A possible sign of critical transition

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Abstract

Forecast of critical transitions in a dynamical system is one of the most important research problems in recent time. In this short communication, we discuss a possible novel sign of critical transitions in nonlinear systems. We have shown that the higher order terms of the Taylor series play an important role in determining critical transitions in a system. Moreover, we explain our approach using the Logistic map.

Keywords: `Critical Transition; Tipping point; Taylor series expansion; Logistic map`

1. Introduction

Nonlinear phenomena like bifurcation, chaos, etc. are frequently observed in various dynamical systems. In recent times, one of the most important and challenging research problems is the prediction of critical transitions in such nonlinear systems. Research in this topic is still in its infancy. Only very few theories exist in the literature in this regard. The most important among them is the theory of 'critical slowing down'[1, 2, 3, 4, 5, 6]. This theory states that the recovery rate from any external perturbation tends to zero as a system approaches some critical transition. Although this theory is very simple to understand but at the same time it has been proven to be the most powerful existing theory of predicting bifurcation or tipping points in complex systems. Besides, there are also some other theories in the literature[1]. However, there are also some limitations of these existing theories[7]. The aim of this communication is to reveal a new signature of critical transition in complex systems. In the next section, first we shall discuss our theory and then we present an illustrative example.

2. Main theory

To study the dynamics of any nonlinear system, very often we analyse the dynamics of the corresponding linear system in a sufficiently small neighbourhood of a point, as an approximation of the dynamics of the original system. This is known as the method of linearization, i.e. we linearize the system about a given point and consider essentially the dynamics of that linearised system. Although this linearization remains valid in some sufficiently small neighbourhood of the point. Now a significant observation is that, as we change one or more system parameters continuously, the nonlinearity of the system changes and as a result of that the volume or radius of the neighbourhood of linearization also changes continuously. For example, if the nonlinearity increases continuously then the volume of the neighbourhood of linearization about a point will become smaller and smaller. In either case, the volume is expanded.

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Therefore we ask an immediate question: **" Is it possible to forecast bifurcation or tipping points well in advance, by measuring the change in the volume of the neighbourhoods of linealization in a dynamical system?"**

Let us first describe the issue mathematically. Consider a discrete dynamical system given by

$$X_{n+1} = F(X_n; \lambda_1, \lambda_2, \cdots, \lambda_p)$$

where $F : \mathbb{R}^n \to \mathbb{R}^m$ is a smooth function and $\lambda_1, \lambda_2, \cdots, \lambda_p$ are parameters. Let $X = (x_1, x_2, \cdots, x_n)$. Again suppose $a = (a_1, a_2, \cdots, a_m)$ be a stable fixed point, then we can expand $F(X)$ in a Taylor series like

$$F(X) = F(a) + \sum_{i=1}^{n} F_{x_i}(a)(x_i - a_i) + \frac{1}{2}\sum_{i,j=1}^{n} F_{x_i x_j}(a)(x_i - a_i)(x_j - a_j) + \cdots$$

$$+\frac{1}{k!}\sum_{i_1,i_2,\cdots,i_k=1}^{n} F_{x_{i_1}x_{i_2}\cdots x_{i_n}}(x_{i_1} - a_{i_1})(x_{i_2} - a_{i_2})\cdots(x_{i_k} - ai_k) + \cdots$$

where $F_{x_{i_1}x_{i_2}\cdots x_{i_n}} = \frac{\partial}{\partial x_{i_1}}\frac{\partial}{\partial x_{i_2}}\cdots\frac{\partial}{\partial x_{i_k}}(F)$

Let us call

$$T_k = \frac{1}{k!}\sum_{i_1,i_2,\cdots,i_k=1}^{n} F_{x_{i_1}x_{i_2}\cdots x_{i_n}}(a), \ \ k = 2, 3, \cdots$$

Then each T_k is function of the system parameters $\lambda_1, \lambda_2, \cdots \lambda_p$. Now in general, in a small neighbourhood $N_\delta(a)$ (this is a neighbourhood of radius δ around the stable fixed point $X = a$) of $X = a$ we can well approximate the dynamics of the original nonlinear system by analysing the dynamics of the corresponding linearised system

$$F(X) = F(a) + \sum_{i=1}^{n} F_{x_i}(a)(x_i - a_i).$$

Here we introduce another approach of looking at the higher order neglected terms of the Taylor series which play a very important role in the future dynamics of the system. The coefficients of these higher order terms i.e. T_k are functions of the system parameters. Therefore as we change any system parameter these coefficients are also changed and due to this change the volume (or the radius δ of $N_\delta(a)$) of the neighbourhood of linearisation also changes. Actually this change occurs due to the increasing or decreasing nonlinearity in the system determined completely by the coefficients T_k of the higher order nonlinear terms of the Taylor series. In this case the radius δ of the neighbourhood $N_\delta(a)$ simply becomes a function of the system parameter $\lambda_1, \lambda_2, \cdots, \lambda_p$, i.e. $\delta = \delta(\lambda_1, \lambda_2, \cdots, \lambda_p)$. Therefore as we change some system parameter λ_i, $\delta \to \delta_0$ (δ_0 is some fixed number) as $\lambda_i \to \lambda_0$, where $\lambda_i = \lambda_0$ is the critical value of the parameter λ_i at which a critical transition occurs.

3. Example

Let us take a simple example. Consider the Logistic map

$$x_{n+1} = f(x_n) = \lambda x_n(1 - x_n)$$

Here if we vary the value of the parameter λ inside the interval (0, 4), we observe period doubling root to chaos. $x = 0$ is a stable fixed point of this map in $0 \le \lambda < 1$. Then we consider the Taylor series expansion of $f(x)$ about $x = 0$

$$f(x) = f(x_0) + \lambda(1 - 2x_0)(x - x_0) + \frac{1}{2}(-2\lambda)(x - x_0)^2 = \lambda x - \lambda x^2$$

Hence the coefficient of the second order term of the Taylor series expansion is $-\lambda$, which is a function of λ. Now coefficient of the second order nonlinear term changes as λ moves from 0 to 4. As a result of that, the nonlinearity increases and as a result of that, the neighbourhood, inside which the linear approximation remains valid, shrinks in size gradually and ultimately tends to some critical value, as we vary the parameter λ.

Here we have considered fixed points only in our discussion. However this may be extended for stable higher periodic orbits also as we know that any periodic orbit can be seen as a fixed point of higher composition of a map.

4. Discussion

Here we briefly summarize the whole idea. Suppose we have a complicated system of larger dimension involving several parameters. Now the coefficients of the higher order terms of the Taylor series expansion are function of all these system parameters, therefore as we change those parameter values, the volume of the neighbourhood of the linear approximation will also changes. In other words, as some system parameters approaches a critical value, the volume or radius of the neighbourhood of linearisation also changes and approaches towards some critical value. This phenomena certainly gives an indication of the forthcoming critical transition in a system. Moreover, we can extract some functional relationship between the parameters and volume of the neighbourhood of linearisation, which may help us to draw some significant conclusion about the future dynamics of the system. One of the important issue related to this proposed theory is that the mathematical description or the functional form of the system is needed, which may not be available in every situation. So now the question is how to apply this theory where only a time series is available. This question leads to a new direction of research regarding this theory. In such cases it may be possible to apply the proposed theory if we can approximately figure out the functional form of the system from the available time series.

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