexing of the motion quantities may be omitted in the functions if the functions are accordingly indexed.

After an execution of a $\lim \boldsymbol{t_\epsilon} \to 0$ process, such as

$$\lim_{\boldsymbol{t_\epsilon}\to 0} f_{\boldsymbol{t_\epsilon}}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\omega}}, \vec{\boldsymbol{r}}) = f(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\omega}}, \vec{\boldsymbol{r}}) \tag{5.2}$$

f and $(\vec{\boldsymbol{\omega}}, \vec{\mathbf{r}})$ are understood as results of an exact measuring process.

The change of motion quantities in point $(\vec{\boldsymbol{x}}, t)$

$$\Big(\vec{\boldsymbol{\omega}}'_{t_\epsilon}(\vec{\boldsymbol{x}} - \Delta\vec{\boldsymbol{x}}, t - t_\epsilon), \vec{\boldsymbol{r}}'_{t_\epsilon}(\vec{\boldsymbol{x}} - \Delta\vec{\boldsymbol{x}}, t - t_\epsilon)\Big) \longrightarrow \Big(\vec{\boldsymbol{\omega}}_{t_\epsilon}(\vec{\boldsymbol{x}}, t), \vec{\boldsymbol{r}}_{t_\epsilon}(\vec{\boldsymbol{x}}, t)\Big)$$

is controlled by the transition probability density $W_{t_\epsilon} = W_{t_\epsilon}(\vec{\boldsymbol{x}}, t, \vec{\boldsymbol{\omega}}, \vec{\boldsymbol{r}}, \vec{\boldsymbol{\omega}}', \vec{\boldsymbol{r}}')$. [1] with

$$\boxed{\begin{aligned} \lim_{t_\epsilon\to 0} W_{t_\epsilon} &= \delta(\vec{\boldsymbol{\omega}}, \vec{\mathbf{r}}; \vec{\boldsymbol{\omega}}', \vec{\mathbf{r}}') \\ f_{t_\epsilon}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\omega}}, \vec{\mathbf{r}}) &= \int\limits_{\vec{\mathbf{r}}} \int\limits_{\vec{\boldsymbol{\omega}}} W_{t_\epsilon}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\omega}}, \vec{\mathbf{r}}, \vec{\boldsymbol{\omega}}', \vec{\mathbf{r}}') \cdot f_{t_\epsilon}(\vec{\mathbf{x}} - \Delta\vec{\mathbf{x}}, t - t_\epsilon, \vec{\boldsymbol{\omega}}', \vec{\mathbf{r}}') d\vec{\boldsymbol{\omega}}' d\vec{\mathbf{r}}' \\ \Delta\vec{\mathbf{x}} &= t_\epsilon \cdot \vec{\boldsymbol{\omega}}' \times \vec{\mathbf{r}}' \end{aligned}} \, . \tag{5.3}$$

These equations characterize stochastic turbulence of the continuum in the frame of an ensemble theory and represent a Markov Process with natural causality.

f_{t_ϵ} is developed in (5.3) until the 1st order around $(\vec{\mathbf{x}}, t) \Longrightarrow$

$$f_{t_\epsilon}(\vec{\mathbf{x}} - \triangle\vec{\mathbf{x}}, t - t_\epsilon, \vec{\boldsymbol{\omega}}', \vec{\mathbf{r}}') = f_{t_\epsilon}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\omega}}', \vec{\mathbf{r}}') - \frac{\partial f'_{t_\epsilon}}{\partial t} \cdot t_\epsilon - \triangle\vec{\mathbf{x}} \cdot \vec{\boldsymbol{\nabla}} f_{t_\epsilon}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\omega}}', \vec{\mathbf{r}}') + \boldsymbol{O}({t_\epsilon}^2) \tag{5.4}$$

[1] The otherwise in distribution theory used test functions in this connection have an immediate physical meaning with the formulation of the transition probability density.

with $f'_{t_\epsilon} = f_{t_\epsilon}(\vec{\mathbf{x}}, t, \vec{\boldsymbol{\omega}}', \vec{\mathbf{r}}')$ and one obtains

$$\int_{\vec{\mathbf{r}}}\int_{\vec{\boldsymbol{\omega}}} W_{t_\epsilon}[\frac{\partial f'_{t_\epsilon}}{\partial t} + \vec{\boldsymbol{\omega}}' \times \vec{\mathbf{r}}' \cdot \vec{\boldsymbol{\nabla}} f'_{t_\epsilon}] d\vec{\boldsymbol{\omega}}' d\vec{\mathbf{r}}' + \boldsymbol{O}({t_\epsilon}^2) = \frac{\int_{\vec{\mathbf{r}}}\int_{\vec{\boldsymbol{\omega}}} W_{t_\epsilon} f'_{t_\epsilon} d\vec{\boldsymbol{\omega}}' d\vec{\mathbf{r}}' - f_{t_\epsilon}}{t_\epsilon}. \quad (5.5)$$

$\lim \boldsymbol{t_\epsilon} \to 0$ applied to (5.5) leads to

$$\frac{\partial f}{\partial t} + \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}} \cdot \vec{\boldsymbol{\nabla}} f = \lim_{t_\epsilon \to 0} \frac{\int_{\vec{\mathbf{r}}}\int_{\vec{\boldsymbol{\omega}}} W_{t_\epsilon} f'_{t_\epsilon} d\vec{\boldsymbol{\omega}}' d\vec{\mathbf{r}}' - f_{t_\epsilon}}{t_\epsilon}. \quad (5.6)$$

The right side must contain the characteristics of the turbulent fluid.

$$\lim_{t_\epsilon \to 0} \frac{\int_{\vec{\mathbf{r}}}\int_{\vec{\boldsymbol{\omega}}} W_{t_\epsilon} f'_{t_\epsilon} d\vec{\boldsymbol{\omega}}' d\vec{\mathbf{r}}' - f_{t_\epsilon}}{t_\epsilon} = F \quad (5.7)$$

F has to be chosen such, that the deterministic vortex equations result under the influence of the assumed acceleration field. Further on the ansatz

$$F = \frac{1}{2}\left[\frac{\vec{\boldsymbol{\omega}}}{\omega^2} \cdot \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{q}}\right] f \quad (5.8)$$

is shown precisely fulfilling this condition. Thus one obtains

$$\frac{\partial f}{\partial t} + \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}} \cdot \vec{\boldsymbol{\nabla}} f = \frac{1}{2}\left[\frac{\vec{\boldsymbol{\omega}}}{\omega^2} \cdot \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{q}}\right] f. \quad (5.9)$$

Limiting ourselves to one system of the ensemble the distribution function f degenerates to a δ-function.

$$f \longrightarrow \delta(\vec{\boldsymbol{\omega}}_{(\vec{\boldsymbol{x}},t)}, \vec{\mathbf{r}}_{(\vec{\boldsymbol{x}},t)}; \vec{\boldsymbol{\omega}}, \vec{\mathbf{r}}) \quad (5.10)$$

The indexing of quantities like $\vec{\boldsymbol{\omega}}_{(\vec{\boldsymbol{x}},t)}$ by $(\vec{\boldsymbol{x}}, t)$ means the vector $\vec{\boldsymbol{\omega}}$ in the space-time point $(\vec{\boldsymbol{x}}, t)$ [2] whereas $\vec{\boldsymbol{\omega}}(\vec{\boldsymbol{x}}, t)$ represents the spatiotemporal field $\vec{\boldsymbol{\omega}}$ in dependence on $(\vec{\boldsymbol{x}}, t)$.

It results in the key equation for the transition stochastic-deterministic

$$\boxed{\frac{\partial}{\partial t}\delta + \vec{\boldsymbol{\omega}}_{(\vec{\boldsymbol{x}},t)} \times \vec{\mathbf{r}}_{(\vec{\boldsymbol{x}},t)} \cdot \vec{\boldsymbol{\nabla}}\delta = \frac{1}{2}\left[\frac{\vec{\boldsymbol{\omega}}_{(\vec{\boldsymbol{x}},t)}}{\omega^2_{(\vec{\boldsymbol{x}},t)}} \cdot \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{q}}_{(\vec{\boldsymbol{x}},t)}\right]\delta}. \quad (5.11)$$

[2]That is the situation considering stochastically.

Definition of the operator $\Xi[...]$:
From the vector $\vec{\mathbf{A}}_{(\vec{\boldsymbol{x}},t)}$ respectively the scalar function value $\mathbf{f}_{(\vec{\boldsymbol{x}},t)}$ existing in the space-time-point $(\vec{\boldsymbol{x}},t)$ of the system a vector function respectively a scalar function arises by the operator Ξ

$$\Xi\left[\vec{\mathbf{A}}_{(\vec{\boldsymbol{x}},t)}\right] = \vec{\mathbf{A}}(\vec{\boldsymbol{x}},t) \tag{5.12}$$

respectively

$$\Xi\left[\mathbf{f}_{(\vec{\boldsymbol{x}},t)}\right] = \mathbf{f}(\vec{\boldsymbol{x}},t) \tag{5.13}$$

an appropriate field existing around the point $(\vec{\boldsymbol{x}},t)$. The Operator $\Xi[...]$ evokes this functionality to "life".
Accordingly the following relationships are noted:

$$\begin{aligned}
&\Xi\left[\int_{\vec{\mathbf{r}}}\int_{\vec{\boldsymbol{\omega}}} \delta(\vec{\boldsymbol{\omega}}_{(\vec{\boldsymbol{x}},t)}, \vec{\mathbf{r}}_{(\vec{\boldsymbol{x}},t)}; \vec{\boldsymbol{\omega}}, \vec{\mathbf{r}}) d\vec{\boldsymbol{\omega}} d\vec{\mathbf{r}}\right] = \mathbf{1}\\
&\Xi\left[\int_{\vec{\mathbf{r}}}\int_{\vec{\boldsymbol{\omega}}} \delta(\vec{\boldsymbol{\omega}}_{(\vec{\boldsymbol{x}},t)}, \vec{\mathbf{r}}_{(\vec{\boldsymbol{x}},t)}; \vec{\boldsymbol{\omega}}, \vec{\mathbf{r}}) \vec{\boldsymbol{\omega}} d\vec{\boldsymbol{\omega}} d\vec{\mathbf{r}}\right] = \Xi\left[\vec{\boldsymbol{\omega}}_{(\vec{\boldsymbol{x}},t)}\right] = \vec{\boldsymbol{\omega}}(\vec{\boldsymbol{x}},t)\\
&\Xi\left[\int_{\vec{\mathbf{r}}}\int_{\vec{\boldsymbol{\omega}}} \delta(\vec{\boldsymbol{\omega}}_{(\vec{\boldsymbol{x}},t)}, \vec{\mathbf{r}}_{(\vec{\boldsymbol{x}},t)}; \vec{\boldsymbol{\omega}}, \vec{\mathbf{r}}) \vec{\mathbf{r}} d\vec{\boldsymbol{\omega}} d\vec{\mathbf{r}}\right] = \Xi\left[\vec{\mathbf{r}}_{(\vec{\boldsymbol{x}},t)}\right] = \vec{\mathbf{r}}(\vec{\boldsymbol{x}},t)
\end{aligned} \tag{5.14}$$

or

$$\Xi\left[\int_{\vec{\mathbf{r}}}\int_{\vec{\boldsymbol{\omega}}} \delta(\vec{\boldsymbol{\omega}}_{(\vec{\boldsymbol{x}},t)}, \vec{\mathbf{r}}_{(\vec{\boldsymbol{x}},t)}; \vec{\boldsymbol{\omega}}, \vec{\mathbf{r}}) \boldsymbol{\omega}^2 \vec{\mathbf{r}} d\vec{\boldsymbol{\omega}} d\vec{\mathbf{r}}\right] = \Xi\left[\boldsymbol{\omega}^2_{(\vec{\mathbf{x}},t)} \vec{\mathbf{r}}_{(\vec{\mathbf{x}},t)}\right] = \boldsymbol{\omega}^2(\vec{\mathbf{x}},t)\vec{\mathbf{r}}(\vec{\mathbf{x}},t). \tag{5.15}$$

5.3. The deterministic equations of turbulence

From the general momentum equation

$$\frac{\partial \vec{\mathbf{v}}}{\partial t} + (\vec{\mathbf{v}} \cdot \vec{\boldsymbol{\nabla}})\vec{\mathbf{v}} = \vec{\boldsymbol{q}} \tag{5.16}$$

the vortex equation may be developed using the $\vec{\boldsymbol{\nabla}}\times$-operator

$$\frac{\partial}{\partial t}\vec{\boldsymbol{\omega}} - \vec{\boldsymbol{\nabla}} \times (\vec{\mathbf{v}} \times \vec{\boldsymbol{\omega}}) - \frac{1}{2}\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{q}} = 0. \tag{5.17}$$

The relations of deterministic and stochastic description are established the same vortex equation opening up from the above key equation. In the following the method is presented designing the dual pair of deterministic vector equations from the key equation

$$\frac{\partial}{\partial t}\delta + \vec{\boldsymbol{\omega}}_{(\vec{\boldsymbol{x}},t)} \times \vec{\mathbf{r}}_{(\vec{\boldsymbol{x}},t)} \cdot \vec{\boldsymbol{\nabla}}\delta = \frac{1}{2}\left[\frac{\vec{\boldsymbol{\omega}}_{(\vec{\boldsymbol{x}},t)}}{\omega^2_{(\vec{\boldsymbol{x}},t)}} \cdot \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{q}}_{(\vec{\boldsymbol{x}},t)}\right]\delta. \tag{5.18}$$

In this situation the vectors of the motion quantities may be pushed before and after the differential operators. The Term

$$\frac{1}{2}\left[\frac{\vec{\boldsymbol{\omega}}_{(\vec{\boldsymbol{x}},t)}}{\omega^2_{(\vec{\boldsymbol{x}},t)}} \cdot \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{q}}_{(\vec{\boldsymbol{x}},t)}\right]\delta \tag{5.19}$$

guarantees the finding of equation (5.17) and its dual one. It is

$$\vec{\mathbf{v}} \perp \vec{\boldsymbol{\omega}} \perp \vec{\mathbf{r}}. \tag{5.20}$$

and setting

$$\vec{\mathbf{a}} = \vec{\mathbf{v}} \times \vec{\boldsymbol{\omega}} \tag{5.21}$$

this results in

$$\vec{\mathbf{r}} \parallel \vec{\mathbf{a}}. \tag{5.22}$$

Such $\vec{\mathbf{a}}$ and $\vec{\mathbf{r}}$ are linked as follows[3]

$$\vec{\mathbf{r}} = \frac{\vec{\mathbf{a}}}{\omega^2}. \tag{5.23}$$

$\Longrightarrow$

$$\text{with} \qquad \delta = \delta(\vec{\boldsymbol{\omega}}_{(\vec{\boldsymbol{x}},t)}, \vec{\mathbf{r}}_{(\vec{\boldsymbol{x}},t)}; \vec{\boldsymbol{\omega}}, \vec{\mathbf{r}})$$

$$\begin{aligned}
\vec{\boldsymbol{\omega}}_{(\vec{\boldsymbol{x}},t)} \times \vec{\mathbf{r}}_{(\vec{\boldsymbol{x}},t)} \cdot \vec{\boldsymbol{\nabla}}\delta &= -\vec{\mathbf{r}}_{(\vec{\boldsymbol{x}},t)} \times \vec{\boldsymbol{\omega}}_{(\vec{\boldsymbol{x}},t)} \cdot \vec{\boldsymbol{\nabla}}\delta \\
&= -\vec{\boldsymbol{\omega}}_{(\vec{\boldsymbol{x}},t)} \cdot \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{r}}_{(\vec{\boldsymbol{x}},t)}\delta \\
&= -\frac{\vec{\boldsymbol{\omega}}_{(\vec{\boldsymbol{x}},t)}}{\omega^2_{(\vec{\boldsymbol{x}},t)}} \cdot \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{a}}_{(\vec{\boldsymbol{x}},t)}\delta.
\end{aligned}$$

[3] Symbols as ω, r, a, v etc. always mean amounts of the corresponding vectors.

Inserting in (5.18) gives

$$\begin{aligned}
&\frac{\partial}{\partial t}(\frac{\vec{\boldsymbol{\omega}}_{(\vec{x},t)}\cdot\vec{\boldsymbol{\omega}}_{(\vec{x},t)}}{\omega^2_{(\vec{x},t)}}\delta) - \frac{\vec{\boldsymbol{\omega}}_{(\vec{x},t)}}{\omega^2_{(\vec{x},t)}}\cdot\vec{\boldsymbol{\nabla}}\times(\vec{\mathbf{a}}_{(\vec{x},t)}\delta) - \frac{1}{2}\left[\frac{\vec{\boldsymbol{\omega}}_{(\vec{x},t)}}{\omega^2_{(\vec{x},t)}}\cdot\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{q}}_{(\vec{x},t)}\right]\delta = 0\\
&\Longrightarrow \frac{\vec{\boldsymbol{\omega}}_{(\vec{x},t)}}{\omega^2_{(\vec{x},t)}}\cdot\left[\frac{\partial}{\partial t}(\vec{\boldsymbol{\omega}}_{(\vec{x},t)}\delta) - \vec{\boldsymbol{\nabla}}\times(\vec{\mathbf{a}}_{(\vec{x},t)}\delta) - \frac{1}{2}\left[\cdot\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{q}}_{(\vec{x},t)}\right]\delta\right] = 0 \qquad (5.24)\\
&\Longrightarrow \frac{\partial}{\partial t}(\vec{\boldsymbol{\omega}}_{(\vec{x},t)}\delta) - \vec{\boldsymbol{\nabla}}\times(\vec{\mathbf{a}}_{(\vec{x},t)}\delta) - \frac{1}{2}\left[\cdot\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{q}}_{(\vec{x},t)}\right]\delta = 0
\end{aligned}$$

One obtains

$$\boldsymbol{\Xi}\left[\int_{\vec{\mathbf{r}}}\int_{\vec{\boldsymbol{\omega}}}\left[\frac{\partial}{\partial t}(\vec{\boldsymbol{\omega}}_{(\vec{x},t)}\delta) - \vec{\boldsymbol{\nabla}}\times(\vec{\mathbf{a}}_{(\vec{x},t)}\delta) - \frac{1}{2}\left[\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{q}}_{(\vec{x},t)}\right]\delta = 0\right]d\vec{\boldsymbol{\omega}}d\vec{\mathbf{r}}\right] \qquad (5.25)$$

because integration and differentiation beeing exchangeable follows

$$\left[\frac{\partial}{\partial t}\boldsymbol{\Xi}\left[\vec{\boldsymbol{\omega}}_{(\vec{x},t)}\right] - \vec{\boldsymbol{\nabla}}\times\boldsymbol{\Xi}\left[\vec{\mathbf{a}}_{(\vec{x},t)}\right] - \frac{1}{2}\vec{\boldsymbol{\nabla}}\times\boldsymbol{\Xi}\left[\vec{\mathbf{q}}_{(\vec{x},t)}\right] = 0 \qquad (5.26)$$

and we have the first of the dual turbulence equations

$$\frac{\partial}{\partial t}\vec{\boldsymbol{\omega}} - \vec{\boldsymbol{\nabla}}\times\vec{\mathbf{a}} - \frac{1}{2}\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{q}} = 0 \qquad (5.27)$$

accordingly

$$\frac{\partial}{\partial t}\vec{\boldsymbol{\omega}} - \vec{\boldsymbol{\nabla}}\times(\vec{\mathbf{v}}\times\vec{\boldsymbol{\omega}}) - \frac{1}{2}\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{q}} = 0.$$

Hereby the connection of stochastics and deterministics is achieved. From the key-equation above a second equation, the dual one, may be derived.

Back to the initial equation (5.18)

$$\frac{\partial}{\partial t}\delta + \vec{\boldsymbol{\omega}}_{(\vec{x},t)}\times\vec{\mathbf{r}}_{(\vec{x},t)}\cdot\vec{\boldsymbol{\nabla}}\delta = \frac{1}{2}\left[\frac{\vec{\boldsymbol{\omega}}_{(\vec{x},t)}}{\omega^2_{(\vec{x},t)}}\cdot\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{q}}_{(\vec{x},t)}\right]\delta$$

Simple conversions give

$$\frac{\partial}{\partial t}\left(\vec{\mathbf{r}}_{(\vec{x},t)}\cdot\frac{\vec{\mathbf{r}}_{(\vec{x},t)}}{r^2_{(\vec{x},t)}}\delta\right)+\vec{\mathbf{r}}_{(\vec{x},t)}\cdot\vec{\boldsymbol{\nabla}}\times(\vec{\boldsymbol{\omega}}_{(\vec{x},t)}\delta)-\frac{\vec{\mathbf{r}}_{(\vec{x},t)}\cdot\vec{\mathbf{r}}_{(\vec{x},t)}}{r^2_{(\vec{x},t)}}\frac{1}{2}\left[\frac{\vec{\boldsymbol{\omega}}_{(\vec{x},t)}}{\omega^2_{(\vec{x},t)}}\cdot\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{q}}_{(\vec{x},t)}\right]\delta=0$$
$$\longrightarrow\vec{\boldsymbol{r}}_{(\vec{x},t)}\left[\frac{\partial}{\partial t}\frac{\vec{\mathbf{r}}_{(\vec{x},t)}}{r^2_{(\vec{x},t)}}\delta+\vec{\boldsymbol{\nabla}}\times(\vec{\boldsymbol{\omega}}_{(\vec{x},t)}\delta)-\frac{\vec{\mathbf{r}}_{(\vec{x},t)}}{r^2_{(\vec{x},t)}}\frac{1}{2}\left[\frac{\vec{\boldsymbol{\omega}}_{(\vec{x},t)}}{\omega^2_{(\vec{x},t)}}\cdot\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{q}}_{(\vec{x},t)}\right]\delta\right]=0 \tag{5.28}$$

Using the curvature vector field of the fluid trajectories $\vec{\mathbf{b}}=\frac{\vec{\mathbf{r}}}{r^2}$ the equation is written

$$\frac{\partial}{\partial t}(\vec{\mathbf{b}}_{(\vec{x},t)}\delta)+\vec{\boldsymbol{\nabla}}\times(\vec{\boldsymbol{\omega}}_{(\vec{x},t)}\delta)-\frac{1}{2}\vec{\mathbf{b}}_{(\vec{x},t)}\frac{\vec{\boldsymbol{\omega}}_{(\vec{x},t)}}{\omega^2_{(\vec{x},t)}}\cdot\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{q}}_{(\vec{x},t)}\delta=0 \tag{5.29}$$

and applying the operators $\boldsymbol{\Xi}$ arises

$$\boldsymbol{\Xi}\left[\int_{\vec{\mathbf{r}}}\int_{\vec{\boldsymbol{\omega}}}\left[\frac{\partial}{\partial t}(\vec{\mathbf{b}}_{(\vec{x},t)}\delta)+\vec{\boldsymbol{\nabla}}\times(\vec{\boldsymbol{\omega}}_{(\vec{x},t)}\delta)-\frac{1}{2}\vec{\mathbf{b}}_{(\vec{x},t)}\frac{\vec{\boldsymbol{\omega}}_{(\vec{x},t)}}{\omega^2_{(\vec{x},t)}}\cdot\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{q}}_{(\vec{x},t)}\delta=0\right]d\vec{\boldsymbol{\omega}}d\vec{\mathbf{r}}\right] \tag{5.30}$$

respectively

$$\frac{\partial}{\partial t}\boldsymbol{\Xi}[\vec{\mathbf{b}}_{(\vec{x},t)}]+\vec{\boldsymbol{\nabla}}\times\boldsymbol{\Xi}[\vec{\boldsymbol{\omega}}_{(\vec{x},t)}]-\frac{1}{2}\boldsymbol{\Xi}\left[\left(\vec{\mathbf{b}}\frac{\vec{\boldsymbol{\omega}}}{\omega^2}\cdot\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{q}}\right)_{(\vec{x},t)}\right]=0. \tag{5.31}$$

Such the second of the dual turbulence equations is approached

$$\frac{\partial}{\partial t}\vec{\mathbf{b}}+\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{\omega}}-\frac{1}{2}\vec{\mathbf{b}}\left[\frac{\vec{\boldsymbol{\omega}}}{\omega^2}\cdot\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{q}}\right]=0. \tag{5.32}$$

Closing this dual equation system

$$\boxed{\begin{aligned}&\frac{\partial}{\partial t}\vec{\boldsymbol{\omega}}-\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{a}}-\frac{1}{2}\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{q}}=0\\&\frac{\partial}{\partial t}\vec{\mathbf{b}}+\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{\omega}}-\frac{1}{2}\vec{\mathbf{b}}\left[\frac{\vec{\boldsymbol{\omega}}}{\omega^2}\cdot\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{q}}\right]=0\\&\vec{\mathbf{v}}=\vec{\boldsymbol{\omega}}\times\frac{\vec{\mathbf{b}}}{b^2},\ \vec{\mathbf{a}}=\vec{\mathbf{v}}\times\vec{\boldsymbol{\omega}}\end{aligned}} \tag{5.33}$$

further equations are necessary besides the momentum equations. In the case of the Navier-Stokes-equations

$$\frac{\partial \vec{\mathbf{v}}}{\partial t} + (\vec{\mathbf{v}} \cdot \vec{\boldsymbol{\nabla}})\vec{\mathbf{v}} = -\frac{1}{\rho}\vec{\boldsymbol{\nabla}}\mathbf{p} + \vec{\mathbf{g}} + \nu\Delta\vec{\mathbf{v}} + (\zeta + \frac{\nu}{3})\vec{\boldsymbol{\nabla}}\ (\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{v}})$$

i.e.

$$\vec{\mathbf{q}} = -\frac{1}{\rho}\vec{\boldsymbol{\nabla}}\mathbf{p} + \vec{\mathbf{g}} + \nu\Delta\vec{\mathbf{v}} + (\zeta + \frac{\nu}{3})\vec{\boldsymbol{\nabla}}\ (\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{v}})$$

this could happen by simultaneously using the known continuity, energy as well as state equation. But this proves not to be expedient. In chapter 7 the complete equation system is presented and it is shown that the usual Navier-Stokes-equations are not warranting the correct momentum balancing in turbulence.
The term

$$-\frac{1}{2}\vec{\mathbf{b}}\left[\frac{\vec{\boldsymbol{\omega}}}{\omega^2} \cdot \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{q}}\right]$$

may lead to removable singularities in space-time-points $(\vec{\mathbf{x}}, t)$ when turning points occur in the fluid element trajectories $\vec{\boldsymbol{\omega}} = 0$ and $\vec{\mathbf{b}} = \mathbf{0}$ arising simultaneously. In this case the whole term is calculated from its surroundings. The same shall apply for the calculation of the velocity $\vec{\mathbf{v}}$. In such cases there is an alternative way shown in chapter 7, too.

6. Stochastic and deterministic general vector fields

$$f_{t_\epsilon}(\vec{\mathbf{x}},t,\vec{\mathbf{E}},\vec{\mathbf{B}}) = \int\limits_{\vec{\mathbf{B}}}\int\limits_{\vec{\mathbf{E}}} W_{t_\epsilon}(\vec{\mathbf{x}},t,\vec{\mathbf{E}},\vec{\mathbf{B}},\vec{\mathbf{E}}',\vec{\mathbf{B}}')\cdot f_{t_\epsilon}(\vec{\mathbf{x}}-\Delta\vec{\mathbf{x}},t-t_\epsilon,\vec{\mathbf{E}}',\vec{\mathbf{B}}')d\vec{\mathbf{E}}'d\vec{\mathbf{B}}'$$

$$\Updownarrow$$

$$\frac{\partial}{\partial t}\vec{\mathbf{B}}-\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{E}}=0$$
$$\frac{\partial}{\partial t}\left(\frac{B^2}{E^2}\cdot\vec{\mathbf{E}}\right)+\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{B}}=0$$

6.1. Introduction

Subsequently continuum fluctuations of general 3 dimensional vector fields $\vec{\mathbf{A}}(\vec{\mathbf{x}},t)$ with $\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{A}}\neq\mathbf{0}$ are analysed. They have to be sufficiently often continuously differentiable. Defining the vector fields $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ by

$$\begin{aligned}\vec{\mathbf{E}} &= \partial\vec{\mathbf{A}}/\partial t \neq 0\\ \vec{\mathbf{B}} &= \vec{\boldsymbol{\nabla}}\times\vec{\mathbf{A}}\neq 0\end{aligned} \tag{6.1}$$

and owing to the exchangeability of the operators $\partial/\partial t$ und $\vec{\boldsymbol{\nabla}}\times$

$$\frac{\partial\vec{\mathbf{B}}}{\partial t} = \vec{\boldsymbol{\nabla}}\times\vec{\mathbf{E}} \tag{6.2}$$

follows. This is a necessary consequence of the condition of the continuous differentiability of $\vec{\mathbf{A}}(\vec{\mathbf{x}},t)$. This relation is known according to the Maxwell Equations. The for this purpose dual equation is subsequently beeing looked for. In an analogous approach derivating the turbulence equations a stochastic continuum process in the frame of an ensemble theory is formulated such that according to a deterministic theory the already known as well as the related dual equation arise with fluctuating quantities $\vec{\mathbf{E}}$ und $\vec{\mathbf{B}}$.

6.2. The Transition: stochastic theory $\longleftrightarrow$ deterministic theory

This transition takes place in the same way as the derivation of the dual turbulence equation pair. Every space-time-point$(\vec{\mathbf{x}}, t)$ a continuously differentiable distribution density $f_{\boldsymbol{t_\epsilon}}$ is assigned to the motion quantities $\vec{\mathbf{E}}_{t_\epsilon} = \partial \vec{\mathbf{A}}_{t_\epsilon}/\partial t$ and $\vec{\mathbf{B}}_{t_\epsilon} = \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{A}}_{t_\epsilon}$ with

$$f_{\boldsymbol{t_\epsilon}} = f_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, t, \vec{\mathbf{E}}, \vec{\mathbf{B}}). \tag{6.3}$$

In the with $\boldsymbol{t_\epsilon}$ or $\boldsymbol{\epsilon}$ indexed functions $f_{\boldsymbol{t_\epsilon}}$ it is automatically assumed that the included motion quantities $(\vec{\mathbf{E}}, \vec{\mathbf{B}})$ are assigned to a $\boldsymbol{t_\epsilon}$-measurement accuracy. The indexing of the motion quantities may be omitted in functions appropriately indexed themselves.

After the execution of a $\lim \boldsymbol{t_\epsilon} \to 0$-process

$$\lim_{\boldsymbol{t_\epsilon} \to 0} f_{\boldsymbol{t_\epsilon}}(\vec{\mathbf{x}}, t, \vec{\mathbf{E}}, \vec{\mathbf{B}}) = f(\vec{\mathbf{x}}, t, \vec{\mathbf{E}}, \vec{\mathbf{B}}) \tag{6.4}$$

f and $(\vec{\boldsymbol{E}}, \vec{\boldsymbol{B}})$ are understood in the sense of an exact measurement process.

The stochastic transport of the fluctuation quantities

$$\left(\vec{\mathbf{E}}'_{t_\epsilon}(\vec{\boldsymbol{x}} - \Delta\vec{\boldsymbol{x}}, t - t_\epsilon), \vec{\mathbf{B}}'_{t_\epsilon}(\vec{\boldsymbol{x}} - \Delta\vec{\boldsymbol{x}}, t - t_\epsilon)\right) \longrightarrow \left(\vec{\mathbf{E}}_{t_\epsilon}(\vec{\boldsymbol{x}}, t), \vec{\mathbf{B}}_{t_\epsilon}(\vec{\boldsymbol{x}}, t)\right)$$

happens by the transition probability density $W_{t_\epsilon} = W_{t_\epsilon}(\vec{\boldsymbol{x}}, t, \vec{\mathbf{E}}, \vec{\mathbf{B}}, \vec{\mathbf{E}}', \vec{\mathbf{B}}')$ with

$$\begin{aligned}
\lim_{t_\epsilon \to 0} W_{t_\epsilon} =& \delta(\vec{\mathbf{E}}, \vec{\mathbf{B}}; \vec{\mathbf{E}}', \vec{\mathbf{B}}') \\
f_{t_\epsilon}(\vec{\mathbf{x}}, t, \vec{\mathbf{E}}, \vec{\mathbf{B}}) =& \int\limits_{\vec{\mathbf{B}}'} \int\limits_{\vec{\mathbf{E}}'} W_{t_\epsilon}(\vec{\mathbf{x}}, t, \vec{\mathbf{E}}, \vec{\mathbf{B}}, \vec{\mathbf{E}}', \vec{\mathbf{B}}') \cdot f_{t_\epsilon}(\vec{\mathbf{x}} - \Delta\vec{\mathbf{x}}, t - t_\epsilon, \vec{\mathbf{E}}', \vec{\mathbf{B}}') d\vec{\mathbf{E}}' d\vec{\mathbf{B}}' \\
\Delta\vec{\mathbf{x}} =& t_\epsilon \cdot \vec{\mathbf{E}}' \times \frac{\vec{\mathbf{B}}'}{B'^2} \text{ and } \vec{\mathbf{E}}' \times \frac{\vec{\mathbf{B}}'}{B'^2} = \text{velocity of propagation.}
\end{aligned} \tag{6.5}$$

These equations define stochastic continuum fluctuations of the quantities $\vec{\mathbf{E}}$ und $\vec{\mathbf{B}}$ in the sense of an ensemble-theory and represent a Markov Process of natural causality. The test-functions of distribution theory obtain by this formulation of a transition probability density W_{t_ϵ} an immediate physical meaning.

f_{t_ϵ} is developed until the 1st order about $(\vec{\mathbf{x}}, t) \Longrightarrow$

$$\begin{aligned}
f_{t_\epsilon}(t - t_\epsilon, \vec{\mathbf{x}} - \triangle\vec{\mathbf{x}}, \vec{\mathbf{E}}', \vec{\mathbf{B}}') = f'_{t_\epsilon} - \frac{\partial f'_{t_\epsilon}}{\partial t} \cdot t_\epsilon - \triangle\vec{\mathbf{x}} \cdot \vec{\boldsymbol{\nabla}} f'_{t_\epsilon} + \boldsymbol{O}({t_\epsilon}^2) \\
f'_{t_\epsilon} = f_{t_\epsilon}(\vec{\mathbf{x}}, t, \vec{\mathbf{E}}', \vec{\mathbf{B}}')
\end{aligned} \tag{6.6}$$

und one gets

$$\int_{\vec{\mathbf{E}}} \int_{\vec{\mathbf{B}}} W_{t_\epsilon} \left[\frac{\partial f'_{t_\epsilon}}{\partial t} + \vec{\mathbf{E}}' \times \frac{\vec{\mathbf{B}}'}{B'^2} \cdot \vec{\boldsymbol{\nabla}} f'_{t_\epsilon} \right] d\vec{\mathbf{E}}' d\vec{\mathbf{B}}' + \boldsymbol{O}({t_\epsilon}^2) = \frac{\int_{\vec{\mathbf{B}}} \int_{\vec{\mathbf{E}}} W_{t_\epsilon} f'_{t_\epsilon} d\vec{\mathbf{E}}' d\vec{\mathbf{B}}' - f_{t_\epsilon}}{t_\epsilon}. \tag{6.7}$$

By the process $t_\epsilon \to 0$ W_{t_ϵ} degenerates to a δ-function:

$$\lim_{\boldsymbol{t_\epsilon \to 0}} W_{t_\epsilon} = \delta(\vec{\mathbf{E}}, \vec{\mathbf{B}}; \vec{\mathbf{E}}', \vec{\mathbf{B}}') \tag{6.8}$$

$\lim \boldsymbol{t_\epsilon} \to 0$ applied leads to

$$\frac{\partial f}{\partial t} + \vec{\mathbf{E}} \times \frac{\vec{\mathbf{B}}}{B^2} \cdot \vec{\boldsymbol{\nabla}} f = \lim_{t_\epsilon \to 0} \frac{\int_{\vec{\mathbf{E}}} \int_{\vec{\mathbf{B}}} W_{t_\epsilon} f'_{t_\epsilon} d\vec{\mathbf{E}}' d\vec{\mathbf{B}}' - f_{t_\epsilon}}{t_\epsilon}. \tag{6.9}$$

Recovering equation (6.2) after the transition to deterministic consideration the exchange term has to vanish, in this case.

$$\lim_{t_\epsilon \to 0} \frac{\int_{\vec{\mathbf{B}}} \int_{\vec{\mathbf{E}}} W_{t_\epsilon} f'_{t_\epsilon} d\vec{\mathbf{E}}' d\vec{\mathbf{B}}' - f_{t_\epsilon}}{t_\epsilon} = \mathbf{0}. \tag{6.10}$$

This link is an integral part of the considered stochastic process.

Limiting ourselves to one system of the ensemble the function $f(\vec{\mathbf{x}}, t, \vec{\mathbf{E}}, \vec{\mathbf{B}})$ in the space-time-point $(\vec{\boldsymbol{x}}, t)$ degenerates to a $\delta-$function

$$f(\vec{\mathbf{x}}, t, \vec{\mathbf{E}}, \vec{\mathbf{B}}) \longrightarrow \delta(\vec{\mathbf{E}}_{(\vec{\mathbf{x}},t)}, \vec{\mathbf{B}}_{(\vec{\mathbf{x}},t)}; \vec{\mathbf{E}}, \vec{\mathbf{B}})\text{-function.} \tag{6.11}$$

From equation (6.9) arises the key-equation

$$\boxed{\frac{\partial}{\partial t}\delta + \vec{\mathbf{E}}_{(\vec{\boldsymbol{x}},t)} \times \frac{\vec{\mathbf{B}}_{(\vec{\boldsymbol{x}},t)}}{B^2_{(\vec{\boldsymbol{x}},t)}} \cdot \vec{\boldsymbol{\nabla}}\delta = \mathbf{0}}. \tag{6.12}$$

Respectively section 5.2 the $\Xi[...]$-operator is inserted as follows

$$\begin{aligned} \Xi\left[\int_{\vec{\mathbf{E}}}\int_{\vec{\mathbf{B}}} \delta(\vec{\mathbf{B}}_{(\vec{\boldsymbol{x}},t)}, \vec{\mathbf{E}}_{(\vec{\boldsymbol{x}},t)}; \vec{\mathbf{B}}, \vec{\mathbf{E}})\vec{\mathbf{B}} d\vec{\mathbf{B}} d\vec{\mathbf{E}}\right] &= \Xi[\vec{\mathbf{B}}_{(\vec{\boldsymbol{x}},t)}] = \vec{\mathbf{B}}(\vec{\boldsymbol{x}}, t) \\ \Xi\left[\int_{\vec{\mathbf{E}}}\int_{\vec{\mathbf{B}}} \delta(\vec{\mathbf{B}}_{(\vec{\boldsymbol{x}},t)}, \vec{\mathbf{E}}_{(\vec{\boldsymbol{x}},t)}; \vec{\mathbf{B}}, \vec{\mathbf{E}})\vec{\mathbf{E}} d\vec{\mathbf{B}} d\vec{\mathbf{E}}\right] &= \Xi\left[\vec{\mathbf{E}}_{(\vec{\boldsymbol{x}},t)}\right] = \vec{\mathbf{E}}(\vec{\boldsymbol{x}}, t) \end{aligned} \tag{6.13}$$

or

$$\Xi\left[\int_{\vec{\mathbf{E}}}\int_{\vec{\mathbf{B}}} \delta(\vec{\mathbf{B}}_{(\vec{\boldsymbol{x}},t)}, \vec{\mathbf{E}}_{(\vec{\boldsymbol{x}},t)}; \vec{\mathbf{b}}, \vec{\mathbf{E}})\left(\frac{B^2}{E^2}\cdot\vec{\mathbf{E}}\right) d\vec{\mathbf{B}} d\vec{\mathbf{E}}\right] = \Xi\left[\frac{B^2_{(\vec{\boldsymbol{x}},t)}}{E^2_{(\vec{\boldsymbol{x}},t)}}\cdot\vec{\mathbf{E}}_{(\vec{\boldsymbol{x}},t)}\right] = \frac{B^2(\vec{\boldsymbol{x}}, t)}{E^2(\vec{\boldsymbol{x}}, t)}\cdot\vec{\mathbf{E}}(\vec{\boldsymbol{x}}, t), \tag{6.14}$$

developing the deterministic equations from the key equation.

6.3. The deterministic fluctuation-equations

The key-equation (6.12) represents the interface for the transition of stochastic to deterministic consideration. From the perspective of statistics over the states of movement of the parallelly assumed deterministic processes in the respective point $(\vec{\mathbf{x}}, t)$ one is confined to a single system and such to a single state of motion $(\vec{\mathbf{E}}_{(\vec{\mathbf{x}},t)}, \vec{\mathbf{B}}_{(\vec{\mathbf{x}},t)})$.

In this situation the vectors of the motion quantities may be pushed before and behind the differential operators

$$\begin{aligned}\vec{\mathbf{E}}_{(\vec{\boldsymbol{x}},t)} \times \frac{\vec{\mathbf{B}}_{(\vec{\boldsymbol{x}},t)}}{B^2_{(\vec{\boldsymbol{x}},t)}} \cdot \vec{\boldsymbol{\nabla}}\delta &= -\frac{\vec{\mathbf{B}}_{(\vec{\boldsymbol{x}},t)}}{B^2_{(\vec{\boldsymbol{x}},t)}} \times \vec{\mathbf{E}}_{(\vec{\boldsymbol{x}},t)} \cdot \vec{\boldsymbol{\nabla}}\delta \\ &= -\frac{\vec{\mathbf{B}}_{(\vec{\boldsymbol{x}},t)}}{B^2_{(\vec{\boldsymbol{x}},t)}} \cdot \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}}_{(\vec{\boldsymbol{x}},t)}\delta\end{aligned}$$

Further more there is

$$\begin{aligned}&\frac{\partial}{\partial t}(\frac{\vec{\mathbf{B}}_{(\vec{\boldsymbol{x}},t)} \cdot \vec{\mathbf{B}}_{(\vec{\boldsymbol{x}},t)}}{B^2_{(\vec{\boldsymbol{x}},t)}}\delta) - \frac{\vec{\mathbf{B}}_{(\vec{\boldsymbol{x}},t)}}{B^2_{(\vec{\boldsymbol{x}},t)}} \cdot \vec{\boldsymbol{\nabla}} \times (\vec{\mathbf{E}}_{(\vec{\boldsymbol{x}},t)}\delta) = 0 \\ \Longrightarrow\ &\frac{\vec{\mathbf{B}}_{(\vec{\boldsymbol{x}},t)}}{B^2_{(\vec{\boldsymbol{x}},t)}} \cdot [\frac{\partial}{\partial t}(\vec{\mathbf{B}}_{(\vec{\boldsymbol{x}},t)}\delta) - \vec{\boldsymbol{\nabla}} \times (\vec{\mathbf{E}}_{(\vec{\boldsymbol{x}},t)}\delta)] = 0 \\ &\Longrightarrow\ \frac{\partial}{\partial t}(\vec{\mathbf{B}}_{(\vec{\boldsymbol{x}},t)}\delta) - \vec{\boldsymbol{\nabla}} \times (\vec{\mathbf{E}}_{(\vec{\boldsymbol{x}},t)}\delta) = 0.\end{aligned} \tag{6.15}$$

Now the vector fields of the motion quantities $(\vec{\mathbf{E}}_{(\vec{\boldsymbol{x}},t)}, \vec{\mathbf{B}}_{(\vec{\boldsymbol{x}},t)})$ of the one determinstic system are created about the point $(\vec{\boldsymbol{x}}, t)$ and such the transition to the deterministic equations of the one system has succeeded.

One obtains

$$\boldsymbol{\Xi}\Big[\int_{\vec{\mathbf{B}}}\int_{\vec{\mathbf{E}}}\Big[\frac{\partial}{\partial t}(\vec{\mathbf{B}}_{(\vec{\boldsymbol{x}},t)}\delta) - \vec{\boldsymbol{\nabla}} \times (\vec{\mathbf{E}}_{(\vec{\boldsymbol{x}},t)}\delta) = 0\Big]d\vec{\mathbf{E}}d\vec{\mathbf{B}}\Big]. \tag{6.16}$$

As integration and differentiation are exchangeable $\Longrightarrow$

$$\frac{\partial}{\partial t}\boldsymbol{\Xi}[\vec{\mathbf{B}}_{(\vec{\boldsymbol{x}},t)}] - \vec{\boldsymbol{\nabla}} \times \boldsymbol{\Xi}[\vec{\mathbf{E}}_{(\vec{\boldsymbol{x}},t)}] = 0 \tag{6.17}$$

and it results in the 1.st of the dual fluctuation equations

$$\frac{\partial}{\partial t}\vec{\mathbf{B}} - \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}} = 0. \tag{6.18}$$

Hereby the stochastic-deterministic connection is established.

Back to the key-equation (6.12)

$$\frac{\partial}{\partial t}\delta + \vec{\mathbf{E}}_{(\vec{\boldsymbol{x}},t)} \times \frac{\vec{\mathbf{B}}_{(\vec{\boldsymbol{x}},t)}}{B^2_{(\vec{\boldsymbol{x}},t)}} \cdot \vec{\boldsymbol{\nabla}}\delta = \mathbf{0}$$

one obtains by simple conversion

$$\begin{aligned} \frac{\partial}{\partial t}\left(\vec{\mathbf{E}}_{(\vec{x},t)} \cdot \frac{\vec{\mathbf{E}}_{(\vec{x},t)}}{E^2_{(\vec{x},t)}}\delta\right) + \vec{\mathbf{E}}_{(\vec{x},t)} \cdot \vec{\nabla} \times \left(\frac{\vec{\mathbf{B}}_{(\vec{x},t)}}{B^2_{(\vec{x},t)}}\delta\right) = 0 \\ \frac{\partial}{\partial t}\left(\frac{B^2_{(\vec{x},t)}}{E^2_{(\vec{x},t)}} \cdot \vec{\mathbf{E}}_{(\vec{x},t)}\delta\right) + \vec{\nabla} \times (\vec{\mathbf{B}}_{(\vec{x},t)}\delta) = 0 \end{aligned} \tag{6.19}$$

and

$$\Xi\left[\int_{\vec{\mathbf{B}}}\int_{\vec{\mathbf{E}}}\left[\frac{\partial}{\partial t}\left(\frac{B^2_{(\vec{x},t)}}{E^2_{(\vec{x},t)}} \cdot \vec{\mathbf{E}}_{(\vec{x},t)}\delta\right) + \vec{\nabla} \times (\vec{\mathbf{B}}_{(\vec{x},t)}\delta) = 0\right] d\vec{\mathbf{E}} d\vec{\mathbf{B}}\right] \tag{6.20}$$

respectively

$$\frac{\partial}{\partial t}\Xi\left[\frac{B^2_{(\vec{x},t)}}{E^2_{(\vec{x},t)}} \cdot \vec{\mathbf{E}}_{(\vec{x},t)}\right] + \vec{\nabla} \times \Xi[\vec{\mathbf{B}}_{(\vec{x},t)}] = 0. \tag{6.21}$$

So we have the second of the two dual equations

$$\frac{\partial}{\partial t}(\frac{B^2}{E^2} \cdot \vec{\mathbf{E}}) + \vec{\nabla} \times (\vec{\mathbf{B}}) = 0. \tag{6.22}$$

The result is recapitulated by the following equation system:

$$\boxed{\begin{aligned} &\frac{\partial}{\partial t}\vec{\mathbf{B}} - \vec{\nabla} \times \vec{\mathbf{E}} = 0 \\ &\frac{\partial}{\partial t}\left(\frac{B^2}{E^2} \cdot \vec{\mathbf{E}}\right) + \vec{\nabla} \times \vec{\mathbf{B}} = 0 \\ &\vec{\mathbf{E}} \times \frac{\vec{\mathbf{B}}}{B^2} = \text{propagation speed} \end{aligned}} \tag{6.23}$$

with $|\vec{\mathbf{E}} \times \frac{\vec{\mathbf{B}}}{B^2}| \leq |\vec{\mathbf{E}}| \cdot |\frac{\vec{\mathbf{B}}}{B^2}|$. I.e. $\frac{E^2}{B^2}$ is not the quadratic propagation speed. Interestingly, this only becomes clear after the involvement of the stochastic ensemble theory.

The equation system (6.23) is in such general terms that the physical significance depends on the interpretation of the starting field $\vec{\mathbf{A}}$, the boundary conditions as well as on the initial conditions. Hereunder, a deformation vector field, the velocity vector field of turbulence motions or the fluctuations of any other continuously differentiable vector field may be understood. These equations possess with boundary- and suitable initial conditions exactly one solution after the theorem of Cauchy-Kowalewskaja [2]. This statement is at first restricted to the calculation of the fields $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$. Calculating the field $\vec{\mathbf{A}}$ with the mere knowledge of

$$\frac{\partial \vec{\mathbf{A}}}{\partial \boldsymbol{t}} = \vec{\mathbf{E}} \tag{6.24}$$

is not possible in all cases. A negative example is the calculation of $\vec{\mathbf{v}}$ with the knowledge of $\frac{\partial \vec{\mathbf{v}}}{\partial \boldsymbol{t}}$ related to turbulent velocity fluctuations as shown in chapter 7. However, in this case these relations are applied completing the turbulence equations. The particular definition of turbulence fluctuation elements (chapter 2) makes this problem almost vividly comprehensible.
Considering turbulent motions this can be done from a different perspective. With the equation system (6.23) the motion quantities

$$\vec{\mathbf{E}} = \frac{\partial}{\partial t}\vec{\mathbf{v}} \qquad \text{and} \qquad \vec{\mathbf{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{v}}$$

are transported with the propagation speed

$$\vec{\mathbf{v}} = \vec{\mathbf{E}} \times \frac{\vec{\mathbf{B}}}{B^2}.$$

The equation system (5.33) describes the mass transport by the velocity $\vec{\mathbf{v}}$. In consideration of $\vec{\mathbf{b}} = \frac{\vec{\mathbf{a}}}{\vec{\mathbf{v}}^2}$ (5.33) may be formulated omitting the viscosity and assuming $\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{q}} = 0$ as follows:

$$\begin{aligned}
&\frac{\partial}{\partial t}\vec{\boldsymbol{\omega}} - \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{a}} = 0 \\
&\frac{\partial}{\partial t}\left(\frac{\vec{\mathbf{a}}}{v^2}\right) + \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{\omega}} = 0 \\
&\vec{\mathbf{v}} = \vec{\boldsymbol{\omega}} \times \frac{\vec{\mathbf{b}}}{b^2},\ \vec{\mathbf{a}} = \vec{\mathbf{v}} \times \vec{\boldsymbol{\omega}},\ \vec{\mathbf{v}} = \text{propagation speed}
\end{aligned} \tag{6.25}$$

In doing so $\vec{\mathbf{v}} \perp \vec{\boldsymbol{\omega}} \perp \vec{\mathbf{a}}$ holds. The equations (6.23) and (6.25) do not formally differ apart from orthogonality conditions. But it is not expected, that the fluctuations generated by a conservative accelleration field ($\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{q}} = 0$) may describe hydrodynamic turbulences. This is discussed in chapter 7.

6.3.1. The vacuum Maxwell Equations

The propagation speed having the constant amount of light velocity one obtains the known equations of vacuum-electrodynamics in the coordinate system of the observer:

$$\boxed{\begin{array}{l} \frac{\partial}{\partial t}\vec{\mathbf{B}} - \vec{\mathbf{\nabla}} \times \vec{\mathbf{E}} = 0 \\ \frac{1}{c^2}\frac{\partial}{\partial t}\vec{\mathbf{E}} + \vec{\mathbf{\nabla}} \times \vec{\mathbf{B}} = 0 \qquad \text{mit} \qquad \vec{\mathbf{E}} \perp \vec{\mathbf{B}} \\ \vec{\mathbf{E}} \times \frac{\vec{\mathbf{B}}}{B^2} = \vec{\mathbf{c}} = \text{propagation speed of light.} \end{array}} \tag{6.26}$$

Hereby a formal analogy is established between electrodynamics and turbulent fluid dynamics. It is only based on the analogy of the propagation of the motion quantities $(\vec{\mathbf{E}}, \vec{\mathbf{B}})$ and $(\frac{\partial}{\partial t}\vec{\mathbf{v}}, \vec{\mathbf{\nabla}} \times \vec{\mathbf{v}})$. But a turbulent mass transport with the local velocity $\vec{\mathbf{v}}$ cannot be sufficiently described in this way as stated in chapter 7.

So the electrodynamic equations of vacuum are generally derived, too. Usually, they are seen in the above equations with $-\vec{\mathbf{E}}$. It is more than pure supposition, that they describe properties of space-time without a unification of General Relativity and electromagnetic field in vacuum having succeeded, though many physicists not least Einstein [3], Jordan [5] and many others having endeavoured.

There is still the explanation of the associated initial field $\vec{\mathbf{A}}$ it generally happening in the frame of vector potential considerations, without recognizing $\vec{\mathbf{A}}$ as definite physical object. Only by a direct comprehension of the vector potential the electromagnetic field may be explained without means of mechanical quantities.[1]

[1]Electrodynamics is introduced in physics via mechanical effects.

7. The complete equation system of Turbulence

$$\vec{\mathbf{E}} + \frac{1}{2}\vec{\nabla}\vec{\mathbf{v}}^2 - 2\vec{\mathbf{v}} \times \vec{\boldsymbol{\omega}} = \vec{\mathbf{q}}$$

$$\frac{\partial}{\partial t}\vec{\boldsymbol{\omega}} - \vec{\nabla} \times \vec{\mathbf{a}} = \frac{1}{2}\vec{\nabla} \times \vec{\mathbf{q}}$$

$$\frac{\partial}{\partial t}\vec{\mathbf{b}} + \vec{\nabla} \times \vec{\boldsymbol{\omega}} = \frac{1}{2}\vec{\mathbf{b}}\left[\frac{\vec{\boldsymbol{\omega}}}{\boldsymbol{\omega}^2} \cdot \vec{\nabla} \times \vec{\mathbf{q}}\right]$$

$$\frac{\partial}{\partial t}\vec{\mathbf{F}} = -2\vec{\nabla} \times \vec{\boldsymbol{\omega}} \text{ mit } \vec{\mathbf{F}} = \frac{4\omega^2}{E^2} \cdot \vec{\mathbf{E}}$$

7.1. Introduction

For a fluctuating continuum field

$$\frac{d}{dt}\vec{\mathbf{v}}(\vec{\mathbf{x}}, t) = \frac{\partial \vec{\mathbf{v}}}{\partial t} + (\vec{\mathbf{v}} \cdot \vec{\nabla})\vec{\mathbf{v}} = \vec{\mathbf{q}}(\vec{\mathbf{x}}, t) \tag{7.1}$$

may be formally comprehended as a momentum equation. As soon as hydrodynamics is involved where a local thermodynamic balance is assumed, the Eulerian equations

$$\vec{\mathbf{q}} \stackrel{?}{=} -\frac{1}{\rho}\vec{\nabla}\mathbf{p} \tag{7.2}$$

are noted with the indication of the 2nd Newtonian law. They are only justified under restrictive rules like incompressibility of fluids or $\frac{1}{\rho}\vec{\nabla}\mathbf{p} = \vec{\nabla}\mathbf{h}$ (h=spec. enthalpy) and or negligible rubbing viscosity. So only limiting cases of fluid dynamics are characterized.

But generally, $\vec{\nabla} \times \vec{\mathbf{q}} \neq \mathbf{0}$ is to be presumed. $\vec{\mathbf{q}}$ is in contrast to Newtonian mechanics a non-conservative acceleration field. $\vec{\mathbf{q}}$ has transversal and longitudinal parts

$$\vec{\mathbf{q}} = \vec{\mathbf{q}}_{\perp} + \vec{\mathbf{q}}_{\|}. \tag{7.3}$$

The same applies for the velocity field $\vec{\mathbf{v}}$

$$\vec{\mathbf{v}} = \vec{\mathbf{v}}_{\perp} + \vec{\mathbf{v}}_{||} = \vec{\boldsymbol{\omega}} \times \vec{\mathbf{R}}. \tag{7.4}$$

The disassembly of the velocity field is adequately taken into account by the development of the dual turbulence equation system. In the momentum equation (7.1) 12 unknowns are "hiddenly" contained and with the turbulence equation only 9 coupled equations are available. For the field $\boldsymbol{\rho}\vec{\mathbf{q}}$ a disassembly in longitudinal und transversal part has to be considered, too.

$$\boldsymbol{\rho}\frac{d}{dt}\vec{\mathbf{v}}(\vec{\mathbf{x}},t) = \boldsymbol{\rho}\vec{\mathbf{q}} = (\boldsymbol{\rho}\vec{\mathbf{q}})_{\perp} + (\boldsymbol{\rho}\vec{\mathbf{q}})_{||} \tag{7.5}$$

Using the Navier-Stokes-equations this leads to

$$\boldsymbol{\rho}\vec{\mathbf{q}} = (\boldsymbol{\rho}\vec{\mathbf{q}})_{\perp} + (\boldsymbol{\rho}\vec{\mathbf{q}})_{||} \stackrel{?}{=} -\vec{\boldsymbol{\nabla}}\mathbf{p} + \boldsymbol{\rho}\cdot\vec{\mathbf{g}} + \eta\Delta\vec{\mathbf{v}} + (\xi + \frac{\eta}{3})\vec{\boldsymbol{\nabla}}\ (\vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{v}})$$

$\Longrightarrow$ [1]

$$(\boldsymbol{\rho}\vec{\mathbf{q}})_{\perp} \stackrel{?}{=} -\eta\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{v}} \tag{7.6}$$

and

$$\begin{aligned}(\boldsymbol{\rho}\vec{\mathbf{q}})_{||} &\stackrel{?}{=} -\ \vec{\boldsymbol{\nabla}}\mathbf{p} + \boldsymbol{\rho}\cdot\vec{\mathbf{g}} + (\xi + \eta\frac{4}{3})\vec{\boldsymbol{\nabla}}\ (\vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{v}}).\\ \vec{\mathbf{g}} &= \text{earth acceleration}\end{aligned} \tag{7.7}$$

As turbulent motions of sufficiently high reynolds number create negligible viscosity effects and on the other hand $\vec{\mathbf{q}}_{\perp}$ represents the decisive propulsion of the vortex motions turbulences are not correctly calculated by the usual equation system consisting of Navier-Stokes-equations, equation of continuity and energy equation. Equation (7.6) can not be correct. $\vec{\mathbf{q}}_{||}$ contributes nothing to the propulsion of the vortex motions. The turbulent dissipation can not be attributed to viscosity but to the matter exchange of the fluid elements and involved thermodynamic changes of state, if a local thermodynamic state is possible. Then the turbulent dissipation decisively decomposes the kinetic energy. $\Longrightarrow$

$$\boldsymbol{\rho}\vec{\mathbf{q}} = (\boldsymbol{\rho}\vec{\mathbf{q}})_{\perp} + (\boldsymbol{\rho}\vec{\mathbf{q}})_{||} \neq -\vec{\boldsymbol{\nabla}}\mathbf{p} + \boldsymbol{\rho}\cdot\vec{\mathbf{g}} + \eta\Delta\vec{\mathbf{v}} + (\xi + \frac{\eta}{3})\vec{\boldsymbol{\nabla}}\ (\vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{v}}) \tag{7.8}$$

[1] $\Delta\vec{\mathbf{v}} = \vec{\boldsymbol{\nabla}}\ (\vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{v}}) - \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{v}}$

The equations, often called conservation laws [1](Navier-Stokes-equations, equation of continuity and energy equation), do not meet these requirements for turbulence with the exception of the equation of continuity.

7.2. The composition of the complete equation system

In the turbulence equations (5.33) the viscous terms according to high reynolds numbers may be omitted whereas for sufficienly small reynolds numbers (laminar motions) they obtain significance.

The equation system

$$\frac{\partial \vec{\mathbf{v}}}{\partial t} + \frac{1}{2}\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}^2 - 2\vec{\mathbf{v}} \times \vec{\boldsymbol{\omega}} = \vec{\mathbf{q}} \tag{7.9}$$

$$\frac{\partial}{\partial t}\vec{\boldsymbol{\omega}} - \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{a}} = \frac{1}{2}\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{q}} \tag{7.10}$$

$$\frac{\partial}{\partial t}\vec{\mathbf{b}} + \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{\omega}} = \frac{1}{2}\vec{\mathbf{b}}\left[\frac{\vec{\boldsymbol{\omega}}}{\boldsymbol{\omega}^2} \cdot \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{q}}\right] \tag{7.11}$$

with

$$\vec{\mathbf{v}} = \vec{\boldsymbol{\omega}} \times \frac{\vec{\mathbf{b}}}{\mathbf{b}^2},\ \vec{\mathbf{a}} = \vec{\mathbf{v}} \times \vec{\boldsymbol{\omega}},\ \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{v}} \perp \vec{\mathbf{v}} \tag{7.12}$$

is not complete and as the Navier-Stokes-equations as momentum balance are refuted, the usual energy equation, derived from Navier-Stokes-equations and equation of continuity, is rejected, too. So the customarily for completion used energy equation, equation of continuity and state equation can not fill this gap.

There is the possibility observing the evolution of the velocity field not only by mass transport via the equations (7.9), (7.10) and (7.11) but via the progress of their fluctuation quantities $\frac{\partial \vec{\mathbf{v}}}{\partial t}$ and $\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{v}}$, too. Assuming the equation system (6.25)

$$\frac{\partial}{\partial t}\vec{\mathbf{B}} - \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}} = 0 \tag{7.13}$$

$$\frac{\partial}{\partial t}\left(\frac{B^2}{E^2} \cdot \vec{\mathbf{E}}\right) + \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}} = 0 \tag{7.14}$$

$$\vec{\mathbf{E}} \times \frac{\vec{\mathbf{B}}}{B^2} = propagationspeed \tag{7.15}$$

with

$$|\vec{\mathbf{E}} \times \frac{\vec{\mathbf{B}}}{B^2}| \leq |\vec{\mathbf{E}}| \cdot |\frac{\vec{\mathbf{B}}}{B^2}|$$

and

$$\vec{\mathbf{E}} = \frac{\partial \vec{\mathbf{v}}}{\partial t} \text{ and } \vec{\mathbf{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{v}}, \text{as well as} \vec{\mathbf{F}} = \frac{B^2}{E^2} \cdot \vec{\mathbf{E}},$$

one obtains the further equation

$$\boxed{\frac{\partial}{\partial t}\vec{\mathbf{F}} + 2\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{\omega}} = 0}. \tag{7.16}$$

Equation 7.13 with $\vec{\mathbf{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{v}} = 2\vec{\boldsymbol{\omega}}$ results in

$$\frac{\partial}{\partial t} 2\vec{\boldsymbol{\omega}} - \vec{\boldsymbol{\nabla}} \times \frac{\partial \vec{\mathbf{v}}}{\partial t} = 0.$$

It corresponds to (7.10) on account of

$$\vec{\boldsymbol{\nabla}} \times \frac{\partial \vec{\mathbf{v}}}{\partial t} = 2 \cdot \left(\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{a}} + \frac{1}{2} \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{q}} \right) = 2 \cdot \frac{\partial \vec{\boldsymbol{\omega}}}{\partial t}$$

with

$$\begin{aligned}
\vec{\mathbf{v}} &= \vec{\boldsymbol{\omega}} \times \frac{\vec{\mathbf{b}}}{b^2}, \\
\vec{\mathbf{a}} &= \vec{\mathbf{v}} \times \vec{\boldsymbol{\omega}}, \\
\vec{\mathbf{v}} &\perp \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{v}} \\
\vec{\mathbf{E}} &= \frac{\partial \vec{\mathbf{v}}}{\partial t} \\
\vec{\mathbf{E}} &= 4\omega^2 \vec{\mathbf{F}}^{-1}.
\end{aligned}$$

The invers vector respectively the scalar product means $\vec{\mathbf{F}}^{-1} = \vec{\mathbf{F}}/\vec{\mathbf{F}}^2 \Longrightarrow \vec{\mathbf{F}}^{-1} \cdot \vec{\mathbf{F}} = \mathbf{1}$. This corresponds to the relation of a curvature vector $\vec{\mathbf{b}}$ and its associated radius vector $\vec{\mathbf{r}}$ of a continuously differentiable trajectory in one point $(\vec{\mathbf{x}}, t)$ with $\vec{\mathbf{b}} \cdot \vec{\mathbf{r}} = \mathbf{1}$.

The motion of a turbulence field is characterised by a vortex field $\vec{\boldsymbol{\omega}}(\vec{\mathbf{x}}, t)$ and a curvature vector field[2] $\vec{\mathbf{b}}(\vec{\mathbf{x}}, t)$.

[2] Generally, one meets in physics curvature tensor fields at least of 2nd degree as in deformation theory or General Relativity.

So one obtains the complete equation system

$$\boxed{\begin{array}{l} \vec{\mathbf{E}} + \frac{1}{2}\vec{\boldsymbol{\nabla}}\vec{\mathbf{v}}^2 - 2\vec{\mathbf{v}} \times \vec{\boldsymbol{\omega}} = \vec{\mathbf{q}} \\ \hline \frac{\partial}{\partial t}\vec{\boldsymbol{\omega}} - \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{a}} = \frac{1}{2}\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{q}} \\ \frac{\partial}{\partial t}\vec{\mathbf{b}} + \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{\omega}} = \frac{1}{2}\vec{\mathbf{b}}\left[\frac{\vec{\boldsymbol{\omega}}}{\boldsymbol{\omega}^2} \cdot \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{q}}\right] \\ \frac{\partial}{\partial t}\vec{\mathbf{F}} = -2\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{\omega}} \text{ with } \vec{\mathbf{F}} = \frac{4\omega^2}{E^2} \cdot \vec{\mathbf{E}} \end{array}} . \tag{7.17}$$

At this a pairwise orthogonality of the vectors $(\vec{\mathbf{v}}, \vec{\boldsymbol{\omega}}, \vec{\mathbf{b}})$ i.e.: $\vec{\mathbf{v}} \perp \vec{\boldsymbol{\omega}}$, $\vec{\mathbf{v}} \perp \vec{\mathbf{b}}$, $\vec{\mathbf{b}} \perp \vec{\boldsymbol{\omega}}$ exists. Pursueing the trajectory of a fluid element beeing possible only after the calculation of the deterministic turbulence field the trajectory is accompanied by a frame of $\vec{\mathbf{v}}, \vec{\boldsymbol{\omega}}$ and $\vec{\mathbf{b}}$ except in points where $\vec{\boldsymbol{\omega}} = \mathbf{0}$ and $\vec{\mathbf{b}} = \mathbf{0}$ (turning points). Nevertheless, in this case $\vec{\mathbf{v}} \neq \mathbf{0}$ has to be otherwise the turbulence has come to an end.

7.3. Comments on the application of the complete equation system

On account of the theorem of Cauchy-Kowalewskaja [2] a unique solution is existing. The equation system may be numerically solved for the fields $\vec{\boldsymbol{\omega}}$, $\vec{\mathbf{b}}$, $\vec{\mathbf{q}}$ and $\vec{\mathbf{E}} = \frac{\partial \vec{\mathbf{v}}}{\partial t}$ (this is treated as an independent field as well as $\vec{\boldsymbol{\omega}}$,$\mathbf{b}$ und $\vec{\mathbf{q}}$) simultaneously obtaining the fields $\vec{\mathbf{a}}$ and $\vec{\mathbf{v}}$. The special approach of [10] enables 2 times continuously differentiable solutions not meaning analytical results. The order of differentiability may be principally driven forward. This particularly goes at the expense of the calculation effort.
Numerically solving this equation system [10] inflexible difference schemes are forbidden as beeing usual according to DNS-calculations (Direct Numerical Simulations related to Navier-Stokes-, continuum- and energy equation), as in the above equation system from the field environment removable singularities of $\vec{\mathbf{v}} = \vec{\boldsymbol{\omega}} \times \frac{\vec{\mathbf{b}}}{\mathbf{b}^2}$, $\frac{1}{2}\vec{\mathbf{b}}\left[\frac{\vec{\boldsymbol{\omega}}}{\boldsymbol{\omega}^2} \cdot \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{q}}\right]$ and $(2\vec{\omega})^2\vec{\mathbf{F}}^{-1} = \frac{\partial \vec{\mathbf{v}}}{\partial t}$ in different space-time-points $(\vec{\mathbf{x}}, t)$ are to be recognized. This outcome is a result of possible turning points of the fluid element trajectories leading to simultaneous values of $\vec{\boldsymbol{\omega}} = 0$ and $\vec{\boldsymbol{b}} = 0$. Die fineness of the time discretisations is determined by the vortex field $\vec{\boldsymbol{\omega}}$.
The in some turbulence models mentioned space- and time-scaling in this theory is

led back to the fluctuations of curvature fields $\vec{\mathbf{b}}$ and vortex fields $\vec{\boldsymbol{\omega}}$. Quantitative dependencies become accessible through numerical calculations.
Though friction losses according to heavy turbulent motions (high reynolds numbers) may be omitted the kinetic energy density may significantly decrease. Thus a part has to be converted into inner energy of thermodynamics if a local thermodynamic balance is existent. It is recalled, that turbulent fluid motions are characterized the surroundings of fluid elements continuously exchanging their matter and thus their thermodynamic state quantities, too.
The equation system (7.17) stands out only consisting of motion quantities, i.e. velocities and their temporal and spatial differentiations, a vector curvature field, its assigned vortex field and an abstract accelleration field $\vec{\mathbf{q}}$. Mass distributions respectively densities and thermodynamic quantities as pressure and inner energy do not appear. This fact finds its application in the general-relativistic considerations, too. The density distributions may be calculated by subsequent evaluation via the known velocity fields and the equation of continuity

$$\frac{\partial}{\partial t}\boldsymbol{\rho} = -\vec{\boldsymbol{\nabla}} \cdot (\boldsymbol{\rho}\vec{\mathbf{v}}). \tag{7.18}$$

The complete turbulence equation system may be solved even if no local thermodynamics is existent. Then the subsequent evaluation is limited to density calculations. One obtains the thermodynamic pressure distribution if existent by the subsequently calculated density field $\boldsymbol{\rho}$ and the accelleration field $\vec{\mathbf{q}}$ assuming

$$(\boldsymbol{\rho}\vec{\mathbf{q}})_{\|} = -\vec{\boldsymbol{\nabla}}\mathbf{p} + \boldsymbol{\rho}\vec{\mathbf{g}} + (\xi + \eta\frac{4}{3})\vec{\boldsymbol{\nabla}}\ (\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{v}}). \tag{7.19}$$

via Poisson-equation [3] :

$$\Delta\mathbf{p} = -\vec{\boldsymbol{\nabla}} \cdot (\boldsymbol{\rho}\vec{\mathbf{q}}) + \vec{\boldsymbol{\nabla}} \cdot \boldsymbol{\rho}\vec{\mathbf{g}} + \vec{\boldsymbol{\nabla}} \cdot (\xi + \eta\frac{4}{3})\vec{\boldsymbol{\nabla}}\ (\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{v}}). \tag{7.20}$$

At high reynolds numbers

$$\Delta\mathbf{p} = -\vec{\boldsymbol{\nabla}} \cdot (\boldsymbol{\rho}\vec{\mathbf{q}}) + \vec{\boldsymbol{\nabla}} \cdot \boldsymbol{\rho}\vec{\mathbf{g}} \tag{7.21}$$

is certainly sufficient. But it is not obvious, whether $(\boldsymbol{\rho}\vec{\mathbf{q}})_{\|}$ may be represented this way. Upon positive comparison density- and pressure evolution are determined without knowledge of a related state equation. Knowing the state equation all desired thermodynamic state quantities of a single-phase system result. On the other hand a physical process is to be found to create the used inital conditions.

[3] The transversal part $(\boldsymbol{\rho}\vec{\mathbf{q}})_{\perp}$ disappears with divergence formation

The Turbulence depends on an initially assumed motion field

$$\left(\vec{\boldsymbol{\omega}}(\vec{\mathbf{x}}, t_0), \vec{\mathbf{b}}(\vec{\mathbf{x}}, t_0), \frac{\partial \vec{\mathbf{v}}}{\partial t}|_{t_0}\right) \Longrightarrow \vec{\mathbf{q}}(\vec{\mathbf{x}}, t_0),^{4} \tag{7.22}$$

determining the further course, alone. Evaluating $\vec{\mathbf{q}}(\vec{\mathbf{x}}, t_0)$ happens by summation of the terms in the momentum equation. An interaction of geometrodynamics and thermodynamics, maybe assumed in accordance with the Navier-Stokes-equations, does not apply. The geometrodynamics coincidently determines turbulent motion and thermodynamics (pressure, density etc.). But this turbulent geometrodynamics is possible too , if no local thermodynamic equilibrium is existent.

[4]Inserting in equation (7.9)

8. Conclusion

With the installation of the equation system (7.17) a geometrodynamics of turbulence is expressed only obtaining motion quantities i.e. it only consists of velocities and their time and space derivatives. A corresponding statement is made for their initial- and boundary conditions. Special material properties of a fluid (that are the state variables of thermodynamics) may only influence solutions via initial- and boundary conditions. Initial- and boundary conditions determine uniquely the space- and time-scaling of the turbulence field. Thus it may be important to formulate a suitable process of the genesis of such initial conditions (for example the infinitesimal disturbances of the stationary fluid motion by infinitesimal thermodynamic fluctuations as the beginning of turbulence).

The formulation of the geometrodynamics of turbulence does not need an existent local thermodynamic equilibrium.

In the case of fluid turbulence there is no requirement for establishing chaos theories.

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